Entanglement, detection, and geometry of nonclassical states

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Nonclassical states that are characterized by their nonpositive quasiprobabilities in phase space are known to be the basis for various quantum effects. In this work, we investigate the interrelation between the nonclassicality and entanglement, and then characterize the nonclassicality that precisely corresponds to entanglement. The results naturally follow from two findings: one is the general structure among nonclassical, entangled, separable, and classical states over Hermitian operators, and the other a general scheme to detect nonclassical states.

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I. INTRODUCTION

In the seminal paper of the quantum theory of light, Glauber has shown that quantum systems reveal their *nonclassicality* by nonpositive quasiprobability distributions in phase space that classical systems fail to describe [1]. By not relying only on the correlational effects among quantum systems, the nonclassicality turned out to be the basis for various quantum effects. In past decades, quantum states with correlations that could not be prepared by local operations and classical communications (i.e., entanglement) had been extensively investigated; their essential role was to outperform classical counterparts in information processing. The presence of the nonclassicality is more primitive than entanglement as the nonclassicality has to exist if entangled states are to be generated [2]. Or equivalently, multimode classical states can *never* be entangled, and all entangled states are already nonclassical. For instance, the nonclassicality of the initial system is a quantity preserved under transformations via linear optical elements [3] that are often used in entanglement engineering, such as quantum computation [4]. Consequently, the preexisting nonclassicality dictates or already limits the entanglement that can be manipulated. Therefore, to have a precise estimate of the nonclassicality in connection to entanglement is, not only of theoretical interest, a line that allows us to decide the intrinsic capability of given quantum systems in information processing.

In this work, we characterize entanglement in terms of the nonclassicality with the nonclassicality measure in Ref. [5]. The nonclassicality that precisely corresponds to entanglement is refined. The result is derived from the geometry of quantum states over quasiprobability distributions, which is based on the nonclassical states detection method that we shall show later. These findings are also of fundamental importance, devoted to discovering the convex geometry of physical, separable, and classical states out of Hermitian operators.

This paper is organized as follows. We first introduce a map that detects *all* nonclassical states. The map can also be translated to witness operators, which may be called as nonclassicality witnesses. By using the map, we identify positive operators (i.e., physical states) out of positive s-ordered quasiprobability distributions. Remarkably, all the positive s-ordered quasiprobability distributions, except the normally ordered one, do not necessarily correspond to physical states. This then leads to the geometric refinement of the nonclassicality that finally defines the entanglement parameter: the nonclassicality depth of entanglement.

II. DETECTING NONCLASSICAL STATES

Let us begin by introducing nonclassical states and the nonclassicality measure. Quasi-probability distributions in phase space are in general parametrized by the ordering parameter s which takes values in $[0,1]$ and have the range from the Glauber-Sudarshan P function ($s=0$) to the Husimi Q function ($s=1$). For the symmetric ordering $s=1/2$, the quasiprobability distribution corresponds to the Wigner function. For a given quantum state, when its quasiprobability distribution for any s has negative values, the state is referred to as nonclassical. For instance, negativity in the Wigner function has been often used as the signature of nonclassicality [6]. Then, in general, it holds that if a quantum state has negative probabilities in some s-ordered representation, its P function must have negative probabilities. This can be easily seen by the regularization processing of nonpositive P functions, which is shown in the following.

A multimode state ρ can be uniquely written in terms of the P function as [7]

$$\rho = \int \prod_{i=1}^{n} d^2 z_i \, P(z_1,\ldots,z_n) \bigotimes_{i=1}^{n} |z_i\rangle\langle z_i|, \tag{1}$$

where $|z_i\rangle$ in the ith mode are coherent states. When the P function has negative values, the state is referred to as nonclassical. A nonpositive P function can be transformed into a true (i.e., non-negative) probability distribution by the following regularization processing:

$$R_\tau[P](\alpha_1,\ldots,\alpha_n) = \int \prod_{i=1}^{n} \frac{d^2\alpha_i'}{\pi\tau} e^{-|\alpha-\alpha_i'|^2/\tau} P(\alpha_1,\ldots,\alpha_n), \tag{2}$$

where $0<\tau\leqslant 1$ and $R_0[P](\alpha)=P(\alpha)$. Note that once $R_\tau[P]\geqslant 0$ it holds that $R_{\tau'}[P]\geqslant 0$ for all $\tau'\geqslant\tau$, from which the minimum τ that regularizes a given P function, denoted by $\tau_m[\rho]$ throughout the paper, has been defined as a measure

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to quantify the nonclassicality and called the nonclassicality depth [5,8]. In fact, the regularization with s corresponds to the transformation of the quasiprobability distribution from the normally ordered representation to an s-ordered one. The Q function is given when $s = 1$, and can be expressed as $Q(\alpha) = \langle\alpha|\rho|\alpha\rangle \geqslant 0$ with coherent states basis $|\alpha\rangle$. It is clear that Q functions are always non-negative. Therefore, if a given Hermitian operator ρ cannot be regularized with $s \leqslant 1$, then one can conclude that the operator is nonpositive. This also means, due to the fact that the Q function is nonpositive, that there exists some coherent state $|\beta\rangle$ to detect a negative expectation of the Q function [i.e., $Q(\beta) = \langle\beta|\rho|\beta\rangle < 0$].

Lemma 1. A Hermitian operator of unit trace is nonpositive if it cannot be regularized by the transformation in Eq. (2).

It is clear that classical states form a convex set, since a convex combination of non-negative P functions is automatically non-negative and thus constitutes a new P function of the corresponding mixed state. The convex structure implies that the characterization of the set of classical states can be greatly simplified by the so-called witness operators developed significantly in the entanglement theory. For Hermitian operators, the Hahn-Banach theorem can be applied so that one can always find a Hermitian operator W such that for all classical states σ, $\mathrm{tr}(W\sigma) \geqslant 0$ while $\mathrm{tr}(W\rho) < 0$ for some nonclassical states ρ, which may, therefore, be called nonclassicality witnesses [9].

We now introduce the map that detects all nonclassical states. Here, detection means that nonclassical states are mapped to nonpositive operators so that the nonclassical states are detected by negative expectation values. The map can be defined on the P function as follows:

$$\Lambda_a[P(z_i)] = P_a(z_i) = \frac{1}{a^2} P\left(\frac{z_i}{a}\right), \tag{3}$$

which in fact describes, for $a \in [0,1]$, the state that has transmitted the beam splitter with transmittance $T = a^2$. (See Fig. 1.) The expression of the map can also be obtained on the level of states by considering state transformation under the beam splitter and then taking $a > 1$. Note that non-negative P functions remain non-negative under the map Λ_a.

Proposition 1. For a nonclassical state ρ, there exists $a > 1$ such that $\Lambda_a[\rho] \ngeqslant 0$.

Proof. For simplicity, let us introduce the characteristic function of ρ through the Fourier transformation of the P function, $\chi(x) = \int d^2\alpha\, P(\alpha) e^{\alpha x^* - \alpha^* x}$. By regularizing the P function with τ, the characteristic function is transformed to $K_\tau(\beta) = e^{-\tau|\beta|^2}\chi(\beta)$. The Bochner's theorem tells that the regularized function is positive if and only if the characteristic function is positive definite [11].

Suppose that the P function of a nonclassical state ρ can be regularized to a positive distribution with τ_m. The map Λ_a is applied to the state, and then the characteristic function denoted by $\chi_a(\beta)$ can be expressed as $\chi_a(\beta) = \chi(a\beta)$. Now, we want to see the $\tau_{a,m}$ that regularizes $\Lambda_a[\rho]$ on the level of the characteristic function, $K_{a,\tau}(\beta) = e^{-\tau_a|\beta|^2}\chi_a(\beta)$. Since $K_\tau(\beta)$ is positive if and only if $\tau \geqslant \tau_m$, $K_{a,\tau}(\beta)$ is positive if and only if $\tau_a \geqslant a^2\tau_m$, and so is $R_\tau[\Lambda_a(P)]$, from which $\tau_{a,m} = a^2\tau_m$. If a is large enough that $\tau_{a,m} > 1$, this implies that $\Lambda_a[\rho]$ cannot be regularized and therefore, according to Lemma 1, $\Lambda_a[\rho]$ is nonpositive. In particular, a map Λ_a with $a > 1/\sqrt{\tau_m}$ (which is larger than 1 since $\tau_m \leqslant 1$) can detect nonclassical states having τ_m. Since $\tau_m > 0$ for all nonclassical states, the map Λ_a can detect all nonclassical states. ■

The preceding proof can also derive the useful relation, $\Lambda_a \circ R_\tau[P] = R_{a^2\tau} \circ \Lambda_a[P]$, which provides a geometrical structure of s-ordering representation as follows.

Lemma 2. Let $\tau_m[\rho]$ denote the nonclassicality depth of state ρ. By using the map Λ_a, the nonclassicality depth is mapped to $\tau_m[\![\Lambda_a[\rho]]\!] = a^2\tau_m[\rho]$.

The idea behind the map comes from the fact that nonphysical operations may cause certain effects which cannot be interpreted as being physical. For instance, in the entanglement theory, nonphysical operations, such as positive but not completely positive maps, detect all entangled states, exploiting negative expectation values. As such, the map Λ_a with $a > 1$ transforms nonclassical states into operators, providing negative expectation values. Together with Lemma 2, it can also be seen that the map is nonphysical in that Λ_a with $a > 1$ describes the reverse direction from the output to the input states, increasing the nonclassicality depth, which is of course nonphysical as a time-reversal process. In this way, only classical states are not detected since their nonclassicality depth is constantly zero.

Example. A nonclassical state that is not detected by the criteria shown in Ref. [12] was presented in Ref. [13], and its P function is $P(\alpha) = \frac{2}{\pi}e^{-|\alpha|^2} - \delta(\alpha)$. We now apply the map Λ_a to detect that ρ is nonclassical,

$$\langle\beta|\Lambda_a[\rho]|\beta\rangle = \frac{2}{a^2+1}\exp\left(-\frac{|\beta|^2}{1+a^2}\right) - \exp(-|\beta|^2),$$

which is nonpositive for a sufficiently large a. ■

The map in Eq. (3) can be translated into what we may call nonclassicality witnesses. For $\Lambda_a[\rho]$ that cannot be regularized, the Q function is not positive, meaning that there exists coherent state $|\beta\rangle$ such that $Q(\beta) = \langle\beta|\Lambda_a[\rho]|\beta\rangle < 0$. Being constrained to keep the expectation value the same, the dual map Λ_a^* can be obtained and applied to evolution of the coherent state, $W_\beta = \Lambda_a^*[|\beta\rangle\langle\beta|]$, such that the following holds:

$$\mathrm{tr}(\rho W_\beta) = \mathrm{tr}(|\beta\rangle\langle\beta|\Lambda_a[\rho]).$$

Note that the collection of all W_β can completely characterize the set of classical states, since (i) classical states form a

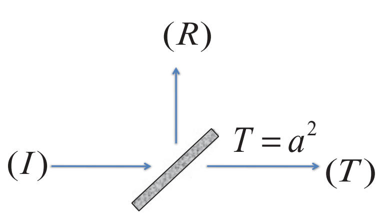

FIG. 1. (Color online) The (I), (T), and (R) are input, transmitted, and reflected states. The map in Eq. (3) when $0 \leqslant a \leqslant 1$ describes the relation between the input and the transmitted states. For $a > 1$, the map in Eq. (3) can be thought of as a "nonphysical" direction from (T) to (I).

convex set, and (ii) Λ_a detects all nonclassical states. Although coherent states are applied here due to the Q function, in general, any Hermitian operators that overlap with negative ranges of $\Lambda_a[\rho]$ can be in the case.

III. GEOMETRY OF NONCLASSICAL STATES

So far, we have seen quantum states in terms of positive distributions in s-ordering representation, based on which the map Λ_a with $a>1$ is shown to increase the nonclassicality depth so that physical states are sent away to nonpositive ones. However, positive quasiprobability distributions do not mean physical operators in general [14–16]. For instance, the Q function of the following operator, $A = k|0\rangle\langle 2| + |1\rangle\langle 1| + k^*|2\rangle\langle 0|$, which is the unit trace and Hermitian but not positive, has the positive Q function, $Q_A(\beta) = |k|^2(\beta+\beta^*)^2 \geqslant 0$. In which value of s do positive quasiprobability distributions mean positive operators? In the following text, $s=0$ is shown to be the only case.

First, positive operators form a convex set, including separable and classical states. Let us then show that there exists a nonpositive operator that can be regularized with very small τ in Eq. (2). The nonclassical state, $\rho = (1-\epsilon)|0\rangle\langle 0| + \epsilon|2\rangle\langle 2|$, is arbitrarily close to the vacuum as ϵ tends to 0. Applying the map, $\Lambda_a[\rho] = \sum_{i=0}^{2} r_i |i\rangle\langle i|$ with $r_0 = (1-2\epsilon a^2 + \epsilon a^4)$, $r_1 = 2\epsilon a^2(1-a^2)$, and $r_2 = \epsilon a^4$. Note that the coefficients r_i are the eigenvalues of $\Lambda_a[\rho]$, meaning that $\Lambda_a[\rho]$ becomes nonpositive whenever $a>1$ (since $r_1<0$). The regularization of $\Lambda_a[\rho]$ is

$$R_\tau[P](z) = \frac{1}{\tau}\exp\left(-\frac{|z|^2}{\tau}\right)\left\{r_0 + r_1\left(\frac{|z|^2}{\tau^2} - \frac{1-\tau}{\tau}\right) + r_2\left[\frac{|z|^4}{2\tau^4} - \frac{2(1-\tau)}{\tau^3}|z|^2 + \left(\frac{1-\tau}{\tau}\right)^2\right]\right\}.$$

From the above, the depth of the nonclassicality is

$$\tau_m[\Lambda_a[\rho]] = \frac{a^2\sqrt{\epsilon}}{\sqrt{1-\epsilon}+\sqrt{\epsilon}},$$

which can be arbitrarily small for $a>1$ by taking ϵ, which tends to be a very small number. In summary, this shows that outside the set of positive states there exists a nonpositive operator which can still have a very small nonclassicality depth. This leads to the following conclusion.

Proposition 2. For all $s \in (0,1]$, there exist positive quasiprobability distributions that may correspond to nonpositive Hermitian operators.

Based on Proposition 2, plus the fact that separable and classical states form convex sets, respectively, the positive operators can be drawn over the axis of the nonclassicality depth as is shown in Fig. 2. Note that, in general, quantum states of the same nonclassicality depth form a convex set. Figure 2 is drawn as well based on the following facts. First, there are separable states having the unit nonclassicality depth as $\tau_m[\rho_A\otimes\rho_B]=1$ if and only if either $\tau_m[\rho_A]=1$ or $\tau_m[\rho_B]=1$. Separable states also consist of all classical states since classical states are already separable. Next, there are entangled states having a nonunit nonclassicality depth, which can be seen by the state

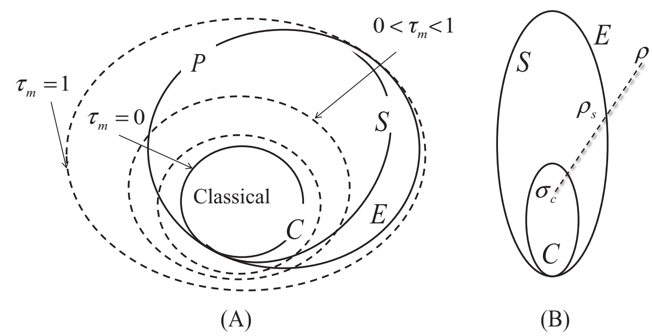

FIG. 2. (A) Physical states are shown over the axis of the nonclassicality depth. Positive operators (P) are only a subset from $\tau=0$ to $\tau=1$, including the set of separable states (S) which consists of all classical states. (B) The convex geometry of separable and classical states leads the entanglement parameter that takes into account the nonclassicality, the NcDE in Eq. (7).

$$\rho_p = p|\phi^+\rangle\langle\phi^+| + (1-p)\frac{I}{4}, \tag{4}$$

where $|\phi^+\rangle = (|00\rangle+|11\rangle)/\sqrt{2}$ and $I=\sum_{i,j=0}^{1}|ij\rangle\langle ij|$. The regularization is given as

$$\begin{aligned} R_\tau[P](z_A,z_B) &= \frac{1}{4\tau^2}\exp\left(-\frac{|z_A|^2+|z_B|^2}{\tau}\right)\mathcal{A}_\tau(z_A,z_B),\\ \mathcal{A}_\tau(z_A,z_B) &= 2p\left|1+\frac{z_Az_B}{\tau^2}\right|^2 + (1-p)\frac{|z_Az_B|^2}{\tau^4}\\ &\quad + 2p\left(\frac{1-\tau}{\tau}\right)^2 + (1-p)\left(\frac{2\tau-1}{\tau}\right)^2\\ &\quad + \frac{|z_A|^2+|z_B|^2}{\tau^3}(2\tau-1-p), \end{aligned} \tag{5}$$

from which $\tau_m[\rho_p] = (1+p)/2$ for the state. It is known that the state ρ_p is separable if and only if $p\leqslant 1/3$ [17]. Hence, as the value p decreases from 1 to 0, $\tau_m[\rho_p]$ decreases from 1 to 1/2, at which the state $\tau_m[\rho_p]$ passes through the border between the separable and the entangled states when $\tau=2/3$ (or, equivalently, $p=1/3$). All this constitutes the geometry of physical, entangled, separable, and classical states, as shown in Fig. 2.

IV. ENTANGLEMENT OF NONCLASSICAL STATES

Based on the convex geometry shown in Fig. 2, we are now ready to geometrically characterize what part of the nonclassicality corresponds to the correlational property, entanglement. This is inspired by a geometric entanglement measure, the robustness of entanglement in Ref. [18] that was based on the convexity of separable states. As such, here we are based on the convexity of classical states. For a multimode

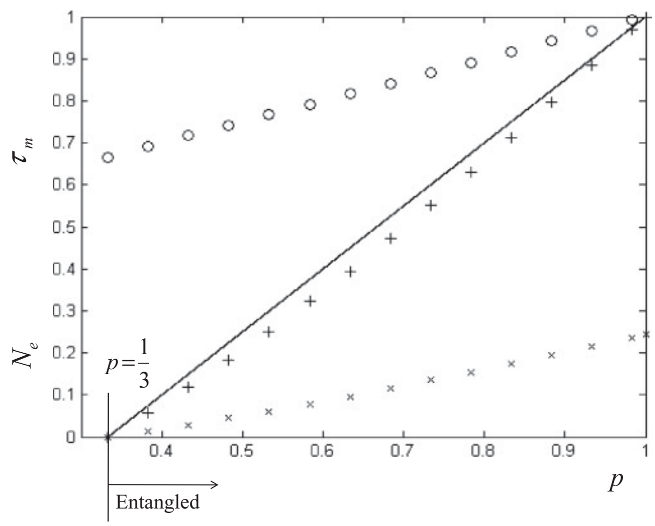

FIG. 3. The NcDE (plotted by $\times$) N_e of the state ρ_p is fitted. Note that ρ_p is entangled if and only if $p \geqslant 1/3$. The circle shows the nonclassicality depth τ_m of the state for each p: $\tau_m[\rho_p] = (1+p)/2$. The normalized NcDE is plotted by $+$, which is proportional to the entanglement parameter, negativity (solid line), $\mathcal{N}(\rho_p) = (3p-1)/2$.

nonclassical state ρ, since classical states are a subset of separable states there always exists a separable state ρ_s by admixing ρ with a classical σ, as follows:

$$\rho_s = \min_{\kappa} \frac{1}{\kappa+1}(\rho + \kappa\sigma). \tag{6}$$

Note that ρ_s lies on the boundary of separable states in Fig. 2. Using the state ρ_s, one can divide the nonclassicality depth into two: one from entangled to separable states and the other from separable to classical states.

Definition of the nonclassicality depth of entanglement (NcDE). The NcDE of ρ with respect to the classical state σ, denoted by $N_e(\rho\|\sigma)$, is $N_e(\rho\|\sigma) = \tau_m[\rho] - \tau_m[\rho_s]$, with the state ρ_s found by Eq. (6). The NcDE of a state ρ is

$$N_e(\rho) = \min_{\sigma\in C} N_e(\rho\|\sigma), \tag{7}$$

where the minimization runs over the set of all classical states C.

It is clear that by definition the NcDE of separable states is zero. Then, for an entangled state ρ, the nonclassical depth is strictly larger than the ρ_s defined in Eq. (6) for all classical states σ. This can be seen by the inequality that for nonclassical ρ and classical σ, it holds $\tau_m[\rho] > \tau_m[(1-\epsilon)\rho + \epsilon\sigma]$ for all $0 < \epsilon \leqslant 1$. The details are shown in the Appendix. Hence, the NcDE is an entanglement parameter first derived from the nonclassicality of quantum states.

In general, to obtain the NcDE, one should minimize the $N(\rho\|\sigma)$ over all classical states, and then apply a separability criteria to obtain a separable state ρ_s that is interpolated by ρ and σ. Moreover, the optimization processing in the NcDE runs for all bosonic systems, including non-Gaussian states, for which little is known about the separability criteria. It is thus generally hard to explicitly evaluate.

To illustrate the NcDE with an example, let us now explicitly compute the NcDE for the state ρ_p in Eq. (4). Since the optimization generally difficult, the NcDE is computed here with respect to the ansatz state, σ, as follows. As can be seen in Eq. (5), the state σ that classicalizes ρ_p should remove the nonpositive part of the P function of ρ_p. That is, the P function of σ is

$$P_\sigma(z_A, z_B) = \frac{|z_A|^2 + |z_B|^2}{2\pi^2\tau^3} \exp\left(-\frac{|z_A|^2 + |z_B|^2}{\tau}\right). \tag{8}$$

Let $\rho(\beta)$ denote the mixture of ρ_p with the classical state σ, $\rho(\beta) = (\rho_p + \beta\sigma)/(1+\beta)$, which is classical if and only if $\beta \geqslant \pi^2\tau^{-2}(1+p-2\tau)/2$. Then, one can find the minimal value of β_s such that $\rho(\beta_s)$ is separable. Note that ρ_p is defined in a $(2\otimes 2)$-dimensional Hilbert space and σ is separable. Therefore, decomposing the state σ with a two-dimensional number basis $\{|0\rangle, |1\rangle\}$ and the rest [i.e., $(2\otimes 2)$ subsystems are only relevant], one can apply the known separability criteria [17]. Finally,

$$\beta_s = \frac{(3p-1)(1+\tau)^4}{6\tau}. \tag{9}$$

The nonclassicality depth of the state, $\tau_m[\rho(\beta_s)]$, can be obtained by numerics, and the NcDE of the state is plotted in Fig. 3. The NcDE behaves similarly with the known entanglement measure, the negativity.

V. CONCLUSION

To conclude, we provide the general method of detecting nonclassical states and find the geometry of physical states over positive s-ordered quasiprobability distributions. It is shown that the positive s-ordered quasiprobability distribution can generally correspond to nonpositive operators. Together with the convexity of positive operators, the set of positive (i.e., physical) states are characterized. Based on the geometry, we have finally derived the entanglement parameter from the nonclassicality, the NcDE.

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APPENDIX

We prove the inequality for a nonclassical state ρ and a classical one σ,

$$\tau_m[\rho] > \tau_m[(1-\epsilon)\rho + \epsilon\sigma] \tag{A1}$$

for $0 < \epsilon \leqslant 1$, where τ_m is defined as the minimum amount of thermal noise in the regularization processing,

$$R_\tau[P](\alpha) = \int \frac{d^2\alpha'}{\pi\tau} e^{\frac{|\alpha-\alpha'|^2}{\tau}} P(\alpha'). \tag{A2}$$

First, let $P_\rho(z)$ denote the P function of a nonclassical state ρ, for which there exists the minimum value τ_1 that the P function is regularized [i.e., $R_{\tau_1}[P_\rho](z) \geqslant 0$]. Hence, we have $\tau_1 = \tau_m[\rho]$.

Let τ_2 denote $\tau_m[(1-\epsilon)\rho + \epsilon\sigma]$ for $0 < \epsilon \leqslant 1$ and a classical state σ, i.e., $R_{\tau_2}[(1-\epsilon)P_\rho + \epsilon P_\sigma] \geqslant 0$, where P_σ is the P function of σ. Note that since σ is classical $R_\tau[\sigma]$ is positive for all $\tau \geqslant 0$. Also note that the regularization is linear, $R_{\tau_2}[(1-\epsilon)P_\rho + \epsilon P_\sigma] = (1-\epsilon)R_{\tau_2}[P_\rho] + \epsilon R_{\tau_2}[P_\sigma]$. This means that, by τ_2 in (A2), the function P_ρ is not yet regularized but transformed such that

$$R_{\tau_2}[P_\rho] \geqslant -\frac{\epsilon}{1-\epsilon} R_{\tau_2}[P_\sigma]. \tag{A3}$$

The right-hand side of (A3) is negative, meaning again that by τ_2 in (A2) the P function P_ρ is not regularized. Therefore, there exists the minimum value $\tau > 0$ that regularizes $R_{\tau_2}[P_\rho]$, i.e.,

$$R_\tau[\![R_{\tau_2}[P_\rho]]\!] \geqslant 0. \tag{A4}$$

Let us recall the identity $R_{a+b}[P] = R_a[\![R_b[P]]\!]$. Hence, we arrive at the identity, for $\tau > 0$, $\tau_1 = \tau + \tau_2$. It is thus proved that $\tau_m[\rho] > \tau_m[(1-\epsilon)\rho + \epsilon\sigma]$.

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