Quantum Fisher information flow and non-Markovian processes of open systems

Xiao-Ming Lu,[1] Xiaoguang Wang,[1,*] and C. P. Sun[2,†]
[1]*Zhejiang Institute of Modern Physics, Department of Physics, Zhejiang University, Hangzhou 310027, China*
[2]*Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100080, China*
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We establish an information-theoretic approach for quantitatively characterizing the non-Markovianity of open quantum processes. Here, the quantum Fisher information (QFI) flow provides a measure to statistically distinguish Markovian and non-Markovian processes. A basic relation between the QFI flow and non-Markovianity is unveiled for quantum dynamics of open systems. For a class of time-local master equations, the exactly analytic solution shows that for each fixed time the QFI flow is decomposed into additive subflows according to different dissipative channels.

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I. INTRODUCTION

Any system in the realistic world is open since it inevitably interacts with its environment. The time evolutions of open systems are simply classified into Markovian and non-Markovian ones according to the ways in which they lose energy or information [1]. In most situations, the Markovian process uniquely determines its final steady state as a thermal equilibrium [2], which is independent of its initial one. In this sense a Markovian process is essentially an information erasure process, thus, it tends to continuously reduce the distinguishability between any two initial states [3].

However, the Markovian description for an open quantum system is only an approximation of most of the realistic processes, which are non-Markovian. With many recent investigations about non-Markovian dynamics making use of various analytical approaches and numerical simulations, a computable measure of “Markovianity” for quantum channels was introduced in Ref. [4]. Most recently, it was also recognized that the difference between them can be measured through the continuous increment of the state distinguishability [3]. This increment, then, is intuitively interpreted as the revival of information flow between the bath and the system though no quantitative information measure is utilized. Based on this measure of non-Markovianity, a method for direct measurement of the non-Markovian character was proposed [5]. Another approach based on entanglement is proposed in Ref. [6]. In this paper, the quantum Fisher information (QFI) flow is introduced to directly characterize the non-Markovianity of the quantum dynamics of open systems.

Actually, in the system-plus-bath approach for open quantum systems, the effective dynamics of the reduced density matrix ρ is induced by tracing over the environment [1]. The simplest reduced dynamics is the quantum Markovian process described by dynamical semigroups [7]. There, the reduced density matrix ρ at time $t + dt$ is determined completely by the one at time t. Contrarily, the general reduced dynamics may be non-Markovian when the surrounding environment may retain a memory of the information about states at earlier times, and transfer it back to the system to affect its evolution. In this sense the Markovian process only happens when the environmental correlation time is relatively short so that memory effects can be neglected. These memory-based considerations for the Markovian approximation also mean that the information-theoretical characterization of non-Markovianity is a quite natural fashion. However, it is still an open question about how to treat the information flow in open quantum systems based on a solid information-theoretic foundation.

In this paper, we establish such a framework by adopting the QFI flow as the quantitative measure for the information flow. The QFI characterizes the statistical distinguishability of the reduced density matrix [8,9]. An intuitive picture of the memory effect of a non-Markovian behavior then immediately follows from the dynamic return of the QFI, which is depicted by the inward QFI flow. For a class of the non-Markovian master equations in time-local forms, we exactly calculate the information flows. The analytic results show that the total QFI flow can be decomposed into the split contributions from different dissipative channels for each fixed time. On the other hand, the QFI plays an essential role in quantum metrology [10], where the highest precision of estimating an unknown parameter we may achieve is related to the inverse of the QFI. We point out that this QFI flow approach is feasible for understanding the problems of quantum metrology.

II. QUANTUM FISHER INFORMATION IN NON-MARKOVIAN DYNAMICS

We consider the quantum processes described by the following time-local master equation [3,11]:

$$\frac{\partial}{\partial t}\rho(t) = \mathcal{K}(t)\rho(t), \tag{1}$$

where $\mathcal{K}(t)$ is a superoperator acting on the reduced density matrix $\rho(t)$ as [11–13]

$$\mathcal{K}(t)\rho = -i[H,\rho] + \sum_i \gamma_i \left[A_i \rho A_i^\dagger - \frac{1}{2}\{A_i^\dagger A_i,\rho\}\right], \tag{2}$$

with $H(t)$ is the Hermitian Hamiltonian for the open quantum system without the couplings to the bath. $\{\cdot,\cdot\}$ denotes the anticommutator. If all γ_i and A_i are time independent, and all γ_i are positive, Eq. (2) is the conventional master equation of the Lindblad form [12], which describes the conventional Markovian process. However, by making use of a variety of

*xgwang@zimp.zju.edu.cn
†suncp@itp.ac.cn

methods, such as the time-convolutionless projection operator technique [14], the Feynman-Vervon influence functional theory [15], and some others [16], the parameters $\gamma_i = \gamma_i(t)$ and $A_i = A_i(t)$ in the time-local master equation may explicitly depend on time, and γ_i even may become negative sometimes. Obviously, the non-Markovian character resides in these time-dependent coefficients.

Taking some real number θ in the reduced density matrix $\rho(\theta;t)$ as the inference parameter, we write down the QFI [17]:

$$\mathcal{F}(\theta;t) := \mathrm{Tr}[L^2(\theta;t)\rho(\theta;t)], \tag{3}$$

where $L(\theta;t)$ is the so-called symmetric logarithmic derivative (SLD), which are Hermitian operators determined by [17]

$$\frac{\partial}{\partial\theta}\rho(\theta;t) = \frac{1}{2}\left[L(\theta;t)\rho(\theta;t) + \rho(\theta;t)L(\theta;t)\right]. \tag{4}$$

An important essential feature of the QFI is that we can obtain the achievable lower bound of the mean-square error of unbiased estimators for the parameter θ through the quantum Cramér-Rao (QCR) theorem:

$$\mathrm{Var}(\theta;t) \geqslant \frac{1}{M\mathcal{F}(\theta;t)}, \tag{5}$$

where M represents the times of the independent measurements [17]. From the QCR theorem, we can see that the QFI is indeed a measure of a certain kind of information with respect to the precision of estimating the inference parameter. The relations between the QFI and the statistical distinguishability of $\rho(\theta;t)$ and its neighbor has been pointed out in some previous works [8,9,17].

A. Flow of the QFI and its decomposition

Here we use the QFI to characterize the non-Markovianity of the open quantum system by introducing the QFI flow, which is defined as the change rate $\mathcal{I} := \partial\mathcal{F}/\partial t$ of the QFI. As a central result in this paper, a proposition about the decomposition of the QFI flow is given as follows.

Proposition. For an open quantum system described by the time local master equation (1), the QFI flow $\mathcal{I} = \sum_i \mathcal{I}_i$ is explicitly written as a sum of subflows $\mathcal{I}_i = \gamma_i \mathcal{J}_i$ with

$$\mathcal{J}_i := -\mathrm{Tr}\{\rho[L,A_i]^\dagger[L,A_i]\} \leqslant 0. \tag{6}$$

Proof. From the differential of Eq. (4) with respect to time t, we have

$$\partial_t\partial_\theta\rho(\theta) = \tfrac{1}{2}[\dot{L}\rho + L\dot{\rho} + \dot{\rho}L + \rho\dot{L}]. \tag{7}$$

It gives

$$\mathrm{Tr}[\rho\dot{L}L + \rho L\dot{L}] = \mathrm{Tr}[2L\partial_t\partial_\theta\rho(\theta)] - \mathrm{Tr}[2\dot{\rho}L^2]. \tag{8}$$

From the differential of Eq. (3) with respect to time t, we obtain the QFI flow as

$$\mathcal{I} = \mathrm{Tr}\left[\mathcal{L}\left(\frac{\partial\rho}{\partial t}\right)\right], \tag{9}$$

where the operator $\mathcal{L} := L(2\partial/\partial\theta - L)$ is defined. By using the concrete expression of the master equation (2), we split the QFI flow into those individuals corresponding to the different dissipative channels as $\mathcal{I} = \mathrm{Tr}\,[\mathcal{L}\mathcal{K}(t)\rho(t)]$ or $\mathcal{I} = \sum_i \gamma_i \mathrm{Tr}[\mathcal{L}(A_i\rho A_i^\dagger) - \frac{1}{2}\mathcal{L}\{A_i^\dagger A_i,\rho\}]$. After some algebra, we get the decomposition $\mathcal{I} = \sum_i \gamma_i\mathcal{J}_i$, where $\mathcal{J}_i$ is just given in Eq. (6). It finally proves the proposition.

The previous proposition and its proof contain rich implications in physics. Firstly, the decomposition of the QFI flow corresponding to the different dissipative channels is due to the linearity of the QFI flow equation (9) with respect to $\partial\rho/\partial t$ and the concrete form of the time-local master equation (2). This is not a simple decomposition since each subflow depends on the whole SLD $L(\theta;t)$; meanwhile, $L(\theta;t)$ is deduced from $\rho(\theta;t)$, whose evolution depends on every dissipative channel and the unitary part of the master equation. So this kind of decomposition does not mean different dissipative channels are separable to influence the change of the QFI for a period of time. However, for each fixed time $t > 0$, the QFI flow at the present moment are decomposed into the split contributions from different dissipative channels. In this sense, we interpret $\mathcal{I}_i(t) = \gamma_i(t)\mathcal{J}_i(t)$ as a subflow of the QFI at time t caused by the dissipative channel described by $A_i(t)$ and $\gamma_i(t)$. The magnitude of the QFI subflow is determined by a state-independent factor γ_i and a state-dependent factor $\mathcal{J}_i$.

Secondly, one of the advantages of such decomposition comes from the link between the direction of each QFI subflow $\mathcal{I}_i$ and the sign of the decay rate γ_i. Because $\mathcal{J}_i$ is nonpositive, we conclude that a negative $\gamma_i(t)$ implies an inward QFI subflow ($\mathcal{I}_i > 0$), except the trivial case of $\mathcal{J}_i = 0$. The temporary appearance of negative decay rates is already considered as the essential feature of the non-Markovian behaviors [18]; here this is justified through the return of the QFI. For the case that all $\gamma_i(t)$ are positive, the master equation (2) describes a so-called time-dependent Markovian quantum process [4,13,19,20], in which cases, $\mathcal{I}$ always decreases. If the total QFI flow $\mathcal{I}(t)$ is positive at time t, it signifies at least one of $\gamma_i(t)$ is negative. In such cases, the QFI flows back to the open system and the non-Markovian behavior emerges.

Actually, like the trace distance used in Ref. [3], the dynamical return of the QFI is linked to the divisibility property of the dynamical map of quantum processes. If the master equation is of the form (2), the corresponding dynamical map is infinitely divisible provided that all γ_i are positive [21]. In such cases, for arbitrary time $t > 0$, the dynamical map from time t to $t + dt$ is a completely positive and trace-preserving map. Thus, the QFI decreases during this time interval since the QFI is monotonic with respect to a completely positive and trace-preserving map [22].

Thirdly, there should be some restriction on the evolution of the QFI. It is seen from the previous proof of the proposition that the coherent part of Eq. (2), that is, $-i\,[H(t),\rho(t)]$, does not contribute to the total QFI flow directly. This observation directly leads to the no-cloning theorem in quantum information, which states that we cannot use unitary operations to evolve the states $|\psi(\theta)\rangle \otimes |0\rangle$ into $|\psi(\theta)\rangle \otimes |\psi(\theta)\rangle$ as a quantum copy [23]. This is because the QFI of the target states is twice that of the source states, due to the additivity of the QFI for the product states. Moreover, if the total system (system plus environment) is assumed closed and the QFI is only distributed in the system initially, then the QFI of the reduced density matrix during evolution should not be greater than the one at the initial time because of the invariance of the QFI under unitary evolution and the nonincrease of the QFI under partial trace operation. This restriction should be reflected in the QFI flow obtained from a proper master equation.

III. TWO-LEVEL SYSTEM

Now we use an example of a two-level system (qubit) to explicitly illustrate our discovery about the intrinsic relation between the QFI flow and the non-Markovianity of the open quantum system and its impact on parameter estimation. In the quantum metrology context, the QFI gives a theoretical-achievable limit on the precision when estimating an unknown parameter, according to the QCR theorem (5). To estimate the parameter as precisely as possible, we should optimize input states to maximize the QFI, and then optimize measurements to achieve the Cramér-Rao bound [10]. However, due to the interaction with the environment, the QFI will change and affect the precision of the parameter estimation.

Here, the QFI-based parameter is assumed to be induced by a single-qubit phase gate $U_\phi := |g\rangle\langle g| + \exp(i\phi)|e\rangle\langle e|$ acting on the qubit, where $\phi = \theta$ is an inference parameter (see Fig. 1). To estimate the unknown parameter ϕ as precisely as possible, the optimal input state may be chosen as $|\psi_{\rm opt}\rangle = (|g\rangle + |e\rangle)/\sqrt{2}$, which maximizes the QFI of the output state $U_\phi|\psi_{\rm opt}\rangle$; see Ref. [10]. In the following model, after the phase gate operation and before the measurement performed, the qubit is assumed as an atom coupled to a reservoir consisting of harmonic oscillators in the vacuum. The total Hamiltonian of this typical model [1,24,25] reads

$$H = \omega_0\sigma_+\sigma_- + \sum_k \omega_k b_k^\dagger b_k + (\sigma_+ B + \sigma_- B^\dagger), \quad (10)$$

with $B = \sum_k g_k b_k$, where ω_0 denotes the transition frequency of the atom with ground and excited states $|g\rangle$ and $|e\rangle$, and $\sigma_\pm$ the raising and lowering operators of the atom; $b_k^\dagger$ and b_k are, respectively, the creating and annihilation operators of the bath mode of frequencies ω_k. g_k denotes the coupling constant. We then consider Lorentzian spectral density $J(\omega) = \lambda W^2/\{\pi[(\omega_0 - \omega)^2 + \lambda^2]\}$, where W is the transition strength, and λ defines the spectral width of the coupling, which is related to the reservoir correlation time scale $\tau_{\rm B}$ by $\tau_B = \lambda^{-1}$ [1,24]. The Lorentzian spectral density describes the reservoir composed of lossy cavity; see Ref. [1]. The time-local master equation of the form (2) can be obtained exactly as follows [1]:

$$\frac{\partial}{\partial t}\rho_{\rm S}(t) = \gamma(t)\left(\sigma_-\rho_{\rm S}(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-,\rho_{\rm S}(t)\}\right), \quad (11)$$

where $\gamma(t) = -2\dot{h}(t)/h(t)$ with a crucial characteristic function [1]:

$$h(t) = \begin{cases} e^{-\lambda t/2}\left[\cosh\left(\frac{dt}{2}\right) + \frac{\lambda}{d}\sinh\left(\frac{dt}{2}\right)\right], & W \leqslant \frac{\lambda}{2}, \\ e^{-\lambda t/2}\left[\cos\left(\frac{dt}{2}\right) + \frac{\lambda}{d}\sin\left(\frac{dt}{2}\right)\right], & W > \frac{\lambda}{2}, \end{cases} \quad (12)$$

where $d = \sqrt{|\lambda^2 - 4W^2|}$.

Taking the initial state $U_\phi|\psi_{\rm opt}\rangle$, the reduced density matrix of the atom obeys the master equation (11). Its solution is $\rho_{\rm S}(t) = (I + \mathbf{B}\cdot\boldsymbol{\sigma})/2$, where $\mathbf{B} = [h(t)\cos\phi, -h(t)\sin\phi, h(t)^2 - 1]$ and $\boldsymbol{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$. In order to calculate the QFI flow, we first diagonalize this reduced density matrix as $\rho_{\rm S}(t) = \sum_i p_i(t)|\psi_i(t)\rangle\langle\psi_i(t)|$. In this diagonal representation, the SLD with matrix elements $L_{ij} = 2\langle\psi_i|\partial_\phi\rho_S|\psi_j\rangle/(p_i + p_j)$ is obtained explicitly as

$$L(t) = ih(t)\left[|\psi_1(t)\rangle\langle\psi_2(t)| - |\psi_2(t)\rangle\langle\psi_1(t)|\right]. \quad (13)$$

Furthermore, we have $\mathcal{J} = -{\rm Tr}(\rho\,[L,\sigma_-]^\dagger\,[L,\sigma_-]) = -h(t)^2$; then the exact solution for the QFI flow,

$$\mathcal{I}_\phi(t) = \gamma(t)\mathcal{J}(t) = 2h(t)\dot{h}(t), \quad (14)$$

is obtained, which leads to $\mathcal{F}_\phi = h(t)^2$.

Therefore, the characteristic of the QFI flow is determined by the function $h(t)$, which has two very different kinds of behaviors. The corresponding properties of the QFI flow are shown in Fig. 2. In the weak coupling regime ($W < \lambda/2$), the function $\gamma(t)$ is always positive, thus the QFI is always lost during the time evolution of the open system. In the strong coupling regime ($W > \lambda/2$), the function $\gamma(t)$ takes on negative values within certain intervals of time; see Fig. 2(d), which displays the non-Markovianity. Obviously, in these time intervals, the QFI flow is inward. It is remarkable that although $\gamma(t)$ diverges at certain times, the QFI flow does not; see

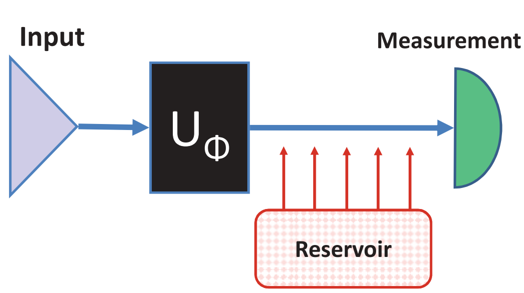

FIG. 1. (Color online) Estimation of parameter ϕ in an unitary operation. After the phase-gate operation, the system interacts with a reservoir. The precision of the estimation is impacted by characteristics of both the reservoir and the interaction.

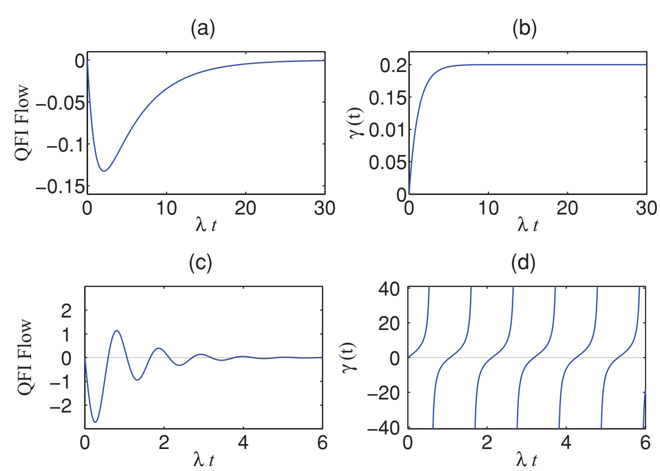

FIG. 2. (Color online) Two-level atom coupled to reservoir with Lorentzian spectral density. (a) QFI flow as a function of rescaled time, plotted in the weak coupling regime ($W = 0.3\lambda$); (b) γ as a function of rescaled time, $W = 0.3\lambda$; (c) QFI flow as a function of rescaled time, plotted in the strong coupling regime ($W = 3\lambda$); (d) γ as a function of rescaled time, $W = 3\lambda$.

Figs. 2(c) and 2(d). This is because the QFI flow is determined by two factors, $\gamma(t)$ and $\mathcal{J}(t)$.

IV. CONCLUSION

In summary, based on the QFI flow, we have proposed an information-theoretical approach for characterizing the time-dependent memory effect of the environment on its surrounding open quantum systems. In this approach the Markovian process is considered as a QFI erasure process, and the return of the QFI (i.e., an inward QFI flow) clearly signatures the non-Markovian process. Using the time-local master equations, we have shown that for each fixed time $t > 0$ the QFI flow is decomposable according to different dissipative channels, and the direction of each subflow is determined by the sign of decay rates. With this decomposition form, the relationship between the temporary appearance of a negative decay rate and the non-Markovian characteristic is justified. Although, in the present work, the analysis of the QFI flow is based on a time-local master equation, the concept of the QFI flow may still be available in more general cases.

The present approach is associated with the current development of quantum metrology, which is concerned with finding an optimal fashion to make high-resolution and highly sensitive measurements of physical parameters [10]. Due to the interaction with the environment in experiments, like the photon losses in the optical interferometry or the presence of quantum noise [26], the QFI will change and affect the precision of the parameter estimation. Therefore, it is worthy to study the dynamical evolution of the QFI in the context of quantum metrology, especially for non-Markovian processes.

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