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Optical transport and statistics of radiative losses in disordered chains of microspheres

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Optical transport in a one-dimensional chain of microspherical resonators with size disorder is studied in the spectral range of high-Q whispering gallery modes. An *ab initio* approach is used to develop a theoretical framework for analysis of steady-state transport parameters with main emphasis on properly defined radiative loss coefficient. Probability distribution and scaling properties of the latter are established and explained.

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Collective optical excitations arising in the chains of microresonators due to evanescent coupling of high-Q whispering gallery modes (WGM) have been attracting recently significant interest. The initial proposal to use such structures as low-loss waveguides in Ref. [1] initiated a number of experimental [2,3] and theoretical [3–5] papers, in which properties of collective WGM optical excitations in coupled microspheres and microdisks have been studied. While understanding the effects of unavoidable size fluctuations of the individual resonators on optical properties of this structure is important for device development, theoretical efforts in this direction have been so far limited only to analysis of phenomenological models [6]. However, relations between microscopic parameters of these structures and their transport properties, which is crucial for applications, can only be obtained on the basis of an *ab initio* approach. This approach is also vital for understanding fundamental optical properties of microresonator chains as representatives of a distinct class of one-dimensional disordered optical systems demonstrating Anderson localization. These systems are characterized by intrinsic radiative losses and differ significantly from other models used to study the interplay between localization and losses [7,8]. First, the losses in these systems are intimately connected with formation of the collective modes and cannot be introduced "by hand." They appear naturally at each site of the structure and are characterized by random rates correlated with the underlying disorder. Additionally, the chain of microresonators present a rare example of an one-dimensional optical system defined on a discrete lattice.

In this work we study numerically light transport through such a system with emphasis on statistical properties of radiative losses. The structure under consideration consists of N microspherical resonators arranged in a linear chain. We assume that all spheres have the same refractive index and positioned at the same distance d from each other but allow for their radii to fluctuate. The disordered portion of the chain is assumed to be connected to the segments built of identical spheres, which play the role of incoming and outgoing leads. All radii are drawn independently of each other from a statistical ensemble with uniform distribution characterized by a rms value δ, used as a measure of disorder in the system. The radius of the spheres in the leads is chosen to coincide with the average of this distribution. The description of the system is based on a multisphere Mie approach, which uses presentation of the incident, scattered, and internal fields of each sphere in the form of linear combination of appropriate vector spherical harmonics. The coefficients in this expansion obey an exact system of linear algebraic equations [9] and become the main subject of study. The coupling between spheres is described by translation coefficients $U_{l,m}^{l',m}(z_n - z_{n'})$, which couples spheres located at points with coordinates z_n and $z_{n'}$, and WGM with different polar numbers l and l' and different polarizations. At the same time, modes with different azimuthal numbers m remain uncoupled if the polar axis of the coordinate system is chosen along the axis of the chain. In the case of high-Q modes, which are characterized by $l \gg 1$, $U_{l,m}^{l',m}(z_n - z_{n'})$ decrease very fast with the distance between spheres allowing use of the nearest-neighbor approximation. If, in addition, the spectral distance between modes with different l and different polarizations is large enough, nondiagonal in l and cross-polarization translation coefficients can also be neglected [5]. In this case the system is described by an equation of a tight-binding type

$$\frac{1}{\alpha_n^{(l)}} a_n^{(l,m)} = U_{n,n-1}^{(l,m)} a_{n-1}^{(l,m)} + U_{n,n+1}^{(l,m)} a_{n+1}^{(l,m)}, \qquad (1)$$

where $a_n^{(l,m)}$ is the expansion coefficient of the field scattered by nth sphere and $\alpha_n^{(l)}$ is a single sphere Mie scattering coefficient for nth sphere whose poles define WGM resonances. In what follows, we shall fix polar and azimuthal numbers at $l = 29$, $m = 1$ and abridge our notations by dropping these indexes.

The Mie coefficient α_n has the following exact representation $\alpha_n = -i\eta_n/(g_n + i\eta_n)$, where η_n and g_n are well known real functions [9]. The resonance frequency is determined by the condition $g_n = 0$ so α_n is exactly equal to -1 at the resonance. This important property determines the residue of α_n at the resonance pole. In the vicinity of the pole α_n can be presented as

$$\alpha_n \approx \frac{-i\gamma_n}{\omega - \omega_n + i\gamma_n}, \qquad (2)$$

where ω_n is the resonance frequency of the nth sphere, and γ_n is the respective radiative decay rate. Tight-binding Eq. (1) in

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this approximation can be rewritten as

$$(\omega - \omega_n + i\gamma_n)a_n = -i\gamma_n(U_{n,n-1}a_{n-1} + U_{n,n+1}a_{n+1}). \quad (3)$$

If all spheres are identical, with $\omega_n \equiv \omega_r$ and $\gamma_n \equiv \gamma$, Eq. (3) is solved by $a_n \propto \exp(iqnd)$, where the wave vector q obeys a dispersion relation

$$\omega - \omega_r + i\gamma = -2i\gamma U \cos(qd). \quad (4)$$

Taking into account that translation coefficients $U_{n,n'}$ are complex valued with their imaginary part $U^{(2)}_{n,n'}$ much larger than the real part $U^{(1)}_{n,n'}$ [5], it is clear that the dispersion of these collective excitations is determined by parameter $\gamma U^{(2)}$, while γ and $\gamma U^{(1)}$ are responsible for their decay.

It is important to note that in the case of coupled resonators, parameter γ determines not only the radiative decay of the modes but also their dispersion. As a result one cannot eliminate radiative losses of the collective mode by setting $\gamma = 0$ since it will also destroy the mode itself. For the disordered chain this fact has even more profound consequences. Indeed, in this case the intersite coupling coefficient $t_{n,n-1}$, which, according to Eq. (3), is $t_{n,n-1} = \gamma_n U_{n,n-1}$ does not have the expected symmetry property $t_{n,n-1} = t_{n-1,n}$, which is crucial for defining the energy flux.

To develop a consistent framework for description of the transport properties of this structure we first convert Eq. (3) to the time domain in the slow changing amplitude approximation. Multiplying the resulting equation by a_n^* and combining it with its complex-conjugated counterpart multiplied by a_n we obtain

$$\frac{\partial |a_n|^2}{\partial t} + \gamma_n |a_n|^2 = \gamma_n(J_n - J_{n+1}), \quad (5)$$

where we defined

$$J_n = -\frac{i}{2}U^{(2)}_{n,n-1}(a_{n-1}a_n^* - a_{n-1}^* a_n) \quad (6)$$

and neglected terms of the order of $\gamma_n U^{(1)}_{n,n\pm1}$. The sign of J_n is chosen to reflect the negative group velocity of collective excitations defined by dispersion law (4). While J_n looks very much like a flux, its place in Eq. (5), characterized by prefactor γ_n, does not allow for such a direct interpretation. Dividing Eq. (5) by γ_n and summing up the resulting equations over all spheres, we see that J_n referring to inner sites $1 < n < N$ cancel out yielding the following result:

$$\sum_{n=1}^{N} \frac{1}{\gamma_n}\frac{\partial |a_n|^2}{\partial t} + \sum_{n=1}^{N} |a_n|^2 = -J_{N+1} + J_1. \quad (7)$$

The first term in this expression can be presented as $\sum_{n=1}^{N}(1/\gamma_n)(\partial|a_n|^2/\partial t) \equiv (1/\Gamma)\sum_{n=1}^{N}(\partial|a_n|^2)/\partial t$. This expression defines a collective decay rate constant Γ for the chain and substituting it instead of the time-dependent term in Eq. (7) one can see that it is ΓJ_n that should be interpreted as the flux. However, one can also see that Γ cancels out of all quantities defined as ratios of fluxes, and, therefore, reflection and transmission coefficients can be found directly from J_n. This becomes particularly clear in the stationary regime, which is of the main interest in this Rapid Communication, where parameters γ_n cancels out completely producing exact equation

$$\sum_{n=1}^{N} |a_n|^2 = -J_{N+1} + J_1, \quad (8)$$

valid outside of the resonant approximation.

To study transport properties of the system under consideration a standard transfer-matrix formalism based on Eq. (1) is used. The transfer matrix is defined in the plane-wave representation [10], which is introduced by presenting the excitation within the disordered region as:

$$a_n = a_n^+ e^{iqnd} + a_n^- e^{-iqnd} \quad (9)$$

$$a_{n-1} = a_n^+ e^{iq(n-1)d} + a_n^- e^{-iq(n-1)d}, \quad (10)$$

where $a_n^{\pm}$ are complex amplitudes of the forward/backward traveling wave and complex-valued wave vector $q = k + i\xi$ is defined by dispersion relation Eq. (4). Transfer-matrix relating amplitudes at $(n+1)$th and nth site is given by

$$P_n = \frac{1}{2i\sin qd}\begin{pmatrix} \dfrac{1-\alpha_n e^{-iqd}(U_{n,n+1}+U_{n,n-1})}{\alpha_n U_{n,n+1}} & e^{-2iqnd}\dfrac{1-\alpha_n(e^{-iqd}U_{n,n+1}+e^{iqd}U_{n,n-1})}{\alpha_n U_{n,n+1}} \\ -e^{2iqnd}\dfrac{1-\alpha_n(e^{iqd}U_{n,n+1}+e^{-iqd}U_{n,n-1})}{\alpha_n U_{n,n+1}} & -\dfrac{1-\alpha_n e^{iqd}(U_{n,n+1}+U_{n,n-1})}{\alpha_n U_{n,n+1}} \end{pmatrix}. \quad (11)$$

This transfer matrix can be presented in an alternative form

$$P_n = \begin{pmatrix} 1/t_n^r & -r_n^l/t_n^r \\ r_n^r/t_n^r\ t_n^l & -r_n^l r_n^r/t_n^r \end{pmatrix}, \quad (12)$$

where amplitude reflection and transmission coefficients are introduced according to $r_n^l = a_n^+/a_n^-$; $r_n^r = a_{n+1}^-/a_{n+1}^+$; $t_n^l = a_{n+1}^-/a_n^-$; $t_n^r = a_n^+/a_{n+1}^+$. Comparing Eq. (11) and Eq. (12), coefficients t_n^r, r_n^r, r_n^l, t_n^l can be expressed in terms of microscopic parameters α_n and $U_{n,n\pm1}$. Presenting the total transfer matrix $P^{(N)} = P_N P_{N-1}\cdots P_1$ in the form of Eq. (12) with respective coefficients T_N and R_N, one can derive the following recurrence relations

$$T_N^r = \frac{t_N^r T_{N-1}^r}{1 - r_N^l R_{N-1}^r}, \quad R_N^r = r_N^r + \frac{R_{N-1}^r t_N^l t_N^r}{1 - r_N^l R_{N-1}^r},$$
$$T_N^l = \frac{t_N^l T_{N-1}^l}{1 - r_N^l R_{N-1}^r}, \quad R_N^l = R_{N-1}^l + \frac{r_N^l T_{N-1}^l T_{N-1}^r}{1 - r_N^l R_{N-1}^r}. \quad (13)$$

Parameters T_N and R_N here are just auxiliary quantities, while real intensity reflection and transmission coefficients must be defined in terms of ratios of related fluxes. The respective expressions are derived by identifying amplitudes of incident, reflected, and transmitted waves as $a_1^- = 1$, $a_1^+ = R_N^l$, $a_{N+1}^+ = 0$, and $a_{N+1}^- = T_N^l$. Such identification is consistent with definition of J_n in Eq. (6) and takes into account that due to negative group velocity of excitations in the leads the direction of the flux is opposite to that of the phase velocity. Presenting J_n in the plane wave approximation one can identify incident, reflected, and transmitted fluxes and define respective coefficients

$$T^{(N)} = \frac{U^{(2)}_{N+1,N}}{U^{(2)}_{1,0}} e^{2N\xi d} \left|T^l_{N+1}\right|^2, \tag{14}$$

$$R^{(N)} = e^{-2\xi d}\left|R^l_{N+1}\right|^2 - \frac{2\sinh(\xi d)\left|R^l_{N+1}\right|\sin(kd+\phi_R)\bigr|}{e^{\xi d}\sin(kd)}, \tag{15}$$

where ϕ_R is the phase of R^l_{N+1}. With these definitions Eq. (8) can be given a form of flux conservation equation $R^{(N)} + T^{(N)} + A^{(N)} = 1$, where $A^{(N)}$ is defined as

$$A^{(N)} = \frac{e^{-\xi d}\sum_{n=1}^N |a_n|^2}{U^{(2)}_{1,0}\sin(kd)}. \tag{16}$$

Introduced quantity $A^{(N)}$ can be directly interpreted as the radiative loss coefficient determining the ratio of the flux radiated by the system to that of the incident wave. This quantity differs from radiative lifetimes studied, for instance, in Ref. [11]. While the latter describes leakage of a normal mode excited in an intrinsically Hermitian system through its boundaries, $A^{(N)}$ characterizes intrinsic losses observed in the steady-state transport regime.

We are interested here in transport properties of asymptotically long systems, when the number of spheres in the chain, N, significantly exceeds the dimensionless localization length N_{loc} defined as $N_{\text{loc}}^{-1} \equiv \zeta_\infty = \lim_{N\to\infty}\zeta$, where $\zeta = -\ln T^{(N)}/(2N)$ is a Lyapunov exponent (LE) of the structure. Using recurrence relations of Eq. (13) and definitions given by Eq. (15) we compute transmission and reflection coefficients for a large number of realizations of the structure. These results are used, first, to verify that the distribution of LE is normal in accordance with main properties of strongly localized systems. An average value of LE, which coincides with ζ_∞, and its standard deviation σ are found to obey scaling relation $\tau = f(\beta)$, where $\tau = \sigma^2 N/\zeta_\infty$, and $\beta = |\xi|/\zeta_\infty$. Scaling function f turns out to coincide with the one found in Ref. [7] for one-dimensional continuous model with constant absorption. While this result confirms the broad universality of statistical properties of disordered systems with losses, one should understand a fundamental difference between models studied in [7,8] and in this work. In phenomenological models, $|\xi|$ and ζ_∞ are two independent variables but in the model considered here this is not the case: while one can vary ζ_∞ independently of $|\xi|$ by changing degree of disorder δ, the $|\xi|$ cannot be changed independently of ζ_∞. Any change of parameter $|\xi|$ will also affect coupling between adjacent spheres, and, therefore, the LE.

Using computed values of reflection and transmission coefficients we find the coefficient of radiative losses from expression $A^{(N)} = 1 - R^{(N)} - T^{(N)}$. The histogram representing probability density of $A^{(N)}$ obtained for the disorder strength $\delta = 0.001$ and a fixed frequency $kd = 3\pi/8$ is shown in the main frame of Fig. 1. [The frequencies here and thereafter are given in terms of real parts k of the wave numbers obtained from dispersion Eq. (4).] This figure also shows fit of this distribution by function $f(A) \propto A^{-2}\exp[-(a-A)^2/(b^2A^2)]$ which is generated from the normal distribution of ζ by transforming it to the distribution of $1/\zeta$. The latter can be interpreted as distribution of finite size localization length N^f_{loc}. This intimate relation between distributions of $A^{(N)}$ and N^f_{loc} can be qualitatively understood by assuming that, as expected for localized systems, $|a_n|^2 \propto \exp[-n/N^f_{\text{loc}}]$. With this ansatz the sum in Eq. (16) can be evaluated to $\sum_n |a_n|^2 = 1/(1-\exp[-1/N^f_{\text{loc}}]) \approx N^f_{\text{loc}}$, where at the last step it is assumed that $N^f_{\text{loc}} \gg 1$. This result differs from function $f(A) \propto A^{-2}\exp(-a/A)$ obtained in continuous models with uniform absorption [8]. We verify the obtained result by directly establishing the linear proportionality between the loss coefficient and the localization length. Taking into account that in the asymptotic regime $N \gg N_{\text{loc}}$ the distribution of $A^{(N)}$ is found to be independent of the system size we assume that this proportionality have the form $A^{(N)} \propto \beta$. In order to verify this conjecture we plot $A^{(N)}$ versus β for a single realization of our system (no averaging!) using data obtained for multiple frequencies in the interval $\pi/4 < kd < 3\pi/4$. The resulting plot shown in the inset frame of Fig. 1 provides strong evidence of this relation and, by extension, of the proposed form of $f(A)$.

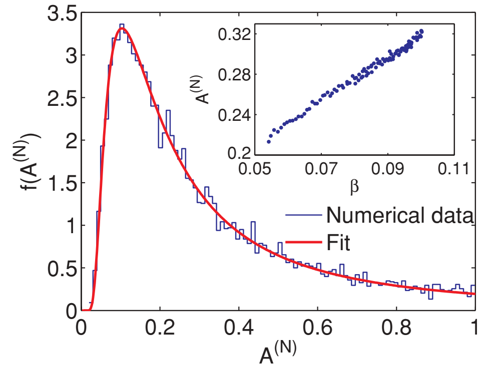

FIG. 1. (Color online) (Main frame) Distribution function of $A^{(N)}$: numerical histogram and its fit by the distribution of the N_{loc}. (Inset) Linear relationship between $A^{(N)}$ and β for a single realization for a wide range of frequencies. Refractive index of the spheres used in calculations is $n = 1.59$.

In order to establish scaling properties of parameters a and b of the probability density of $A^{(N)}$ we note that they can be related to its mean value and variance even though this relation is not as straightforward as in the case of distribution of LE. Computing average value of $A^{(N)}$ and its variance for a number of frequencies we determine that they both depend on a single scaling parameter β. This is clearly seen in the plots presented in Fig. 2, where these quantities are plotted versus β.

To conclude, we developed a theoretical framework based on *ab initio* approach to transport properties of disordered

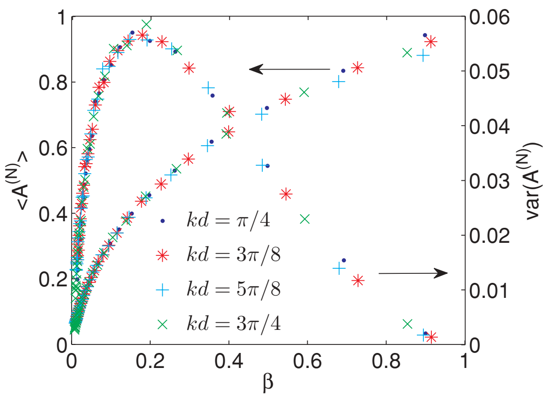

FIG. 2. (Color online) Scaling of mean value and variance of $A^{(N)}$.

chains of spherical microresonators and used it to study statistical properties of the steady-state radiative loss coefficient in the asymptotically long chains. The probability density of this coefficient was found to differ from the distribution found in the continuous models with uniform absorption. This function was shown to coincide with the distribution of the localization length and depend on a single parameter: ratio between the localization length and the loss length in ordered chains.

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