

Entanglement-assisted quantum low-density parity-check codesYuichiro Fujiwara,^{1,*} David Clark,¹ Peter Vandendriessche,² Maarten De Boeck,² and Vladimir D. Tonchev¹¹*Department of Mathematical Sciences, Michigan Technological University, Houghton, Michigan 49931, USA*²*Department of Mathematics, Ghent University, Krijgslaan 281-S22, B-9000 Ghent, Belgium*

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This article develops a general method for constructing entanglement-assisted quantum low-density parity-check (LDPC) codes, which is based on combinatorial design theory. Explicit constructions are given for entanglement-assisted quantum error-correcting codes with many desirable properties. These properties include the requirement of only one initial entanglement bit, high error-correction performance, high rates, and low decoding complexity. The proposed method produces several infinite families of codes with a wide variety of parameters and entanglement requirements. Our framework encompasses the previously known entanglement-assisted quantum LDPC codes having the best error-correction performance and many other codes with better block error rates in simulations over the depolarizing channel. We also determine important parameters of several well-known classes of quantum and classical LDPC codes for previously unsettled cases.

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I. INTRODUCTION

This article develops a general combinatorial method for constructing quantum low-density parity-check (LDPC) codes under the entanglement-assisted stabilizer formalism established by Brun, Devetak, and Hsieh [1]. Our results include many explicit constructions for entanglement-assisted quantum error-correcting codes for a wide range of parameters. We also prove a variety of results for classical error-correcting codes, which directly apply to the quantum setting. Most of the quantum codes designed in this article achieve high error-correction performance and high rates while requiring prescribed amounts of entanglement. These codes can be efficiently decoded by message-passing algorithms such as the sum-product algorithm (for details of iterative probabilistic decoding, see [2]).

The existence of quantum error-correcting codes was one of the most important discoveries in quantum information science [3,4]. Unfortunately, most of the known quantum error-correcting codes lack practical decoding algorithms.

In this article, we focus on the use of LDPC codes in a quantum setting. Classical LDPC codes [5] can be efficiently decoded while achieving information rates close to the classical Shannon limit [6–8]. This extends to the quantum setting: The pioneering works of Hagiwara and Imai [9] and MacKay, Mitchison, and McFadden [10] presented quantum LDPC codes which surpassed, in simulations, all previously known quantum error-correcting codes. Their quantum codes have nearly as low decoding complexity as their classical counterparts.

However, most of the previous results concerning quantum LDPC codes and related efficiently decodable codes have relied on the stabilizer formalism, which severely restricts the classical codes which can be used. The difficulty in developing constructions for nonstabilizer codes was also a substantial obstacle.

Our results use the recently developed theory of entanglement-assisted quantum error-correcting codes (EAQECCs) [1,11–13]. The entanglement-assisted stabilizer formalism allows the use of arbitrary classical binary or quaternary linear codes for quantum data transmission and error correction by using shared entanglement [14,15]. Previous work related to entanglement-assisted quantum LDPC codes is due to Hsieh, Brun, and Devetak [16] and Hsieh, Yen, and Hsu [17].

The major difficulty in using classical LDPC codes in the entanglement-assisted quantum setting is that very little is known about methods for designing EAQECCs requiring desirable amounts of entanglement. While entanglement-assisted quantum LDPC codes can achieve both notable error-correction performance and low decoding complexity, the resulting quantum codes might require too much entanglement to be usable; in general, entanglement is a valuable resource [15]. In some situations, one might wish to effectively take advantage of high-performance codes requiring a larger amount of entanglement [1,12]. To the best of the authors' knowledge, no general methods have been developed which allow the code designer flexibility in choice of parameters and required amounts of entanglement.

Our primary focus in this article is to show that it is possible to create infinite classes of EAQECCs which consume prescribed amounts of entanglement and achieve good error-correction performance while allowing efficient decoding. Our methods are flexible and address various situations, including the extreme case when an EAQECC requires only one pre-existing entanglement bit.

The entanglement-assisted quantum LDPC codes which we construct include quantum analogs of the well-known finite geometry LDPC codes originally proposed by Kou, Lin, and Fossorier [18] (see also [19,20]) and LDPC codes from balanced incomplete block designs that achieve the upper bound on the rate for a classical regular LDPC code with girth 6 proposed independently by several authors (see [21] and references therein). Some classes of our codes outperform previously proposed quantum LDPC codes having the best known error-correction performance [9,10,16,17].

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Our primary tools come from combinatorial design theory, which plays an important role in classical coding theory [22] and also gave several classes of stabilizer codes in quantum coding theory [23–27]. The use of combinatorial design theory allows us to exactly determine or give tighter bounds on the parameters of the finite geometry LDPC codes in both quantum and classical settings. Comprehensive lists of the parameters of these codes are given in Tables XIV and XV in Appendix B.

In Sec. II, we outline our framework for designing entanglement-assisted quantum LDPC codes by using combinatorial design theory. Section III gives explicit constructions for entanglement-assisted quantum LDPC codes based on finite geometries and related combinatorial structures. Additional results concerning the well-known classical finite geometry LDPC codes are also given in this section. Section IV presents simulation results of our entanglement-assisted quantum LDPC codes and discusses their performance over the depolarizing channel. Section V contains concluding remarks and discusses some related problems that can be treated with the techniques developed in this article.

II. COMBINATORIAL ENTANGLEMENT-ASSISTED QUANTUM LDPC CODES

In this section we give a general construction method for entanglement-assisted quantum LDPC codes based on combinatorial designs. We do not describe the theory of classical LDPC codes in detail here; instead, we refer the reader to [2,21] and references therein. Relations between quantum error-correcting codes and LDPC codes are concisely yet thoroughly explained in [10,16]. Basic notions related to LDPC codes and their relations to combinatorial designs can be found in [28]. For a detailed treatment of the entanglement-assisted stabilizer formalism, we refer the reader to [1,12–14].

In Sec. II A we introduce necessary notions from coding theory and combinatorial design theory. A general method for designing entanglement-assisted quantum LDPC codes is presented in Sec. II B.

A. Preliminaries

An $[[n,k;c]]$ EAQEC encodes k logical qubits into n physical qubits with the help of c copies of maximally entangled states. As in classical coding theory, n is the *length* of the EAQEC and k the *dimension*. We say that the EAQEC requires c *ebits*. An $[[n,k;d;c]]$ EAQEC with *distance* d is referred to as an $[[n,k,d;c]]$ code.

The *rate* of an $[[n,k;d;c]]$ EAQEC is defined to be $\frac{k}{n}$. The ratio $\frac{k-c}{n}$ is called the *net rate*. The latter figure describes the rate of an EAQEC when used as a catalytic quantum error-correcting code to create c new bits of shared entanglement [1,12].

Throughout this article, matrix operations are performed over \mathbb{F}_2 , the finite field of order two. The ranks of matrices are also calculated over \mathbb{F}_2 .

We employ the Calderbank-Shor-Steane (CSS) construction [1,4,14,29]. Usually, the CSS construction uses a minimal set of independent generators to construct an EAQEC. Hence, the construction is often described by using a classical

binary linear code with a parity-check matrix of full rank. However, in actual decoding steps, sparse-graph codes may take advantage of redundant parity-check equations to improve error-correction performance. Because the extended syndrome can be obtained in polynomial time without additional quantum interactions, we use the following formulation of the CSS construction for EAQECs.

Theorem 1: Hsieh, Brun, and Devetak [16]. If there exists a classical binary $[n,k,d]$ code with parity-check matrix H , then there exists an $[[n,2k-n+c,d;c]]$ EAQEC, where $c = \text{rank } HH^T$.

Note that H may contain redundant rows which are related only to classical operations to infer the noise by a message-passing algorithm.

We apply Theorem 1 to classical sparse-graph codes. An LDPC code is typically defined as a binary linear code with parity-check matrix H in which every row and column is sparse. In this article we consider LDPC codes with parity-check matrices whose rows and columns contain only small numbers of ones so that simple message-passing algorithms can efficiently give good performance in decoding.

Proposition 1. An LDPC code with parity-check matrix H with n columns and minimum distance d defines a classical binary $[n,n-\text{rank } H,d]$ code, which yields an $[[n,n-2\text{rank } H+\text{rank } HH^T,d;\text{rank } HH^T]]$ EAQEC.

The *Tanner graph* of an $m \times n$ parity-check matrix H is the bipartite graph consisting of n bit vertices and m parity-check vertices, where an edge joins a bit vertex to a parity-check vertex if that bit is included in the corresponding parity-check equation. A *cycle* in a graph is a sequence of connected vertices which starts and ends at the same vertex in the graph and contains no other vertices more than once. The *girth* of a parity-check matrix is the length of a shortest cycle in the corresponding Tanner graph. Short cycles can severely reduce the performance of an otherwise well-designed LDPC code. In fact, one of the greatest obstacles to the development of a general theory of LDPC codes in the quantum setting is the difficulty of avoiding cycles of length four (See, for example, [9,10,30,31]). In order to improve error-correction performance, we generally only treat LDPC codes with girth at least 6.

The *weight* of a row or column of a binary matrix is its Hamming weight, that is, the number of ones in it. An LDPC code is *regular* if its parity-check matrix H has constant row and column weights and *irregular* otherwise. Regular LDPC codes are known to be able to achieve high error-correction performance. Irregular LDPC codes allow the code designer to optimize characteristics of performance by a careful choice of row weights and column weights [6–8].

We now define several combinatorial structures, which we need in Sec. II B and the subsequent sections. For additional facts and design theoretical results, the interested reader is referred to [32].

An *incidence structure* is an ordered pair (V,\mathcal{B}) such that V is a finite set of *points*, and \mathcal{B} is a family of subsets of V , called *blocks*. A *point-by-block incidence matrix* of an incidence structure (V,\mathcal{B}) is a binary $v \times b$ matrix $H = (h_{i,j})$ in which rows are indexed by points, columns are indexed by blocks, and $h_{i,j} = 1$ if the i th point is contained in the j th block, and $h_{i,j} = 0$ otherwise. A *block-by-point incidence*

matrix of (V, \mathcal{B}) is the transposed point-by-block incidence matrix H^T .

Any LDPC code can be associated with an incidence structure by interpreting its parity-check matrix as an incidence matrix. The converse also holds as long as the considered incidence matrix is sparse.

The current article focuses on incidence structures which have been extensively studied in combinatorics. This allows us to effectively exploit combinatorial design theory to develop a framework for designing entanglement-assisted quantum LDPC codes.

A 2 - (v, μ, λ) design is an incidence structure (V, \mathcal{B}) , where V is a set of cardinality v and \mathcal{B} is a family of μ -subsets of V such that each pair of points is contained in exactly λ blocks. We refer to the parameters v , μ , and λ as the *order*, *block size*, and *index* of a 2 -design. Note that the block size of a 2 -design is usually written as k in the combinatorial literature. To avoid any confusion with the dimension of a code, we use μ instead.

The number $b = |\mathcal{B}|$ of blocks in a 2 - (v, μ, λ) design is determined by the design parameters:

$$b = |\mathcal{B}| = \frac{v(v-1)}{\mu(\mu-1)}\lambda. \tag{1}$$

A 2 -design is called *symmetric* if $b = v$.

Every point of a 2 - (v, μ, λ) design occurs in exactly r blocks, where

$$r = \frac{v-1}{\mu-1}\lambda. \tag{2}$$

The number r is called the *replication number* of the design. A point-by-block incidence matrix H of a 2 - (v, μ, λ) design satisfies the equation

$$HH^T = (r - \lambda)I + \lambda J, \tag{3}$$

where I is the identity matrix and J is the $v \times v$ all-one matrix. Since r and b are integers, it follows that the following two conditions,

$$\begin{aligned} \lambda(v-1) &\equiv 0 \pmod{\mu-1}, \\ \lambda v(v-1) &\equiv 0 \pmod{\mu(\mu-1)}, \end{aligned} \tag{4}$$

are necessary conditions for the existence of a 2 - (v, μ, λ) design.

If the block size μ and index λ are relatively small, an incidence matrix of a 2 - (v, μ, λ) design is sparse. Hence, a point-by-block incidence matrix of a 2 - (v, μ, λ) design can be viewed as a parity-check matrix H of a regular LDPC code with constant row weight r and constant column weight μ . Similarly, a block-by-point incidence matrix defines a code with constant row weight μ and constant column weight r . In this article, incidence matrices are generally point-by-block unless it is specifically noted otherwise. In the cases when block-by-point matrices are desirable, the notation H^T is used.

A substantial part of this article deals with one of the most fundamental incidence structures in combinatorial design theory. A *Steiner 2-design*, denoted by $S(2, \mu, v)$, is a 2 - $(v, \mu, 1)$ design. A *Steiner triple system* of order v , denoted by $STS(v)$, is a Steiner 2-design with block size three. The $S(2, \mu, v)$'s are *trivial* Steiner 2-designs if $v \leq \mu$. We generally do not

consider trivial designs to be Steiner 2-designs unless they play an important role.

It is easy to see that both point-by-block and block-by-point incidence matrices of an $S(2, \mu, v)$ give regular LDPC codes with girth 6 (see, for example, [33]).

B. General combinatorial constructions

In this subsection we present a general framework for designing entanglement-assisted quantum LDPC codes based on combinatorial design theory. Specialized construction methods for desirable EAQECCs in this framework are illustrated in Sec. III.

The following propositions are derived from Theorem 1 by using incidence matrices as parity-check matrices of binary LDPC codes.

Proposition 2. Let H be a point-by-block incidence matrix of an incidence structure (V, \mathcal{B}) . Then there exists a $[[|\mathcal{B}|, |\mathcal{B}| - 2\text{rank } H + \text{rank } HH^T; \text{rank } HH^T]]$ EAQECC.

Proposition 3. Let H^T be a block-by-point incidence matrix of an incidence structure (V, \mathcal{B}) . Then there exists a $[[|V|, |V| - 2\text{rank } H + \text{rank } H^T H; \text{rank } H^T H]]$ EAQECC.

We employ the following two theorems.

Theorem 2: Hillebrandt [34]. The rank of an incidence matrix H of an $S(2, \mu, v)$ satisfies the following inequalities:

$$\left\lceil \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{(v-1)(v-\mu)}{\mu}} \right\rceil \leq \text{rank } H \leq v.$$

Theorem 3: Hamada [35]. If H is an incidence matrix of an $S(2, \mu, v)$ with even replication number $r = \frac{v-1}{\mu-1}$, then

$$\text{rank } H = \begin{cases} v-1 & \text{when } \mu \text{ is even,} \\ v \text{ or } v-1 & \text{when } \mu \text{ is odd.} \end{cases}$$

We now give three simple constructions by applying Propositions 2 and 3 to incidence matrices of Steiner 2-designs. These constructions will be specialized and modified to give desirable codes.

Theorem 4: High-rate 1-ebit code. Let H be a point-by-block incidence matrix of an $S(2, \mu, v)$. Suppose $r = \frac{v-1}{\mu-1}$ is odd. Then H has row weight r , column weight μ , girth 6, and the corresponding $[[n, k; c]]$ EAQECC satisfies the following conditions:

$$\begin{aligned} n &= \frac{v(v-1)}{\mu(\mu-1)}, \\ \frac{vr}{\mu} - 2v + 1 &\leq k \leq \frac{vr}{\mu} \\ -2 \left\lceil \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{(v-1)(v-\mu)}{\mu}} \right\rceil + 1, \\ c &= 1. \end{aligned}$$

Proof. By Proposition 2 and Theorem 2, it suffices to prove that $\text{rank } HH^T = 1$. Because r is odd, Eq. (3) reduces to $HH^T = J$, which implies that the rank of HH^T is equal to one. ■

Theorem 5: High-rate high-consumption code. Let H be a point-by-block incidence matrix of an $S(2, \mu, v)$. Suppose $r = \frac{v-1}{\mu-1}$ is even. Then H has row weight r , column weight

μ , girth 6, and the corresponding $[[n, k; c]]$ EAQECC satisfies the following conditions:

$$n = \frac{v(v-1)}{\mu(\mu-1)},$$

$$k = \begin{cases} \frac{vr}{\mu} - v + 1 & \text{when } \mu \text{ is even,} \\ \frac{vr}{\mu} - v + 1 \text{ or } \frac{vr}{\mu} - v - 1 & \text{when } \mu \text{ is odd,} \end{cases}$$

$$c = v - 1.$$

Proof. By Proposition 2 and Theorem 3, it suffices to prove that $\text{rank } HH^T = v - 1$. Because r is even, Eq. (3) reduces to

$$HH^T = \begin{bmatrix} 0 & 1 & & 1 \\ 1 & 0 & \cdots & 1 \\ & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix},$$

that is, a matrix containing zeros on the diagonal and ones in the other entries. Because $r = \frac{v-1}{\mu-1}$ is even, v is odd. Hence, we have $\text{rank } HH^T = v - 1$ as desired. ■

Theorem 6: Low-rate high-redundancy code. Let H^T be a block-by-point incidence matrix of an $S(2, \mu, v)$. Then H has row weight μ , column weight r , girth 6, and the corresponding $[[n, k; c]]$ EAQECC satisfies the following conditions:

$$n = v,$$

$$k \leq v - 2 \left[\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{(v-1)(v-\mu)}{\mu}} \right] + c,$$

$$c \geq 1.$$

Proof. Let H^T be a block-by-point incidence matrix of an $S(2, \mu, v)$. Since any nontrivial $S(2, \mu, v)$ contains a pair of blocks that share exactly one point, we have $\text{rank } H^T H \geq 1$. Applying Proposition 3 to Theorem 2 completes the proof. ■

It is worth mentioning that a weaker version of Theorem 4 was used in the context of integrated optics and photonic crystal technology [25]. Also notable is that Theorems 4 and 5 can be easily extended to the case where pre-existing entanglement is not available. For example, quantum LDPC codes that do not require entanglement can be obtained by applying the extra-column method used in Construction U in [10] and the CSS construction to $S(2, \mu, v)$'s in the same manner as in Proposition 2. Aly's construction for quantum LDPC codes [23] is a special case of this extended method. Djordjevic's construction for quantum LDPC codes [24] can be obtained by applying the CSS construction to 2-designs of even index in the same way as in Proposition 2.

The existence of 2-designs is discussed in Appendix A, which provides Steiner 2-designs necessary to obtain several infinite families of new entanglement-assisted quantum LDPC codes from Theorems 4, 5, and 6. Before applying our theorems to specific $S(2, \mu, v)$'s, we explore general characteristics of our EAQECCs and further develop methods for designing desirable codes.

Theorem 4 yields entanglement-assisted quantum LDPC codes with very high net rates and various lengths while requiring only one ebit. Theorem 5 gives codes which have very high net rates and naturally take advantage of

larger numbers of ebits when there is an adequate supply of entanglement. Since $\text{rank } HH^T \leq \text{rank } H$ holds for any parity-check matrix H , the required amounts of entanglement of high-rate codes in Theorem 5 are expected to be relatively low when compared with randomly chosen codes of the same lengths. Theorem 6 generates entanglement-assisted quantum LDPC codes which can correct many quantum errors by taking advantage of the higher redundancy. The high error-correction performance of these codes is demonstrated in simulations in Sec. IV.

When a parity-check matrix H of an $S(2, \mu, v)$ is of full rank v , the corresponding classical LDPC code in Theorems 4 and 5 achieves an upper bound on the rate for an LDPC code with girth 6.

Theorem 7: MacKay and Davey [36]. Let H be a $v \times n$ parity-check matrix of a classical regular LDPC code of length n , column weight μ , and girth 6. Let also $\text{rank } H = v$. Then it holds that $n \leq \frac{v(v-1)}{\mu(\mu-1)}$, where equality holds if and only if H is an incidence matrix of an $S(2, \mu, v)$.

It follows that EAQECCs based on Steiner 2-designs achieve the highest possible net rates for quantum LDPC codes with girth at least 6 constructed from full-rank parity-check matrices with constant column weights through the CSS construction.

The rank of an incidence matrix of an $S(2, \mu, v)$ may not be full depending on the structure of the design. If one wishes a parity-check matrix to be regular and full rank at the same time, it is important to choose an $S(2, \mu, v)$ with a full-rank incidence matrix. This can always be done for the case when $\mu = 3$, except for $v = 7$ [37]. For a more detailed treatment of the ranks of $S(2, \mu, v)$'s, we refer the reader to [35, 38, 39].

In general, the code minimum distance plays less of a role in the performance of sum-product decoding than maximum-likelihood decoding [10]. Therefore, we explore in detail the distance d of $[[n, k, d; c]]$ EAQECCs based on LDPC codes only when it is of great theoretical interest. Because codes derived from finite geometries are of great importance in coding theory, the distances of EAQECCs obtained from finite geometries are investigated in detail in Sec. III.

Here we briefly review the minimum distances of LDPC codes based on Steiner 2-designs. A pair of $S(2, \mu, v)$'s which are not mutually isomorphic may give different minimum distances. The tightest known upper and lower bounds on the minimum distance of an LDPC code based on an STS(v) can be found in the very large scale integration (VLSI) literature as bounds on even freeness.

Theorem 8: Fujiwara and Colbourn [40]. The minimum distance d of a classical binary linear code whose parity-check matrix forms an incidence matrix of a nontrivial STS(v) satisfies $4 \leq d \leq 8$.

A carefully chosen triple system can have a good topological structure which gives good decoding performance. If conditions require larger minimum distances, the code designer may use either block-by-point incidence matrices or $S(2, \mu, v)$'s of larger block sizes. For known results on minimum distances, girths, and related characteristics of LDPC codes based on combinatorial designs, the reader is referred to [40–42] and references therein.

In what follows, we describe general guidelines for designing entanglement-assisted quantum LDPC codes with

desired parameters and properties by exploiting codes we have presented in this section.

We first consider an $[[n, k; c]]$ EAQECC requiring only a small amount of entanglement. The extreme case is when $c = 1$. The following theorem gives infinitely many such EAQECCs having extremely high rates and low decoding complexity.

Theorem 9. Let v and μ be positive integers satisfying $v - 1 \equiv 0 \pmod{\mu - 1}$ and $v(v - 1) \equiv 0 \pmod{\mu(\mu - 1)}$. Suppose also that $\frac{v-1}{\mu-1}$ is odd. Then for all sufficiently large v and some k satisfying the condition of Theorem 4, there exists an $[[\frac{v(v-1)}{\mu(\mu-1)}, k; 1]]$ EAQECC.

Proof. Use Theorem 42 from Appendix A, which guarantees the existence of an $S(2, \mu, v)$ for all sufficiently large v , and apply Theorem 4. ■

Similarly, applying Theorem 4 to known $S(2, \mu, v)$'s with small v discussed in Appendix A gives $[[n, k; 1]]$ EAQECCs of shorter length n .

In general, the error floor of a well-designed LDPC code is not dominated by low-weight code words. Nonetheless, it is desirable to carefully choose an $S(2, \mu, v)$ when applying our simple constructions so that the resulting code has a promising topological structure. While incidence matrices of $S(2, \mu, v)$'s have long been investigated in various fields, it appears to be difficult to achieve the known upper bounds on the minimum distance of an LDPC code based on an incidence matrix of an $S(2, \mu, v)$. In fact, it is conjectured that the known upper bounds are generally not achievable even for the case $\mu = 3$ [41].

An STS is 4-even-free (or *anti-Pasch*) if its incidence matrix gives a classical LDPC code with minimum distance 5 or greater. A 4-even-free STS(v) exists for all $v \neq 7, 13$ satisfying the necessary conditions (4) [43]. It is conjectured that an incidence matrix of a 4-even-free STS(v) gives the largest possible minimum distance [41].

Theorem 10. There exists a $[[\frac{v(v-1)}{6}, k, d; 1]]$ EAQECC with $k \geq \frac{v(v-1)}{6} - 2v + 1$ and $d \geq 5$ for every $v \equiv 3, 7 \pmod{12}$ except for $v = 7$.

Proof. If $v \equiv 3, 7 \pmod{12}$, then the replication number of an STS(v) is odd. Applying Theorem 4 to a 4-even-free STS(v) completes the proof. ■

A block-by-point incidence matrix of a symmetric $S(2, \mu, v)$ can also be viewed as a point-by-block incidence matrix of a Steiner 2-design of the same parameters [44]. Hence, Theorems 4 and 6 can overlap when symmetric designs are employed. This special case gives the EAQECCs with $c = 1$ and good error-correction performance originally presented in [17]. For completeness, we give a simple proof by using the following two theorems.

Theorem 11. For every integer $t \geq 1$, there exists a symmetric $S(2, 2^t + 1, 4^t + 2^t + 1)$ whose incidence matrix H satisfies $\text{rank } H = 3^t + 1$.

Proof. Take as $S(2, 2^t + 1, 4^t + 2^t + 1)$ the Desarguesian projective plane of order 2^t , whose incidence matrix has rank $3^t + 1$ [45]. ■

Theorem 12: Calkin, Key, and de Resmini [46]. Let H^T be a block-by-point incidence matrix of a symmetric $S(2, 2^t + 1, 4^t + 2^t + 1)$ being the Desarguesian projective plane $\text{PG}(2, 2^t)$. Then H^T defines a classical binary linear $[4^t + 2^t + 1, 4^t + 2^t - 3^t, 2^t + 2]$ code.

Now as a corollary of Theorems 4 and 6 and the preceding two theorems, we obtain the following result.

Theorem 13. For every integer $t \geq 1$, there exists a $[[4^t + 2^t + 1, 4^t + 2^t - 2 \times 3^t, 2^t + 2; 1]]$ EAQECC.

EAQECCs of this kind can be seen as quantum analogs of special type I PG-LDPC codes, which have notable error-correction performance in the classical setting [18–20]. Because of the direct correspondence between entanglement-assisted quantum codes and classical codes, these EAQECCs inherit excellent error-correction performance while consuming only one initial ebit. We further investigate entanglement-assisted quantum LDPC codes based on $S(2, \mu, v)$'s with large minimum distances in Sec. III.

Next we present general combinatorial methods for designing EAQECCs with relatively small c and better error-correction performance. The main idea is that we discard some columns from an incidence matrix of an $S(2, \mu, v)$ and then apply Proposition 2 as we did in Theorem 4. Our methods encompass the rate control technique for classical LDPC codes proposed in [47] as a special case.

Let (V, \mathcal{B}) be an $S(2, \mu, v)$. Take two subsets $V' \subsetneq V$ and $\mathcal{B}' \subsetneq \mathcal{B}$. The pair (V', \mathcal{B}') is called a *proper subdesign* of block size μ if it is an $S(2, \mu, |V'|)$. Since we do not consider other kinds of subdesigns, we simply call a proper subdesign (V', \mathcal{B}') of block size μ a subdesign. A pair of subdesigns (V', \mathcal{B}') and (V'', \mathcal{B}'') of an $S(2, \mu, v)$ are *pointwise disjoint* if $V' \cap V'' = \emptyset$.

Theorem 14. Let (V, \mathcal{B}) be an $S(2, \mu, v)$ with odd $r = \frac{v-1}{\mu-1}$. Assume that (V, \mathcal{B}) contains j pointwise mutually disjoint subdesigns (V_i, \mathcal{B}_i) , $1 \leq i \leq j$, such that $\bigcup_{i=1}^j V_i \subsetneq V$ and each (V_i, \mathcal{B}_i) has odd replication number. Then there exists an $[[n, k; c]]$ EAQECC satisfying the following conditions:

$$n = \frac{v(v-1)}{\mu(\mu-1)} - \left| \bigcup \mathcal{B}_i \right|,$$

$$c = j + 1.$$

Proof. Take an arbitrary incidence matrix H of an $S(2, \mu, v)$ with odd r . Delete j pointwise mutually disjoint subdesigns (V_i, \mathcal{B}_i) , each of which has an odd replication number. It is always possible to reorder the rows and columns of the resulting incidence matrix H' such that $H'H'^T$ has the form

$$H'H'^T = \begin{bmatrix} J & J & \cdots & J \\ J & 0_i & \cdots & J \\ \vdots & \vdots & \ddots & \vdots \\ J & J & \cdots & 0_j \end{bmatrix},$$

where 0_i is a $|V_i| \times |V_i|$ zero matrix and each J is an all-one matrix of appropriate size. It is easy to see that $\text{rank } H'H'^T = j + 1$. Applying Proposition 2 to H' completes the proof. ■

Deleting subdesigns always shortens the length of the corresponding code. Discarding columns will not decrease the minimum distance or the girth. The rank of the parity-check matrix is unlikely to change. In this sense, we expect EAQECCs obtained through subdesign deletion to have better error-correction performance than the original code. We demonstrate this effect in simulations in Sec. IV.

In general, deleting a subdesign makes a parity-check matrix slightly irregular. If this irregularity is not desirable

because of particular circumstances or conditions, it can be alleviated by discarding more pointwise disjoint subdesigns. In fact, if we delete subdesigns of the same order such that each point belongs to one deleted subdesign, we obtain a regular parity-check matrix again. The following construction demonstrates this.

Let (V, \mathcal{B}) be an $S(2, \mu, v)$ and \mathcal{S} a set of Steiner 2-designs $(V_i, \mathcal{B}_i), 1 \leq i \leq |\mathcal{S}|$, where $V_1, \dots, V_{|\mathcal{S}|}$ partition V , that is, $\bigcup V_i = V$ and $V_i \cap V_j = \emptyset$ for all $i \neq j$. Then \mathcal{S} is called a *Steiner spread* in (V, \mathcal{B}) if each (V_i, \mathcal{B}_i) forms a subdesign $S(2, \mu, |V_i|)$ of (V, \mathcal{B}) .

Theorem 15. Let (V, \mathcal{B}) be an $S(2, \mu, v)$ with odd replication number $r = \frac{v-1}{\mu-1}$. Assume that (V, \mathcal{B}) contains a Steiner spread \mathcal{S} , where each subdesign (V_i, \mathcal{B}_i) has an odd replication number. Then there exists an $[[n, k; c]]$ EAQECC satisfying the following conditions:

$$n = \frac{v(v-1)}{\mu(\mu-1)} - \left| \bigcup \mathcal{B}_i \right|,$$

$$c = \begin{cases} |\mathcal{S}| - 1 & \text{when } |\mathcal{S}| \text{ is odd,} \\ |\mathcal{S}| & \text{when } |\mathcal{S}| \text{ is even.} \end{cases}$$

Moreover, if $|V_i| = |V_{i'}| = w$ for all i and i' , then the parity-check matrix of the corresponding LDPC code is regular and has row weight $r - \frac{w-1}{\mu-1}$ and column weight μ .

Proof. Let H be an incidence matrix of an $S(2, \mu, v)$ with odd r which contains a Steiner spread \mathcal{S} . Delete all members of the Steiner spread from (V, \mathcal{B}) . By following the same argument as in the proof of Theorem 14, it is straightforward to see that $\text{rank } HH^T = |\mathcal{S}| - 1$ when $|\mathcal{S}|$ is odd and $|\mathcal{S}|$ otherwise. If $|V_i| = |V_{i'}| = w$ for all i and i' , each subdesign has the same replication number $\frac{w-1}{\mu-1}$. Hence, the resulting code is regular. ■

When there is an adequate supply of entanglement, it may be acceptable to exploit a relatively large amount of entanglement to improve error-correction performance while keeping similar characteristics of high rate codes. Deleting an $S(2, \mu, w)$ with even replication number $\frac{w-1}{\mu-1}$ increases the required amount of entanglement to a slightly larger extent.

Theorem 16. Let (V, \mathcal{B}) be an $S(2, \mu, v)$ with odd replication number $r = \frac{v-1}{\mu-1}$. Assume that (V, \mathcal{B}) contains j pointwise mutually disjoint subdesigns $(V_i, \mathcal{B}_i), 1 \leq i \leq j$, such that $\bigcup_{i=1}^j V_i \subseteq V$ and each (V_i, \mathcal{B}_i) has an even replication number. Then there exists an $[[n, k; c]]$ EAQECC satisfying the following conditions:

$$n = \frac{v(v-1)}{\mu(\mu-1)} - \left| \bigcup \mathcal{B}_i \right|,$$

$$c = \sum_{i=1}^j (|V_i| - 1) + 1.$$

Moreover, if the subdesigns (V_i, \mathcal{B}_i) for $1 \leq i \leq j$ form a Steiner spread with $|V_i| = |V_{i'}| = w$ for all i and i' , then the parity-check matrix of the corresponding LDPC code is regular and has row weight $r - \frac{w-1}{\mu-1}$ and column weight μ .

Proof. Take an arbitrary incidence matrix H of an $S(2, \mu, v)$ with odd r . Delete j point-wise mutually disjoint subdesigns (V_i, \mathcal{B}_i) , each of which has an even replication number. If $\bigcup_{i=1}^j V_i \subsetneq V$, it is always possible to reorder the columns of

the resulting incidence matrix H' such that $H'H'^T$ is of the form

$$H'H'^T = \begin{bmatrix} J & J & & J \\ J & I_1 & \cdots & J \\ & \vdots & \ddots & \vdots \\ J & J & \cdots & I_j \end{bmatrix},$$

where I_i is the $|V_i| \times |V_i|$ identity matrix and each J is an all-one matrix of appropriate size. Because each I_i has V_i independent rows and each $|V_i|$ is odd, $\text{rank } H'H'^T = \sum_{i=1}^j (|V_i| - 1) + 1$. Applying Proposition 2 to H' gives $c = \sum_{i=1}^j (|V_i| - 1) + 1$. If $\bigcup_{i=1}^j V_i = V$, we have identity matrices across the diagonal of $H'H'^T$. Hence, we have $c = \sum_{i=1}^j (|V_i| - 1) + 1$ again. If each V_i is of the same size, it is straightforward to see that the resulting code is regular. ■

When irregularity in a parity-check matrix is acceptable or favorable, the code designer can combine the techniques of Theorems 14, 15, and 16. The required amount of entanglement is readily computed by the same argument as before.

In general, subdesign deletion changes the parameters of a code in a gradual manner. Hence, these techniques are also useful when one would like an EAQECC of specific length or dimension. While we only employed Theorem 4 in the preceding arguments, Theorem 5 can also be used in a straightforward manner to fine tune the parameters of EAQECCs.

In order to exploit the subdesign deletion techniques, one needs Steiner 2-designs having subdesigns or preferably Steiner spreads of appropriate sizes. We conclude this section with a brief review of known general results and useful theorems for finding $S(2, \mu, v)$ with subdesigns and Steiner spreads. For a more thorough treatment, the reader is referred to [32,44] and references therein.

The well-known Doyen-Wilson theorem [48] states that one can always find an $\text{STS}(v)$ containing an $\text{STS}(w)$ as a subdesign as long as both v and w satisfy the necessary conditions for the existence of an STS and $v \geq 2w + 1$. The following is a general asymptotic theorem on Steiner 2-designs having subdesigns.

Theorem 17: Fujiwara [49]. Let $\mu \geq 2$ be a positive integer and $w \equiv 1 \pmod{\mu(\mu-1)}$. Then there exist a constant number w_0 depending on μ and a constant number v_0 depending on w and μ such that if $w > w_0$ and $v > v_0$ satisfies the conditions $v - 1 \equiv 0 \pmod{\mu - 1}$ and $v(v - 1) \equiv 0 \pmod{\mu(\mu - 1)}$, then there exists an $S(2, \mu, v)$ having an $S(2, \mu, w)$ as a subdesign.

Theorem 17 states that one can always find an $S(2, \mu, v)$ having an $S(2, \mu, w)$ as a subdesign as long as v is a sufficiently large integer satisfying the necessary conditions (4) and w is a sufficiently large integer satisfying $w \equiv 1 \pmod{\mu(\mu - 1)}$.

Steiner spreads are closely related to a special kind of combinatorial design. A *group divisible design* (GDD) with index one is a triple $(V, \mathcal{G}, \mathcal{B})$, where

- (i) V is a finite set of elements called *points*;
- (ii) \mathcal{G} is a family of subsets of V , called *groups*, which partition V ;

(iii) \mathcal{B} is a collection of subsets of V , called *blocks*, such that every pair of points from distinct groups occurs in exactly one block;

(iv) $|G \cap B| \leq 1$ for all $G \in \mathcal{G}$ and $B \in \mathcal{B}$.

If all groups are of the same size g , all blocks are of the same size μ , and $|\mathcal{G}| = t$, one refers to the design as a μ -GDD of type g^t .

Theorem 18. The existence of an $S(2, \mu, g)$ and a μ -GDD $(V, \mathcal{G}, \mathcal{B})$ of type g^t with index one implies the existence of an $S(2, \mu, gt)$ having a Steiner spread \mathcal{S} , where each member of \mathcal{S} is an $S(2, \mu, g)$.

Proof. Let $(V, \mathcal{G}, \mathcal{B})$ be a μ -GDD of type g^t with index one and (V', \mathcal{B}') an $S(2, \mu, g)$. For each $G \in \mathcal{G}$, we construct an $S(2, \mu, g)$, (G, \mathcal{B}'_G) , by mapping each point of (V', \mathcal{B}') to an element of G by an arbitrary bijection $\pi_G : V' \rightarrow G$. Define $\mathcal{C} = \bigcup_{G \in \mathcal{G}} \mathcal{B}'_G$. It is straightforward to check that $(V, \mathcal{B} \cup \mathcal{C})$ is an $S(2, \mu, gt)$ having a Steiner spread whose members are all $S(2, \mu, g)$'s. ■

The preceding theorem is useful to obtain regular LDPC codes through Theorems 15 and 16 and similar subdesign deletion techniques based on Theorem 5. One can also modify Theorem 18 for the case when a GDD has different group sizes by a similar argument. The existence of GDDs and their constructions have been extensively investigated in combinatorial design theory. For a comprehensive list of known existence results on GDDs, we refer the reader to [44].

III. FINITE GEOMETRY CODES

In this section, we demonstrate applications of our general designing methods by using combinatorial designs arising from finite geometries.

The classical LDPC codes obtained from finite geometries are known to have remarkable error-correction abilities. By using these codes, we generate infinitely many high-performance entanglement-assisted quantum LDPC codes having numerous Steiner spreads of various sizes. The various Steiner spreads in each code allow the code designer to flexibly fine tune the parameters and error-correction performance.

This section is divided into three subsections. Section III A studies entanglement-assisted quantum LDPC codes of girth 6 obtained from projective geometries. Codes based on affine geometries are investigated in Sec. III B. In Sec. III C we investigate slightly modified affine geometry codes, called Euclidean geometry codes. Classical LDPC codes based on these three kinds of finite geometries are called *finite-geometry LDPC codes* or simply *FG-LDPC codes*.

Many of the results presented in this section can also be seen as new results on classical finite-geometry LDPC codes. In particular, properties of finite geometries have been independently studied in the combinatorial literature, and hence many of the “known” results are new results in the field of LDPC codes. For the convenience of the reader, we summarize our results on fundamental parameters of LDPC codes from finite geometries in Tables XIV and XV in Appendix B. Lengths, dimensions, and minimum distances of the FG-LDPC codes with girth 6 from projective geometry $PG(m, q)$, affine geometry $AG(m, q)$, and Euclidean geometry $EG(2, 2^t)$ are all determined. Specifically for EAQECCs based on FG-LDPC codes, we also determine the required amounts

of entanglement for most cases. For a few cases, we give upper bounds on the required amount of entanglement.

A. Projective geometry codes

We begin with EAQECCs obtained from finite projective geometries. The use of projective geometries for constructing EAQECCs first appeared in the work of Hsieh, Yen, and Hsu [17]. This subsection illustrates how our combinatorial framework generalizes their method and determines fundamental parameters of quantum and classical LDPC codes obtained from $PG(m, q)$.

Points of the m -dimensional projective geometry $PG(m, q)$ over \mathbb{F}_q are the one-dimensional subspaces of \mathbb{F}_q^{m+1} . The i -dimensional projective subspaces of $PG(m, q)$ are the $(i + 1)$ -dimensional vector subspaces of \mathbb{F}_q^{m+1} . The points and lines of $PG(m, q)$ form an $S(2, q + 1, \frac{q^{m+1}-1}{q-1})$, denoted by $PG_1(m, q)$, having $\frac{(q^{m+1}-1)(q^m-1)}{(q^2-1)(q-1)}$ blocks and replication number $\frac{q^m-1}{q-1} = q^{m-1} + q^{m-2} + \dots + q + 1$.

One can obtain two types of EAQECCs from projective geometry designs: type II (using a point-by-block incidence matrix) and type I (using a block-by-point incidence matrix of the design). Applying Proposition 2 to an incidence matrix of $PG_1(m, q)$, we obtain a type II EAQECC. This type of EAQECC belongs to the high rate entanglement-assisted quantum LDPC codes given in Theorems 4 and 5. If we apply Proposition 3 to a block-by-point incidence matrix, we obtain a type I EAQECC. This kind of EAQECC belongs to the high-redundancy entanglement-assisted quantum LDPC codes given in Theorem 6.

The rank of an incidence matrix determines the dimension of the corresponding FG-LDPC code; hence, it is one of the key values in the quantum setting as well. Exact values for many sporadic examples have been computed in the fields of quantum and classical LDPC codes. The following two theorems give the exact rank for all projective geometry designs.

Theorem 19: Hamada [38]. The rank of $PG_1(m, 2^t)$ is given by

$$\begin{aligned} \text{rank } PG_1(m, 2^t) &= \varphi(m, 2^t) \\ &= \sum_{(s_0, s_1, \dots, s_t)} \prod_{j=0}^{t-1} \sum_{i=0}^{L(s_{j+1}, s_j)} (-1)^i \binom{m+1}{i} \\ &\quad \times \binom{m+2s_{j+1}-s_j-2i}{m}, \end{aligned}$$

where the sum is taken over all ordered sets (s_0, s_1, \dots, s_t) with $s_0 = s_t$, $s_j \in \mathbb{Z}$ such that $0 \leq s_j \leq m - 1$ and $0 \leq 2s_{j+1} - s_j \leq m + 1$ for each $j = 0, \dots, t - 1$, and

$$L(s_{j+1}, s_j) = \left\lfloor \frac{2s_{j+1} - s_j}{2} \right\rfloor.$$

We use the notation $\varphi(m, 2^t)$ for the rank of $PG_1(m, q)$ when q is even, that is, $q = 2^t$. When q is odd, the rank of $PG_1(m, q)$ is given by a formula of Frumkin and Yakir [50].

Theorem 20: Frumkin and Yakir [50]. Let q be odd and H an incidence matrix of the design $\text{PG}_1(m, q)$ with $v = \frac{q^{m+1}-1}{q-1}$ points. Then $\text{rank } H = v - 1 = \frac{q^{m+1}-q}{q-1}$.

Hence, the exact dimensions of the corresponding FG-LDPC codes obtained from projective geometries can be calculated for all cases.

The rank of $\text{PG}_1(m, 2^t)$ was conjectured by Hamada [35] to be the lowest rank among all Steiner 2-designs of the same order and block size. This has been confirmed in a number of cases, although in general the conjecture is still open. Thus, we expect that the designs $\text{PG}_1(m, 2^t)$ should provide codes with the best possible dimensions among all nonisomorphic $S(2, 2^t + 1, \frac{2^{t(m+1)}-1}{2^t-1})$'s.

We now examine the codes obtained from $\text{PG}_1(m, q)$ in detail. This subsection is divided into two portions based on the orientation of the incidence matrix.

1. Point-by-block (type II) projective geometry codes

In this portion, we consider the EAQECCs corresponding to a point-by-block incidence matrix of $\text{PG}_1(m, q)$.

We first consider the case $q = 2^t$ for some positive integer t . The following theorem gives an infinite family of entanglement-assisted quantum LDPC codes which consume only one initial ebit and have extremely large net rate.

Theorem 21. For every pair of integers $t \geq 1$ and $m \geq 2$ there exists an entanglement-assisted quantum LDPC codes with girth 6 whose parameters $[[n, k, d; c]]$ are

$$\begin{aligned} n &= \frac{(2^{t(m+1)} - 1)(2^{tm} - 1)}{(2^{2t} - 1)(2^t - 1)}, \\ k &= \frac{(2^{t(m+1)} - 1)(2^{tm} - 1)}{(2^{2t} - 1)(2^t - 1)} - 2\varphi(m, 2^t) + 1, \\ d &= 2^t + 2, \quad \text{and} \\ c &= 1. \end{aligned}$$

To prove Theorem 21, we first prove a new result on the distance of EAQECCs obtained from an incidence matrix of $\text{PG}_1(m, 2^t)$. We use a special set of lines. A *dual hyperoval* \mathcal{H} is a set of $q + 2$ lines of $\text{PG}_1(2, q)$, such that each point of $\text{PG}_1(2, q)$ lies on either zero or two lines of \mathcal{H} . Dual hyperovals exist if and only if q is even. An example is the set of projective lines with equations:

$$\{X_0 + \beta X_1 + \beta^2 X_2 = 0 : \beta \in \mathbb{F}_q\} \cup \{X_1 = 0\} \cup \{X_2 = 0\}.$$

Theorem 22. Let H be an incidence matrix of $\text{PG}_1(m, 2^t)$. The minimum distance of the classical binary linear code with parity-check matrix H is $2^t + 2$.

Proof. First, we note that coordinates of the code words correspond to lines of the geometry, and a code word corresponds to a set S of lines in $\text{PG}_1(m, 2^t)$ such that every point is contained in an even number of lines of S . Assume that c is a nonzero code word, and let $\text{supp}(c)$ denote the support of c , that is, the set of indices of the nonzero coordinates of c . Since $c \neq 0$, the support of c contains at least one line ℓ . Through each point of $\text{PG}(m, 2^t)$ there pass an even number of lines from $\text{supp}(c)$. In particular, each of the $2^t + 1$ points on ℓ lies on at least one other line of $\text{supp}(c)$, and all these lines are different as they have different intersections with ℓ .

Hence, there are at least $1 + (2^t + 1)$ lines in $\text{supp}(c)$; that is, minimum distance d is at least $2^t + 2$. Let π be a plane in $\text{PG}(m, 2^t)$ and S the set of the $2^t + 2$ lines of a dual hyperoval in π . Then S corresponds to a code word of weight $2^t + 2$; hence, $d = 2^t + 2$. ■

Proof of Theorem 21. Let H be an incidence matrix of $\text{PG}_1(m, 2^t)$. The rank of H is $\varphi(m, 2^t)$ given by Theorem 19. The index of $\text{PG}_1(m, 2^t)$ is one. The replication number is odd. By Eq. (3) and Theorem 4, we have $\text{rank } HH^T = 1$. By Theorem 22, the minimum distance of the binary linear code with parity-check matrix H is $2^t + 2$. ■

Next, we examine EAQECCs obtained from an incidence matrix of $\text{PG}_1(m, q)$ with q odd. This case also gives very high rate entanglement-assisted quantum LDPC codes.

Lemma 1. Let H be an incidence matrix of $\text{PG}_1(2, q)$, q odd. Then the classical binary linear code defined by parity-check matrix H consists of only the zero vector and the all-one vector.

Proof. This follows directly from Theorem 20.

A *hyperbolic quadric* Q is a substructure $(\mathcal{P}, \mathcal{L})$ of $\text{PG}_1(3, q)$ with $(q + 1)^2$ points and $2(q + 1)$ lines, such that each point of \mathcal{P} lies on exactly two lines of \mathcal{L} and every plane of $\text{PG}(3, q)$ contains zero or two lines of \mathcal{L} . Hyperbolic quadrics exist for every odd prime power q . ■

Theorem 23. Let H be an incidence matrix of $\text{PG}_1(m, q)$, $m \geq 3$, q odd. Then the minimum distance of the classical binary linear code with a parity-check matrix H is $2(q + 1)$.

Proof. Let Π be a three-dimensional subspace of $\text{PG}(m, q)$ and $(\mathcal{P}, \mathcal{L})$ a hyperbolic quadric in Π . The set of lines \mathcal{L} determines a code word of weight $2q + 2$, since each point of $\text{PG}(m, q)$ is contained in zero or two lines of \mathcal{L} . Hence, minimum distance d is at least $2q + 2$.

We show that there are no code words of weight smaller than $2q + 2$. Assume that there exists a code word c of weight smaller than $2q + 2$; that is, $\text{supp}(c)$ is a set of less than $2q + 2$ lines of $\text{PG}(m, q)$, such that each point lies on an even number of lines of $\text{supp}(c)$. We show that for any two-dimensional subspace π one has either $|\text{supp}(c) \cap \pi| \leq 1$ or $|\text{supp}(c) \cap \pi| \geq q + 2$.

First, let $S = \text{supp}(c) \cap \pi = \{\ell_1, \dots, \ell_i\}$. For each $j \in \{1, \dots, i\}$, each of the points on ℓ_j has to lie on at least one other line of $\text{supp}(c)$, and at most $i - 1$ of them can lie on a line of S . Hence, at least $q + 1 - (i - 1)$ of them are lines in $\text{supp}(c) \setminus S$ and since they all have different intersections with π , this yields $i(q - i + 2)$ lines in $\text{supp}(c) \setminus S$. Together with the i lines of S , we have

$$i(q - i + 2) + i < 2q + 2$$

and solving this quadratic inequality for i gives us that either $i > q + 1$ or $i < 2$. Since i is an integer, hence $i \geq q + 2$ or $i \leq 1$.

Now, let ℓ be any line of $\text{supp}(c)$. Each point of ℓ must lie on at least one other line; hence, there certainly exist planes π with $i \geq 2$, and we have $i \geq q + 2$. Let π be such a plane. We now show that all lines of $\text{supp}(c)$ are contained in π . Assume the contrary, that there exists a line $\ell' \in \text{supp}(c) \setminus \pi$. Through each of the points on $\ell' \setminus \pi$, we need at least one other line of

TABLE I. Sample parameters of type II $[[n,k,d;c]]$ EAQECCs obtained from $PG_1(m,q)$, q even.

m	q	n	k	d	c
3	2	35	14	4	1
4	2	155	104	4	1
5	2	651	538	4	1
6	2	2667	2428	4	1
3	4	357	236	6	1
4	4	5795	5204	6	1
2	8	73	18	10	1
3	8	4745	3944	10	1

$\text{supp}(c)$ which is not contained in π . Since there are at least q points on $\ell' \setminus \pi$, one has

$$|\text{supp}(c)| = |S| + |\text{supp}(c) \setminus S| \geq (q + 2) + (1 + q) > 2q + 2,$$

a contradiction. Hence, ℓ' does not exist and $\text{supp}(c)$ is contained within a single plane π . However, π is a $PG_1(2,q)$ and by Lemma 1 we need $q^2 + q + 1 > 2q + 2$ lines in this case, a contradiction. Hence, there are no code words of weight less than $2q + 2$. ■

We now give another infinite family of type II entanglement-assisted quantum LDPC codes.

Theorem 24. Let q be an odd prime power. Then for every integer $m \geq 3$ there exists an entanglement-assisted quantum LDPC code with girth 6 whose parameters $[[n,k,d;c]]$ are

$$n = \frac{(q^{m+1} - 1)(q^m - 1)}{(q^2 - 1)(q - 1)},$$

$$k = \frac{(q^{m+1} - 1)(q^m - 1)}{(q^2 - 1)(q - 1)} - 2 \frac{q^{m+1} - q}{q - 1} + c,$$

$$d = 2q + 2, \quad \text{and}$$

$$c = \begin{cases} 1 & \text{when } m \text{ is odd,} \\ \frac{q^{m+1} - q}{q - 1} & \text{when } m \text{ is even.} \end{cases}$$

Proof. This follows directly from Proposition 2 and Theorems 4, 20, and 23. ■

Therefore, in the case where m is odd, we have another infinite class of EAQECCs which consume only one ebit. If m is even, we obtain infinitely many high rate codes which consume reasonable numbers of ebit. Tables I and II give a sample of the parameters of the type II codes obtained from $PG_1(m,q)$ with q even and q odd, respectively.

In the remainder of this portion, we examine Steiner spreads of projective geometry designs. These substructures

TABLE II. Sample parameters of type II $[[n,k,d;c]]$ EAQECCs obtained from $PG_1(m,q)$, q odd.

m	q	n	k	d	c
3	3	130	53	8	1
3	5	806	497	12	1
3	7	2850	2053	16	1
4	3	1210	1090	8	120

TABLE III. Summary of type II codes obtained by deleting a Steiner spread of subdesigns isomorphic to $PG_1(2,2)$ from $PG_1(5,2)$.

Subs ^a	n	rank H	k	d	c	Rate
0	651	57	538	4	1	0.8264
1	644	57	532	4	2	0.8370
2	637	57	526	4	3	0.8477
3	630	57	520	4	4	0.8587
4	623	57	514	4	5	0.8700
5	616	57	508	4	6	0.8815
6	609	57	502	4	7	0.8933
7	602	57	496	4	8	0.9053
8	595	57	490	4	9	0.9176
9	588	57	482	4	8	0.9269

^aThis column denotes the number of subdesigns removed.

can be used in Theorems 14, 15, and 16 and their analogous techniques based on Theorem 5 to fine tune the rates and distances of the EAQECCs.

An s -spread of $PG(m,q)$ is a set of s -dimensional projective subspaces which partition the points of the geometry. In other words, an s -spread consists of a set of $(s + 1)$ -dimensional vector subspaces of \mathbb{F}_q^{m+1} which contain every nonzero vector exactly once. It is known that $PG(m,q)$ admits an s -spread if and only if $s + 1$ divides $m + 1$ (see [51] and [52, p. 29]).

Take $PG_1(m,q)$ and suppose $s \geq 2$ is chosen so that $s + 1$ divides $m + 1$. Then an s -spread of $PG(m,q)$ exists. Each s -dimensional subspace in the spread contains an isomorphic copy of $PG_1(s,q)$, and hence this forms a Steiner spread. Note that the blocks of $PG_1(s,q)$ have size $q + 1$ and are also blocks of $PG_1(m,q)$. Therefore, we have the following result.

Theorem 25. Let $s, m \geq 1$ be positive integers such that $s + 1$ divides $m + 1$. Then $PG_1(m,q)$ contains $\frac{q^{m+1} - 1}{q^{s+1} - 1}$ disjoint copies of $PG_1(s,q)$ whose point sets partition the point of $PG_1(m,q)$.

Thus, we can find a set of disjoint subdesigns which partition the points of $PG_1(m,q)$ whenever $m + 1$ has a nontrivial factor. Naturally, we may further subdivide each subdesign of dimension s into smaller subdesigns, based on the nontrivial factors of $s + 1$. Hence, the $S(2,\mu,\nu)$'s from $PG_1(m,q)$ are very flexible in that they have Steiner spreads of various sizes.

In general, the length, dimension, required ebits, and rate each change gradually as we delete subdesigns in a Steiner spread. The minimum distance and rank either remain the same or improve slightly. Table III lists the example parameters of EAQECCs created by deleting subdesigns from $PG_1(5,2)$. The first and last rows correspond to regular LDPC codes.

2. Block-by-point (type I) projective geometry codes

Next we consider EAQECCs obtained via Theorem 6 by using the block-by-point incidence matrix of $PG_1(m,q)$. The codes obtained in this manner correspond to the classical type I LDPC codes. As in the classical setting, type I entanglement-assisted quantum regular LDPC codes can correct many quantum errors. Because an incidence matrix of $PG_1(m,q)$

TABLE IV. Sample parameters of type I $[[n, k, d; c]]$ EAQECCs obtained from $\text{PG}_1(m, q)$, q even.

m	q	n	k	d	c
2	4	21	2	6	1
2	8	73	18	10	1
2	16	273	110	18	1
2	32	1057	570	34	1

for q odd is almost full rank, the corresponding type I code is not of much interest. Hence, in this portion we always assume that $q = 2^t$ for some positive integer t .

Theorem 26. For every pair of integers $t \geq 1$ and $m \geq 2$ there exists an entanglement-assisted quantum LDPC code with girth 6 whose parameters $[[n, k, d; c]]$ are

$$\begin{aligned} n &= \frac{2^{t(m+1)} - 1}{2^t - 1}, \\ k &= \frac{2^{t(m+1)} - 1}{2^t - 1} - 2\varphi(m, 2^t) + c, \\ d &= (2^t + 2)2^{t(m-2)}, \text{ and} \\ c &\leq \varphi(m, 2^t). \end{aligned}$$

Proof. Let H^T be a block-by-point incidence matrix of $\text{PG}_1(m, 2^t)$. Then we have $\text{rank } H^T H \leq \text{rank } H = \varphi(m, 2^t)$, where $\varphi(m, 2^t)$ is given by Theorem 19. By a result of Calkin, Key, and de Resmini [46], the minimum distance of the binary linear code with parity-check matrix H^T is $(2^t + 2)2^{t(m-2)}$. Applying Proposition 3 proves the assertion. ■

Note that here the distance grows exponentially as the dimension of the geometry increases. When $m = 2$, the EAQECCs are based on projective planes. As shown in Sec. II B, the EAQECC obtained from a Desarguesian projective plane of order 2^t consumes only one initial ebit. Basing on Hamada's conjecture, we expect that in general the EAQECCs given in Theorem 26 consume relatively small numbers of ebits.

It is not clear from the formula for $\varphi(m, 2^t)$ whether the net rate of a type I EAQECC based on $\text{PG}_1(m, 2^t)$ is positive. In order to produce useful catalytic quantum codes, it is important to understand when the net rate is positive.

Proposition 4. Let H be an incidence matrix of $\text{PG}_1(2, 2^t)$. Then for all $t \geq 2$ the EAQECC obtained from H^T has a positive net rate.

Proof. By Hamada's formula, we have $\text{rank } H = 3^t + 1$. The number of points in $\text{PG}_1(2, 2^t)$ is $2^{2t} + 2^t + 1$. ■

For $m \geq 3$, we note that as q increases, $\text{rank } H$ grows at a slower rate than the code length. Thus, we may expect that, for q large when compared to m , the net rate will eventually become positive. For example, one can check that the net rate of the type I EAQECC obtained from $\text{PG}_1(3, 2^t)$ is positive for $t \geq 7$. Table IV gives sample parameters of the type I codes obtained from $\text{PG}_1(m, 2^t)$.

B. Affine geometry codes

In this subsection, we study the EAQECCs obtained from affine geometry designs.

The affine geometry $\text{AG}(m, q)$ of dimension m over \mathbb{F}_q is a finite geometry whose points are the vectors in \mathbb{F}_q^m . The i -dimensional affine subspaces (or i -flats) are the i -dimensional vector subspaces of \mathbb{F}_q^m and their cosets. Thus, $\text{AG}(m, q)$ has a natural parallelism.

The points and lines (that is, 1-flats) of an affine geometry form an $S(2, q, q^m)$, denoted by $\text{AG}_1(m, q)$. The design has $q^{m-1} \frac{q^m - 1}{q - 1}$ blocks and replication number $\frac{q^m - 1}{q - 1} = q^{m-1} + q^{m-2} + \dots + q + 1$.

We note that in many articles concerning LDPC codes, the term ‘‘Euclidean geometry’’ and the notation $\text{EG}(m, q)$ are used for affine geometries. Most of the codes studied in relation to Euclidean geometries do not use the zero vector, and hence they do not generally correspond to $S(2, \mu, v)$'s. Because the term ‘‘affine geometry’’ is standard in the recent research on finite geometry in mathematics, we use the notation $\text{AG}_1(m, q)$ when we take all points and lines to form an incidence matrix. The incidence structure obtained by discarding the zero vector and the lines containing the zero vector from $\text{AG}_1(m, q)$ is denoted by $\text{EG}_1(m, q)$, which we study in Sec. III C. Because many of the classical FG-LDPC codes obtained from affine geometries are based on $\text{EG}_1(m, q)$, they are generally not the same as the affine geometry codes presented in this section.

As with projective geometry designs, Propositions 2 and 3 give type II and type I affine geometry codes, respectively. It is notable that the classical ingredients of these codes are quasicyclic LDPC codes similar to other FG-LDPC codes because the elementary abelian group acts transitively on the points of $\text{AG}_1(m, q)$ (see [18, 32]). The rank of an affine geometry design $\text{AG}_1(m, 2^t)$ is directly related to φ given in Theorem 19.

Theorem 27: Hamada [35]. The rank of the affine geometry design $\text{AG}_1(m, 2^t)$ is given by

$$\text{rank } \text{AG}_1(m, 2^t) = \varphi(m, 2^t) - \varphi(m - 1, 2^t).$$

If q is odd, the rank of $\text{AG}_1(m, q)$ over \mathbb{F}_2 is full.

Theorem 28: Yakir [53]. Let H be an incidence matrix of the design $\text{AG}_1(m, q)$ with q odd. Then $\text{rank } H = q^m$.

Thus, the dimensions of the corresponding FG-LDPC codes can be easily calculated.

As in the case of projective designs, Hamada conjectured that the rank of $\text{AG}_1(m, 2^t)$ is minimum among all Steiner 2-designs of the same order and block size. Thus, affine geometry designs with q even may be expected to give codes with the best possible dimensions among all nonisomorphic $S(2, 2^t, 2^{tm})$'s.

We divide this subsection into two portions. In the first portion we examine high-rate type II entanglement-assisted quantum LDPC codes obtained from $\text{AG}_1(m, q)$. Then in the next portion we present type I entanglement-assisted quantum LDPC codes based on $\text{AG}_1(m, q)$, which effectively exploit the redundancy to give excellent error-correction performance.

1. Point-by-block (type II) affine geometry codes

The geometric structure of affine geometry has often been studied independently in various fields. The special substructure we need to give distances has been investigated in connection with the disk failure resilience ability of a class

of redundant arrays of independent disks (RAID). Here we present a known result on RAID related to our codes in coding theoretic terminology.

Theorem 29: Müller and Jimbo [54]. Let H be an incidence matrix of $AG_1(m, q)$. The minimum distance of the classical binary linear code having H as a parity-check matrix is $q + 1$ if q is even, and $2q$ otherwise.

The following two theorems give infinite families of EAQECCs which consume only one initial ebit and have very large net rate.

Theorem 30. For every pair of integers $t \geq 1$ and $m \geq 2$ there exists an entanglement-assisted quantum LDPC code with girth 6 whose parameters $[[n, k, d; c]]$ are

$$\begin{aligned} n &= 2^{t(m-1)} \frac{2^{2^m} - 1}{2^t - 1}, \\ k &= 2^{t(m-1)} \frac{2^{2^m} - 1}{2^t - 1} - 2[\varphi(m, 2^t) - \varphi(m - 1, 2^t)] + 1, \\ d &= 2^t + 1, \quad \text{and} \\ c &= 1. \end{aligned}$$

Proof. Let H be an incidence matrix of $AG_1(m, 2^t)$. By Theorem 27, we have $\text{rank } H = \varphi(m, 2^t) - \varphi(m - 1, 2^t)$. The index of the design $AG_1(m, 2^t)$ is one. Its replication number is always odd. Thus, by Theorem 4, we have $\text{rank } HH^T = 1$. Applying Proposition 2 and Theorem 29 completes the proof. ■

Theorem 31. Let q be an odd prime power. Then for every integer $m \geq 2$ there exists an entanglement-assisted quantum LDPC code with girth 6 whose parameters $[[n, k, d; c]]$ are

$$\begin{aligned} n &= q^{m-1} \frac{q^m - 1}{q - 1}, \\ k &= q^{m-1} \frac{q^m - 1}{q - 1} - 2q^m + c, \\ d &= 2q, \quad \text{and} \\ c &= \begin{cases} 1 & \text{when } m \text{ is odd,} \\ q^m - 1 & \text{when } m \text{ is even.} \end{cases} \end{aligned}$$

Proof. Let H be an incidence matrix of $AG_1(m, q)$ with q odd. By Theorem 28, we have $\text{rank } H = q^m$. The index of the design $AG_1(m, q)$ is one. Its replication number r is a sum of m terms, each being an odd number. Thus, r is odd only when m is odd. By Theorem 4, we have $\text{rank } HH^T = 1$ for m odd. If m is even, we have $\text{rank } HH^T = q^m - 1$ from Theorem 5. Applying Proposition 2 and Theorem 29 proves the assertion. ■

Theorem 31 gives an infinite family of high-rate entanglement-assisted quantum LDPC codes which exploit reasonable amounts of entanglement as well. Tables V and VI give a sample of the parameters of the type II codes obtained from $AG_1(m, q)$ with q even and q odd, respectively.

Next we show that affine geometry designs have numerous subdesigns and Steiner spreads, which make it possible to fine-tune the parameters and error-correction performance of the corresponding EAQECCs.

Theorem 32. If $m \geq 3$, the points of $AG_1(m, q)$ can be partitioned into q disjoint subsets of size q^{m-1} , being the point sets of subdesigns isomorphic to $AG_1(m - 1, q)$.

TABLE V. Sample parameters of type II $[[n, k, d; c]]$ EAQECCs obtained from $AG_1(m, q)$, q even.

m	q	n	k	d	c
3	2	28	15	3	1
4	2	120	91	3	1
5	2	496	435	3	1
6	2	2016	1891	3	1
2	4	20	3	5	1
3	4	336	235	5	1
4	4	5440	4971	5	1
2	8	72	19	9	1
3	8	4672	3927	9	1

Proof. Take a parallel class $\{H_1, \dots, H_q\}$ of q hyperplanes of $AG(m, q)$. Let the point set of H_j be V_j . Clearly, $\cup_{j=1}^q V_j = V$, and the set of all blocks of $AG_1(m, q)$ which are contained entirely in H_j form a subdesign isomorphic to $AG_1(m - 1, q)$. ■

Theorem 32 can be applied recursively to create additional disjoint subdesigns of smaller dimension, giving a variety of EAQECCs via Theorems 14, 15, and 16. Similar subdesign deletion techniques based on Theorem 5 further give infinitely many high-rate EAQECCs. Table VII lists the parameters of the EAQECCs created by spread deletion from $AG_1(3, 4)$.

2. Block-by-point (type I) affine geometry codes

Next we consider the EAQECC obtained from a block-by-point incidence matrix of $AG_1(m, q)$. Because incidence matrices of $AG_1(m, q)$ with q odd are of full rank, here we always assume $q = 2^t$ to obtain interesting codes. The entanglement-assisted quantum LDPC codes presented in this section effectively exploit redundancy. The excellent error-correction performance is demonstrated in simulations in Sec. IV.

Theorem 33: Calkin, Key, and de Resmini [46]. Let H be a block-by-point incidence matrix of $AG_1(m, 2^t)$. Then the minimum distance of the classical binary linear code for which H is a parity-check matrix is $(2^t + 2)2^{t(m-2)}$.

Theorem 34. For every pair of integers $t \geq 1$ and $m \geq 3$ there exists an entanglement-assisted quantum LDPC code with girth 6 whose parameters $[[n, k, d; c]]$ are

$$\begin{aligned} n &= 2^{tm}, \\ k &= 2^{tm} - 2[\varphi(m, 2^t) - \varphi(m - 1, 2^t)] + c, \\ d &= (2^t + 2)2^{t(m-2)}, \quad \text{and} \\ c &\leq \varphi(m, 2^t) - \varphi(m - 1, 2^t). \end{aligned}$$

TABLE VI. Sample parameters of type II $[[n, k, d; c]]$ EAQECCs obtained from $AG_1(m, q)$, q odd.

m	q	n	k	d	c
3	3	117	64	6	1
3	5	775	526	10	1
3	7	2793	2108	14	1
5	3	9801	9316	6	1
4	3	1080	998	6	80

TABLE VII. Summary of type II codes obtained by deleting a Steiner spread of subdesigns isomorphic to $AG_1(2,4)$ from $AG_1(3,4)$.

Subs ^a	n	rank H	k	d	c	Rate
0	336	51	235	5	1	0.6994
1	316	51	216	5	2	0.7468
2	296	51	197	5	3	0.8007
3	276	51	178	5	4	0.8623
4	256	51	158	6	4	0.9297

^aThis column denotes the number of subdesigns removed.

Proof. Let H^T be a block-by-point incidence matrix of $AG_1(m,2^t)$. By Theorem 27, we have $\text{rank } H^T H \leq \text{rank } H = \varphi(m,2^t) - \varphi(m-1,2^t)$. By Theorem 33, the minimum distance of the binary linear code with a parity-check matrix H is $(2^t + 2)2^{t(m-2)}$. The assertion follows from Proposition 3. ■

It is worth mentioning that here the distance grows exponentially with linear increase of the geometry dimension m . Because the rank of $AG_1(m,2^t)$ is conjectured to be the smallest possible among all nonisomorphic $S(2,2^t,2^{tm})$'s, we expect that the EAQECCs obtained from these affine geometry designs consume the smallest possible numbers of ebits attainable by this method with $S(2,2^t,2^{tm})$'s.

When $m = 2$, we can easily determine the required amount of entanglement.

Theorem 35. For every positive integer t there exists an entanglement-assisted quantum LDPC code with girth 6 whose parameters $[[n,k,d;c]]$ are

$$\begin{aligned} n &= 4^t, \\ k &= 4^t + 2^t - 2 \times 3^t, \\ d &= 2^t + 2, \quad \text{and} \\ c &= 2^t. \end{aligned}$$

Proof. Let H^T be a block-by-point incidence matrix of $AG_1(2,2^t)$. We first prove that $\text{rank } H^T H = 2^t$. Two lines of an affine plane are either parallel or intersect in exactly one point. There are $2^t + 1$ parallel classes of lines, each containing exactly 2^t lines, and each line contains 2^t points. Because 2^t is even, it is always possible to reorder the rows of H^T such that $H^T H$ is a block matrix of the following form:

$$H^T H = \begin{bmatrix} 0 & J & & J \\ J & 0 & \cdots & J \\ & \vdots & \ddots & \vdots \\ J & J & \cdots & 0 \end{bmatrix},$$

where J is the $2^t \times 2^t$ all-one matrix. Hence, we have $\text{rank } H^T H = 2^t$. By Theorem 27, we have $\text{rank } H = 3^t$. Applying Proposition 3 and Theorem 33 completes the proof. ■

Table VIII gives sample parameters of the type I EAQECCs obtained from $AG_1(m,2^t)$.

C. Euclidean geometry codes

In this final subsection concerning finite geometry EAQECCs, we examine Euclidean geometry codes.

TABLE VIII. Sample parameters of type I $[[n,k,d;c]]$ EAQECCs obtained from $AG_1(m,q)$, q even.

m	q	n	k	d	c
2	8	64	18	10	8
2	16	256	110	18	16
2	32	1024	570	34	32
2	64	4096	2702	66	64

Given a prime power q and integer $m \geq 2$, we define an incidence structure $EG_1(m,q)$ having as points all points of $AG_1(m,q)$ except the zero vector, and having as blocks (or lines) all lines of $AG(m,q)$ except those lines containing the zero vector. The lines which are excluded from $AG_1(m,q)$ to form $EG_1(m,q)$ consist of all multiples of a single nonzero vector. Thus, $EG_1(m,q)$ has $q^m - 1$ points and $(q^{m-1} - 1) \frac{q^m - 1}{q - 1}$ lines. Each line contains q points, and each point appears in $\frac{q^m - 1}{q - 1} - 1 = q^{m-1} + q^{m-2} + \cdots + q$ lines. Thus, $EG_1(m,q)$ yields regular LDPC codes. Each pair of points appears in *at most* one line. Hence, $EG_1(m,q)$ is a partial Steiner 2-design. Its Tanner graph does not contain 4-cycles.

Applying Proposition 3 to a line-by-point incidence matrix of $EG_1(m,q)$ gives a type I EAQECC. If q is even, the distance is bounded from below by the BCH bound.

Theorem 36: Kou, Lin, and Fossorier [18]. Let H be a line-by-point incidence matrix of $EG_1(m,2^t)$. Then the minimum distance d of the classical binary linear code having H as a parity-check matrix satisfies $d \geq \frac{2^m - 1}{2^t - 1}$. Equality holds if $m = 2$.

We use the following theorem to give the dimensions of FG-LDPC codes obtained from $EG_1(m,2^t)$ and their entanglement-assisted quantum counterparts.

Theorem 37: Hamada [35]. The rank of the incidence structure $EG_1(m,2^t)$, $t > 1$, is given by

$$\text{rank } EG_1(m,2^t) = \varphi(m,2^t) - \varphi(m-1,2^t) - 1.$$

Theorem 38. For every pair of integers $t \geq 1$ and $m \geq 2$ there exists an entanglement-assisted quantum LDPC code with girth 6 whose parameters $[[n,k,d;c]]$ are

$$\begin{aligned} n &= 2^{tm} - 1, \\ k &= 2^{tm} - 2[\varphi(m,2^t) - \varphi(m-1,2^t)] + 1 + c, \\ d &\geq \frac{2^{tm} - 1}{2^t - 1}, \quad \text{and} \\ c &\leq \varphi(m,2^t) - \varphi(m-1,2^t) - 1. \end{aligned}$$

Proof. Let H^T be a line-by-point incidence matrix of $EG_1(m,2^t)$. By Theorem 37, we have $\text{rank } H^T H \leq \text{rank } H = \varphi(m,2^t) - \varphi(m-1,2^t) - 1$. Applying Proposition 3 and Theorem 36 completes the proof. ■

A simple observation gives exact values of all the parameters of the type I codes based on $EG_1(2,2^t)$.

Theorem 39. For every positive integer t there exists an entanglement-assisted quantum LDPC code with girth 6 whose

TABLE IX. Sample parameters of type I $[[n, k, d; c]]$ EAQECCs obtained from $EG_1(2, q)$, q even.

m	q	n	k	d	c
2	8	63	19	9	8
2	16	255	111	17	16
2	32	1023	571	33	32

parameters $[[n, k, d; c]]$ are

$$\begin{aligned} n &= 4^t - 1, \\ k &= 4^t + 2^t - 2 \times 3^t + 1, \\ d &= 2^t + 1, \quad \text{and} \\ c &= 2^t. \end{aligned}$$

Proof. Let H^T be a line-by-point incidence matrix of $EG_1(2, 2^t)$. An incidence matrix of $EG_1(2, 2^t)$ is obtained by removing one row and one column from each block from that of $AG_1(2, 2^t)$. By following the argument in Theorem 35, it is straightforward to see that $\text{rank } H^T H = 2^t$. By Theorem 37, we have $\text{rank } H = \varphi(m, 2^t) - \varphi(m - 1, 2^t) - 1 = 3^t - 1$. Theorem 36 and Proposition 3 prove the assertion. ■

Table IX gives a sample of the parameters of the type I codes obtained from $EG_1(2, 2^t)$.

As with $S(2, \mu, v)$'s, the incidence structure $EG_1(m, q)$ can also generate a high-rate LDPC code with girth 6. Applying Proposition 2 to incidence matrices, we obtain type II EAQECCs. Here we investigate their parameters.

Theorem 40. The minimum distance of a type II EAQECC based on $EG_1(m, q)$ is $q + 1$ if q is even, and $2q$ if q is odd and $m > 2$.

Proof. Consider any set of linearly dependent columns in an incidence matrix of $EG_1(m, q)$. The same columns appear in the corresponding incidence matrix of $AG_1(m, q)$, but with a single zero coordinate added. These columns are still dependent in $AG_1(m, q)$. Hence, the minimum distance is upper bounded by Theorem 29. Thus, we need only to show lower bounds.

We begin with q even. If $q = m = 2$, we can check by hand that the minimum distance is 3. Henceforth assume that $q > 2$ or $m > 2$. Because the minimum distance of the code obtained from $AG_1(m, q)$ is $q + 1$, there exists a set S of $q + 1$ linearly dependent columns of an incidence matrix of $AG_1(m, q)$, corresponding to a set \mathcal{D} of $q + 1$ blocks of $AG_1(m, q)$. Let P be the multiset of points appearing in the blocks of \mathcal{D} . As each block of \mathcal{D} has q points, $|P| = q(q + 1)$. However, because the columns of S are dependent over \mathbb{F}_2 , each point in P must appear with multiplicity 2 or more. Hence, the number of distinct points in P is at most $\frac{q(q+1)}{2} < q^m - 1$ except for $q = m = 2$. Therefore, there is a nonzero point p of $AG(m, q)$ which does not appear in P . Let $\mathcal{D}' = \{B - p : B \in \mathcal{D}\}$, that is, we shift each block of \mathcal{D} by p . Each new block corresponds to a coset of a linear space. Because $p \notin P$, no element of \mathcal{D}' contains the zero vector, and so the elements of \mathcal{D}' are lines of $EG_1(m, q)$. Thus, \mathcal{D}' is a linearly dependent set in $EG_1(m, q)$ of size $q + 1$. Therefore, in all cases, the minimum distance of type II EAQECC based on $EG_1(m, q)$, q even, is

 TABLE X. Sample parameters of type II $[[n, k, d; c]]$ EAQECCs obtained from $EG_1(m, q)$, q even.

m	q	n	k	d	c
3	2	21	15	3	6
4	2	105	91	3	14
5	2	465	434	3	30
6	2	1953	1891	3	62
3	4	315	235	5	20
4	4	5355	4971	5	84
2	8	63	19	9	8
3	8	4599	3927	9	72

$q + 1$. A similar argument proves the case when q is odd and $m \neq 2$. ■

Theorem 41. For every pair of integers $t \geq 1$ and $m \geq 2$ there exists an entanglement-assisted quantum LDPC code with girth 6 whose parameters $[[n, k, d; c]]$ are

$$\begin{aligned} n &= (2^{t(m-1)} - 1) \frac{2^{2m} - 1}{2^t - 1}, \\ k &= (2^{t(m-1)} - 1) \frac{2^{2m} - 1}{2^t - 1} - 2 \text{rank} [EG_1(m, 2^t)] + c, \\ d &= 2^t + 1, \quad \text{and} \\ c &= \frac{2^{2m} - 2^t}{2^t - 1}, \end{aligned}$$

where $\text{rank } EG_1(m, 2^t) = \varphi(m, 2^t) - \varphi(m - 1, 2^t) - 1$.

Proof. Let H be an incidence matrix of $EG_1(m, 2^t)$. Because H is obtained from an incidence matrix of $AG_1(m, 2^t)$ by deleting the row representing the zero vector and the columns that represent the lines containing the zero vector, it is easy to see that the rows and columns of HH^T can be reordered such that the matrix is of the form

$$HH^T = \begin{bmatrix} 0 & J & & J \\ J & 0 & \cdots & J \\ & \vdots & \ddots & \vdots \\ J & J & \cdots & 0 \end{bmatrix},$$

where J is the $(2^t - 1) \times (2^t - 1)$ all-one matrix. Because $2^{2m} - 1$ is odd, $\text{rank } HH^T = \frac{2^{2m} - 1}{2^t - 1} - 1$. Applying Proposition 2 and Theorems 40 and 37 completes the proof. ■

Tables X gives sample parameters for the type II codes obtained from $EG_1(m, 2^t)$.

For the case q odd, Hamada [35] conjectured that an incidence matrix of $EG_1(m, q)$ is of full rank. As shown in Table XI, the conjecture is true for small m and q .

IV. PERFORMANCE

In this section, we present simulation results for EAQECC codes constructed in the previous sections. As in the related works [16, 17], we performed simulations over the depolarizing channel. In this model, each error (X , Y , and Z) occurs independently in each qubit with equal probability f_m . For a given CSS type EAQECC, we performed each decoding in two separate decoding steps, each using the sum-product algorithm. The shared ebits, which do not pass through the

TABLE XI. Sample parameters of type II $[[n, k, d; c]]$ EAQECCs obtained from $EG_1(m, q)$, q odd.

m	q	n	k	d	c
3	3	104	64	6	12
4	3	1040	960	6	80
5	3	9680	9316	6	120
3	5	744	526	10	30
3	7	2736	2108	14	56

noisy channel, are assumed to be error free. Our simulation results are reported in terms of the block error rate (BLER).

We first examine codes obtained from a block-by-point incidence matrix. Figure 1 shows the performance of several such codes based on projective and affine geometry designs. As shown in Sec. III, these codes have very large distances for sparse-graph codes while avoiding short cycles. As expected, these codes perform excellently at relatively high f_m .

To illustrate how well these codes perform, we compare one of our type I LDPC codes with previously known entanglement-assisted quantum LDPC codes with best BLERs.

Theorem 35 gives an EAQECC with parameters $[[256, 110, 18; 16]]$ obtained from $AG_1(2, 16)$. The $[[255, 111, 17; 16]]$ EAQECC in the work of Hsieh, Yen, and Hsu [17] used $EG_1(2, 16)$ and outperformed all previously known quantum codes of similar rate in simulations over the depolarizing channel. Their code based on $PG_1(2, 16)$, which also performed very well, has parameters $[[273, 110, 18; 1]]$.

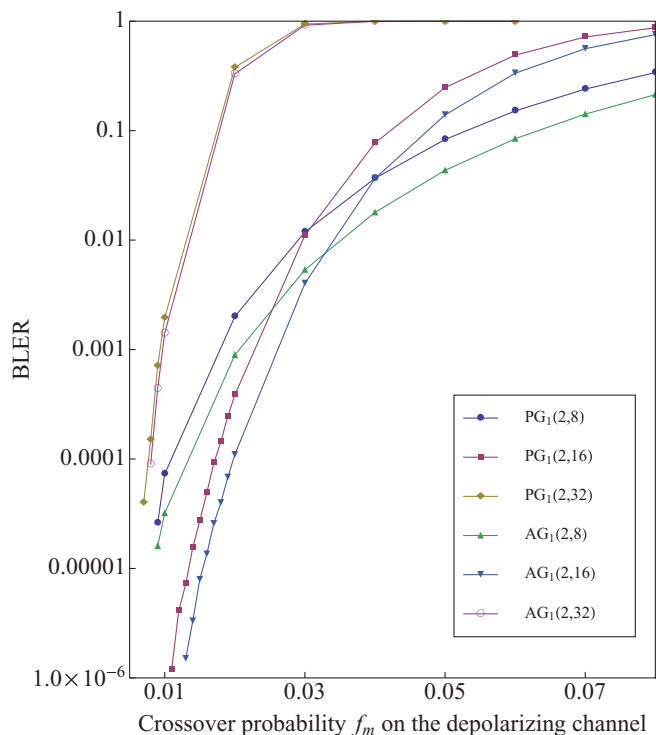


FIG. 1. (Color online) Performance of type I EAQECCs.

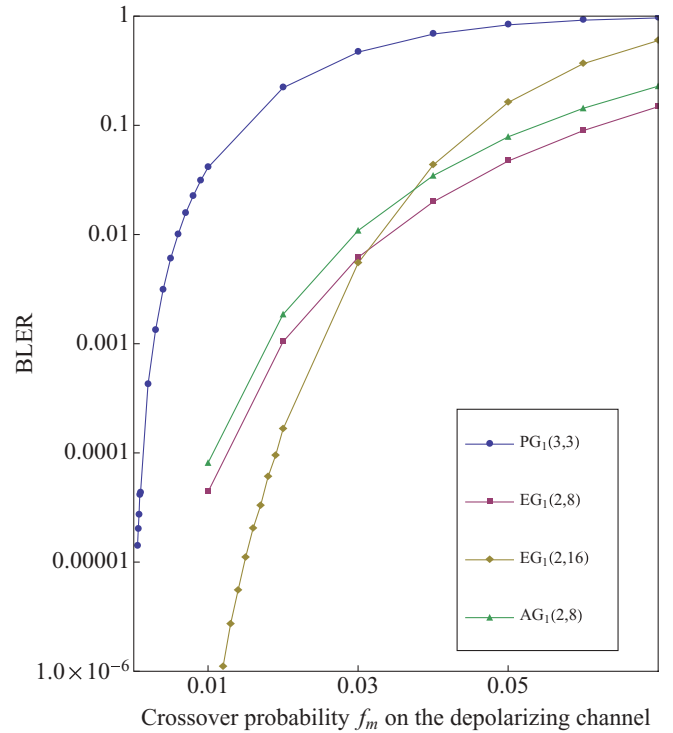


FIG. 2. (Color online) Performance of type II EAQECCs.

Exactly the same EAQECCs as these two can be constructed using Theorems 39 and 13 in our framework without relying on computers to calculate their parameters.

These three EAQECCs based on finite geometries have similar geometrical structures, and they behave quite similarly in simulations. Performance of the $AG_1(2, 16)$ and $PG_1(2, 16)$ codes is directly compared in Fig. 1. The BLER of the $EG_1(2, 16)$ code, which is slightly worse than that of our $AG_1(2, 16)$ code, is plotted in Fig. 2 to compare the three with EAQECCs having different parameters. As shown in the figures, our new $[[256, 110, 18; 16]]$ EAQECC obtained from $AG_1(2, 16)$ shows a better BLER than the other two. The BLERs of $AG_1(2, 16)$, $EG_1(2, 16)$, and $PG_1(2, 16)$ codes at $f_m = 0.02$ are 1.0×10^{-4} , 1.6×10^{-4} , and 3.8×10^{-4} , respectively.

Entanglement-assisted quantum quasicyclic LDPC codes proposed by Hsieh, Brun, and Devetak in [16] have also shown excellent BLERs. In simulations, their $[[128, 58, 6; 18]]$ EAQECCs, called EX1 and EX2, outperformed the previously known best quantum LDPC codes at a similar rate about 0.316. The net rate of EX1 and EX2 is $\frac{58-16}{128} \approx 0.312$. Our $[[256, 110, 18; 16]]$ EAQECC obtained from $AG_1(2, 16)$ has net rate $\frac{110-16}{256} \approx 0.367$, which is higher than that of EX1 and EX2. Their simulation results and our independent simulation results for EX1 and EX2 showed that their BLERs at $f_m = 0.02$ are higher than 1.1×10^{-2} , while our $AG_1(2, 16)$ code has BLER about 1.0×10^{-4} at the same f_m , which is better than EX1 and EX2 by two orders of magnitude. Our EAQECC also requires a smaller amount of entanglement than EX1 and EX2.

Our results here confirm the close linkage between EAQECCs and classical error-correcting codes: good

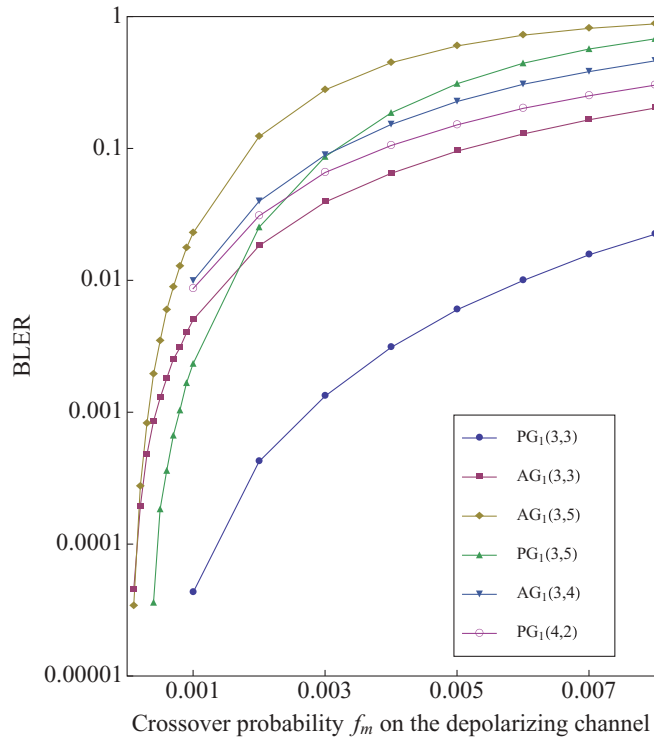


FIG. 3. (Color online) Performance of type II EAQECCs.

performance in the classical setting translates directly into good performance from the corresponding quantum codes.

We next examine codes obtained from a point-by-block incidence matrix. These codes are capable of achieving extremely high rates even at moderate block lengths.

Figures 2 and 3 show the performance of several type II codes based on finite geometries. The type II code from $PG_1(3,3)$ is shown in both figures to serve as a point of reference between the two figures. Figure 4 gives the BLERs for several codes with high rates including $[[301,216,6;1]]$ and $[[1080,998,6;80]]$ codes from cyclic 5-sparse STSs of order 43 and 81, respectively. The incidence matrices of these two Steiner triple systems are constructed from the list of base blocks in [55]. Note that the cyclic automorphisms and sparse configurations immediately give the dimensions and distances of the EAQECCs obtained from the cyclic 5-sparse STSs (see [37,56]). Table XII lists the rates of selected finite geometry codes shown in figures.

As in the classical setting, our codes obtained from point-by-block incidence matrices have waterfall regions at low f_m and transmit at extremely high rates. This direct correlation in performance between the classical and quantum settings can also be seen when codes require only one ebit. It may be worth mentioning that changing geometries or choosing a nongeometric $S(2, \mu, \nu)$ can give slightly different BLER curves. It would be interesting to investigate theoretical methods for finding $S(2, \mu, \nu)$'s with desirable performance curves in given situations.

Finally, we compare EAQECCs obtained by removing subdesigns from the parent design. Here we test a subdesign deletion technique where each deletion step increases the

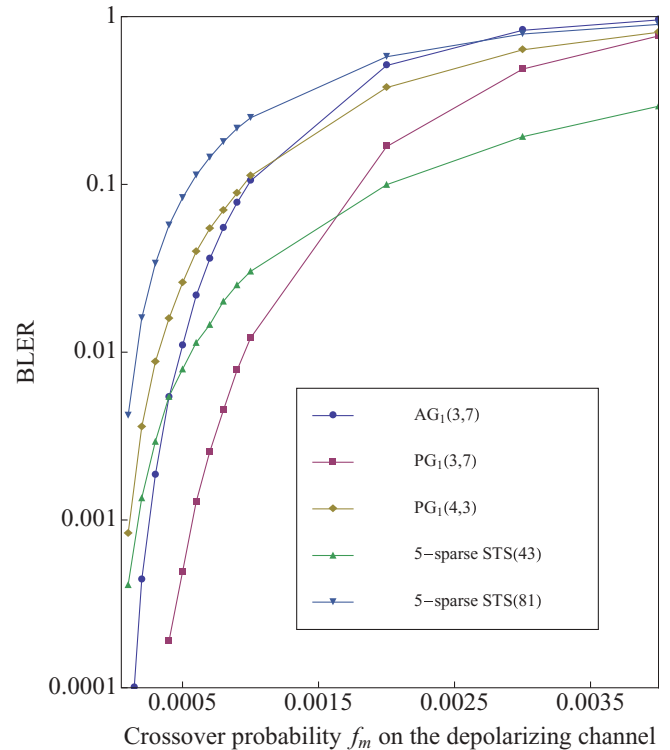


FIG. 4. (Color online) Performance of high-rate type II EAQECCs.

required amount of entanglement to a slightly larger degree than the examples we gave in Sec. III. Each code in Fig. 5 is constructed from a type II code based on $AG_1(3,3)$. Fundamental parameters of these codes are shown in Table XIII.

The original code is also shown for reference. The code labeled “one sub” has had a single subdesign isomorphic to $AG_1(2,3)$ removed. The code labeled “3 subs” has had a Steiner spread removed. This last code is a regular LDPC code. As can be seen from their BLERs, removing subdesigns has improved

TABLE XII. Rates of EAQECCs obtained from finite geometries.

Type	Geometry	m	q	Rate
II	PG	4	3	0.9008
II	PG	3	7	0.7203
II	PG	3	5	0.6166
II	PG	3	3	0.4076
II	AG	3	7	0.7547
II	AG	3	5	0.6787
II	AG	3	3	0.5470
II	AG	2	8	0.2638
II	EG	2	16	0.4352
II	EG	2	8	0.3015
I	PG	2	32	0.5392
I	PG	2	16	0.4029
I	PG	2	8	0.2465
I	AG	2	32	0.5566
I	AG	2	16	0.4296
I	AG	2	8	0.2812

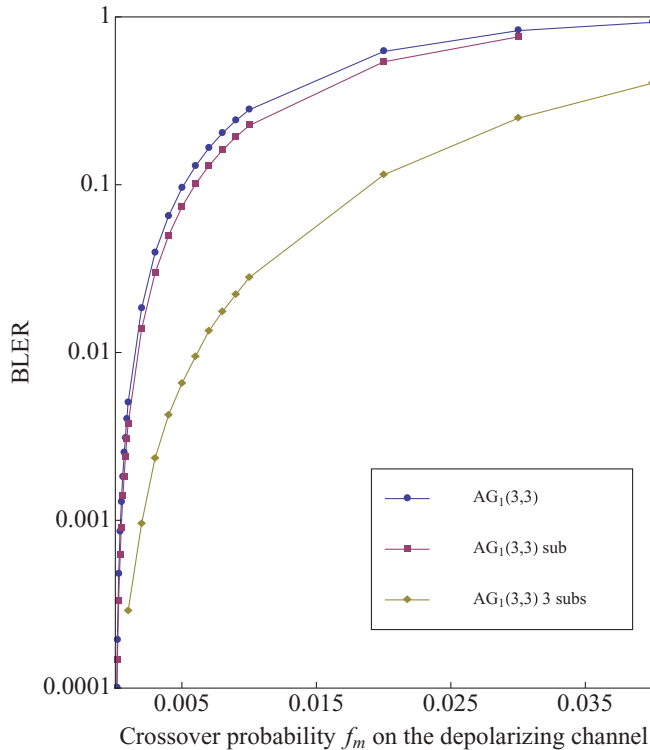


FIG. 5. (Color online) Performance of EAQECCs obtained by deleting subdesigns from $AG_1(3,3)$.

the error-correction performance while increasing the rate and maintaining many of the essential properties.

Because removing subdesigns can increase the required amount of entanglement in a flexible manner, one can generate an EAQECC which effectively exploits pre-existing entanglement. For example, a high net rate code consuming only one ebit can turn into a heavily entanglement-assisted code to achieve better error-correction performance at the same f_m . As illustrated in Table XIII, a $[[117,64,6;1]]$ code with a regular parity-check matrix becomes a $[[81,56,6;25]]$ code with a regular parity-check matrix through gradual steps.

One can also fine-tune parameters and improve error-correction performance while almost keeping the extremely low required amount of entanglement by applying Theorems 14 and 15. As shown in Sec. III, all FG-LDPC codes found in [17] can be constructed using our method. The subdesign deletion techniques further give infinitely many codes by fine-tuning their parameters and error-correction performance. In this sense, our method gives many kinds of excellent EAQECCs in a single framework.

TABLE XIII. Summary of Type II EAQECCs obtained by deleting subdesigns from $AG_1(3,3)$.

Subs ^a	n	rank H	k	d	c	Rate
0	117	27	64	6	1	0.5470
1	105	27	60	6	9	0.5714
2	93	26	58	6	17	0.6236
3	81	25	56	6	25	0.6913

^aThis column denotes the number of subdesigns removed.

V. CONCLUSION

We have developed a general framework for constructing entanglement-assisted quantum LDPC codes using combinatorial design theory. Our constructions generate infinitely many codes with various desirable properties such as high error-correction performance and high rates and requiring only one initial entanglement bit. Our methods are flexible and allow us to design EAQECCs with desirable properties while requiring prescribed amounts of entanglement. All quantum codes constructed in this article can be efficiently decoded through the sum-product algorithm.

We have introduced many families of entanglement-assisted quantum LDPC codes based on combinatorial designs as well as determined all fundamental parameters of the well-known families of LDPC codes based on finite geometries for most cases. Because the entanglement-assisted stabilizer formalism bridges classical and quantum codes in a direct manner, these results on entanglement-assisted quantum LDPC codes are useful both in quantum and classical coding theories.

Our framework encompasses many previously proposed excellent quantum LDPC codes as well. In fact, our method can also be applied to quantum LDPC codes under the standard stabilizer formalism by employing the ideas found in [23,24].

We have focused on the fundamental classes of combinatorial designs. However, other classes of incidence structures may provide interesting results as well. For example, the entanglement-assisted quantum LDPC codes presented in [16] can be seen as incidence structures generated from the so-called difference matrices and their generalizations (see [44] for the definition and basic facts about difference matrices). More general families of combinatorial designs can have nested structures or similar strong orthogonal relations between two incidence matrices. This kind of structure can give asymmetric quantum codes (see [57,58]). Structures in finite geometry we did not employ may also give interesting quantum LDPC codes as well as classical LDPC codes (see, for example, [59,60]). Because LDPC codes and sparse incidence structures are equivalent, we expect that our methods may be further generalized to encompass a wider range of both new and known quantum LDPC codes in future work.

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APPENDIX A: EXISTENCE OF 2-DESIGNS

Here we discuss the existence of 2-designs to be applied to our constructions given in Sec. II B. The following is the well-known asymptotic existence theorem.

TABLE XIV. Parameters of entanglement-assisted quantum LDPC codes from finite geometries.^a

Geometry	Type ^b	m	q	n	k	d	c	Girth
PG	II	Any	2^t	$\frac{(q^{m+1}-1)(q^m-1)}{(q^2-1)(q-1)}$	$\frac{(q^{m+1}-1)(q^m-1)}{(q^2-1)(q-1)} - 2\varphi(m, 2^t) + 1$	$q + 2$	1	6
PG	II	Odd	Odd	$\frac{(q^{m+1}-1)(q^m-1)}{(q^2-1)(q-1)}$	$\frac{(q^{m+1}-1)(q^m-1)}{(q^2-1)(q-1)} - 2\frac{q^{m+1}-q}{q-1} + 1$	$2(q + 1)$	1	6
PG	II	Even	Odd	$\frac{(q^{m+1}-1)(q^m-1)}{(q^2-1)(q-1)}$	$\frac{(q^{m+1}-1)(q^m-1)}{(q^2-1)(q-1)} - \frac{q^{m+1}-q}{q-1}$	$2(q + 1)$	$\frac{q^{m+1}-q}{q-1}$	6
PG	I	2	2^t	$q^2 + q + 1$	$q^2 + q - 2 \times 3^t$	$q + 2$	1	6
PG	I	Any	2^t	$\frac{q^{m+1}-1}{q-1}$	$\frac{q^{m+1}-1}{q-1} - 2\varphi(m, 2^t) + c$	$(q + 2)q^{m-2}$	$\leq \varphi(m, 2^t)$	6
AG	II	Any	2^t	$q^{m-1} \frac{q^m-1}{q-1}$	$q^{m-1} \frac{q^m-1}{q-1} - 2\varrho(m, 2^t) + 1$	$q + 1$	1	6
AG	II	Odd	Odd	$q^{m-1} \frac{q^m-1}{q-1}$	$q^{m-1} \frac{q^m-1}{q-1} - 2q^m + 1$	$2q$	1	6
AG	II	Even	Odd	$q^{m-1} \frac{q^m-1}{q-1}$	$q^{m-1} \frac{q^m-1}{q-1} - q^m - 1$	$2q$	$q^m - 1$	6
AG	I	2	2^t	q^2	$q^2 + q - 2 \times 3^t$	$q + 2$	q	6
AG	I	Any	2^t	$q^{m-1} \frac{q^m-1}{q-1}$	$q^{m-1} \frac{q^m-1}{q-1} - 2\varrho(m, 2^t) + c$	$(q + 2)q^{m-2}$	$\leq \varrho(m, 2^t)$	6
EG	I, II ^c	2	2^t	$q^2 - 1$	$q^2 + q - 2 \times 3^t + 1$	$q + 1$	q	6
EG	II	Any	2^t	$\frac{(q^{m-1}-1)(q^m-1)}{q-1}$	$\frac{(q^{m-1}-1)(q^m-1)}{q-1} - 2\varrho(m, 2^t) + 2 + c$	$q + 1$	$\frac{q^m-q}{q-1}$	6

^aAll codes are $[[n, k, d; c]]$ EAQECCs obtained from $PG_1(m, q)$, $AG_1(m, q)$, or $EG_1(m, q)$. We omit EAQECCs which are created by subdesign deletion techniques or do not have dimension greater than one. $\varphi(m, 2^t)$ is given by Theorem 19 in Sec. III A. $\varrho(m, 2^t)$ is defined as $\varrho(m, 2^t) = \varphi(m, 2^t) - \varphi(m - 1, 2^t)$.

^bType refers to the traditional classification of FG-LDPC codes: Type I uses a line-by-point incidence matrix, while type II uses the transposed (i.e., point-by-line) incidence matrix.

^cThe codes obtained from either orientation of the incidence matrix are identical [18].

Theorem 42: Wilson [61–63]. The necessary conditions for the existence of a 2 - (v, μ, λ) design, $\lambda(v - 1) \equiv 0 \pmod{\mu - 1}$ and $\lambda v(v - 1) \equiv 0 \pmod{\mu(\mu - 1)}$, are also sufficient if $v > v_{\mu, \lambda}$, where $v_{\mu, \lambda}$ is a constant depending only on μ and λ .

For $\mu \in \{3, 4, 5\}$, necessary and sufficient conditions for the existence of an $S(2, \mu, v)$ are known.

Theorem 43: Kirkman [64]. There exists an STS(v) if and only if $v \equiv 1, 3 \pmod{6}$.

Theorem 44: Hanani [65]. There exists an $S(2, 4, v)$ if and only if $v \equiv 1, 4 \pmod{12}$.

Theorem 45: Hanani [66]. There exists an $S(2, 5, v)$ if and only if $v \equiv 1, 5 \pmod{20}$.

For $\mu \geq 6$, the necessary and sufficient conditions on v for the existence of an $S(2, \mu, v)$ are not known in general, although for small values of μ substantial results are known.

For a comprehensive table of known Steiner 2-designs, see [44].

Theorems 42, 43, 44, and 45 were proved by constructive methods. Hence, these existence results allow us to construct infinitely many explicit examples of entanglement-assisted quantum LDPC codes. It is worth mentioning that many of the known proofs of these theorems employ the same construction technique we used in Theorem 18. In fact, most $S(2, \mu, v)$'s in the original proofs of these existence theorems have either Steiner spreads or nontrivial subdesigns.

Numerous other constructions for 2-designs also give explicit examples of $S(2, \mu, v)$'s for a wide range of parameters. A detailed treatment of STS(v)'s is available in [67]. Various constructions for $S(2, \mu, v)$'s for many values of μ are also given in [68].

TABLE XV. Parameters of classical FG-LDPC codes.^a

Geometry	Type	m	q	n	k	d	Girth
PG	II	Any	2^t	$\frac{(q^{m+1}-1)(q^m-1)}{(q^2-1)(q-1)}$	$\frac{(q^{m+1}-1)(q^m-1)}{(q^2-1)(q-1)} - \varphi(m, 2^t)$	$q + 2$	6
PG	II	Any	Odd	$\frac{(q^{m+1}-1)(q^m-1)}{(q^2-1)(q-1)}$	$\frac{(q^{m+1}-1)(q^m-1)}{(q^2-1)(q-1)} - \frac{q^{m+1}-q}{q-1}$	$2(q + 1)$	6
PG	I	Any	2^t	$\frac{q^{m+1}-1}{q-1}$	$\frac{q^{m+1}-1}{q-1} - \varphi(m, 2^t)$	$(q + 2)q^{m-2}$	6
AG	II	Any	2^t	$q^{m-1} \frac{q^m-1}{q-1}$	$q^{m-1} \frac{q^m-1}{q-1} - \varphi(m, 2^t) + \varphi(m - 1, 2^t)$	$q + 1$	6
AG	II	Any	Odd	$q^{m-1} \frac{q^m-1}{q-1}$	$q^{m-1} \frac{q^m-1}{q-1} - q^m$	$2q$	6
AG	I	Any	2^t	q^m	$q^m - \varphi(m, 2^t) + \varphi(m - 1, 2^t)$	$(q + 2)q^{m-2}$	6
EG	I, II ^b	2	2^t	$q^2 - 1$	$q^2 - 3^t$	$q + 1$	6
EG	II	Any	2^t	$(q^{m-1} - 1) \frac{q^m-1}{q-1}$	$(q^{m-1} - 1) \frac{q^m-1}{q-1} - \varphi(m, 2^t) + \varphi(m - 1, 2^t) + 1$	$q + 1$	6
EG	II	≥ 3	Odd	$(q^{m-1} - 1) \frac{q^m-1}{q-1}$	$\geq (q^{m-1} - 1) \frac{q^m-1}{q-1} - q^m + 1^c$	$2q$	6

^aWe omit the cases when codes are created by subdesign deletion techniques or do not have enough dimension.

^bThe codes obtained from either orientation of the incidence matrix are identical [18].

^cIf Hamada's conjecture on the rank of $EG_1(m, q)$ [35] is true, the equation holds.

APPENDIX B: PARAMETERS OF QUANTUM AND CLASSICAL FG-LDPC CODES WITH GIRTH 6

Here we give tables of parameters of LDPC codes with girth 6 based on finite geometries. Table XIV gives parameters

of entanglement-assisted quantum LDPC codes obtained from $PG_1(m, q)$, $AG_1(m, q)$, and $EG_1(m, q)$. Parameters of the corresponding classical FG-LDPC codes are listed in Table XV.

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