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Stable structures with high topological charge in nonlinear photonic quasicrystals

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Stable vortices with topological charges of 3 and 4 are examined numerically and analytically in photonic quasicrystals created by interference of five as well as eight beams, for cubic as well as saturable nonlinearities. Direct numerical simulations corroborate the analytical and numerical linear stability analysis predictions for such experimentally realizable structures.

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The study of vortices has been a principal theme of interest in dispersive nonlinear systems with applications including, among others, Bose-Einstein condensates (BECs) and nonlinear optical media [1–3]. More recently, such states have been studied in settings with some discrete spatial symmetry, that is, nonlinear lattices. There, the notion of “discrete vortices” [4] arose and was subsequently intensely studied in both discrete and quasicontinuum media; see, for example, [5] and [6] for relevant reviews. This led to the experimental realization of unit-charge ($S = 1$) coherent structures in saturably nonlinear photorefractive media [such as SBN:75 ($Sr_{0.75}Ba_{0.25}Nb_2O_6$)] in [7] and [8] and the exploration of higher- charge ($S = 2$) ones in square, hexagonal, and honeycomb lattices [9]. A multipole soliton necklace of out-of-phase neighboring lobes in a square lattice was identified experimentally and theoretically in [10] from the initial condition of a wide $S = 4$ Gaussian beam.

While regular lattices have mostly been studied [5], more recently experimental developments have enabled the study of photonic quasicrystals in photorefractive media [11] and have spurred a correspondingly intense theoretical activity [12]. We also note that recent experiments have been performed on nonsquare optical lattices for ultracold atoms in a BEC [13]. It is then natural to expect that quasicrystals are well within experimental reach in this regard as well.

Motivated by these developments, we illustrate the unique ability of such lattices (with saturable or cubic nonlinearity) to sustain stable vortices of higher topological charge, such as $S = 3$ and $S = 4$. Direct numerical simulations reveal the robustness of such modes. In contrast, perhaps counterintuitively (but as can be analytically predicted), lower-charge vortices are found to be unstable, and this instability is also dynamically monitored.

We introduce the following nondimensionalized evolution equation:

$$\left[i\partial_z + \tfrac{1}{2}\nabla^2 + F(|U|^2) - V(\mathbf{x})\right]U = 0. \quad (1)$$

The (saturable) photorefractive nonlinearity is $F(|U|^2) = -1/(1 + |U|^2) + 1$, where U is the slowly varying amplitude of a probe beam normalized by the dark irradiance of the crystal I_d [3,14], and V an external potential. In a Kerr medium the nonlinearity reads $F(|U|^2) = |U|^2$, and this case also includes the interpretation of U as a mean-field wave function of an atomic BEC [15], while the potential V is either modulation of the refractive index, in the former case, or an externally applied field, in the latter.

The potential V is taken to be of the form $E/[1 + I(\mathbf{x})]$, where $I(\mathbf{x}) = \frac{1}{N^2}|\sum_{j=1}^{N} e^{ik\mathbf{b}_j\cdot\mathbf{x}}|^2$. In the photorefractive paradigm, this is the optical lattice intensity function formed by N interfering beams in the principal directions $\mathbf{b}_j$ with periodicity $2\pi/k$. We consider the cases of $N = 5$ and $N = 8$. Here 1 is the lattice peak intensity, z is the propagation distance, $\mathbf{x} = (x,y)$ are transverse distances, $k = 2\pi/5$ is the wave number of the lattice, and $E = 5$ is proportional to the external voltage. Recently, such a setting has been explored theoretically for positive lattice solitons [12,16], but we extend the considerations here to vortex solutions.

The possible charge S of vortices (the integer number of 2π phase shifts across a discrete contour comprising the solution) is bounded by the symmetry of the lattice [17]. A lattice with n-fold symmetry has natural contours of $2n$ sites. Hence, taking into account the degeneracy of vortex-antivortex pairs $\{S, -S\}$, one has $0 \leqslant S \leqslant n$, with the cases of $S = 0,n$ being the trivial flux cases of in-phase and out-of-phase neighboring lobes, respectively. The quasicrystal with $N = 5$ has $n = 5$, while for $N = 8$, $n = 4$. Hence, the highest possible charge, $S = n - 1$, is $S = 4$ for the case of $N = 5$ and $S = 3$ for $N = 8$.

Considering the quasi-one-dimensional contour of excited sites (depending on the respective amplitudes of the lattice and the probe field), and within the context of coupled-mode theory [18], in which the probe field is expanded in Wannier functions [19], one can obtain insights about the stability of the vortices within the framework of a discrete nonlinear Schrödinger equation [6], $i\dot{u}_n = -\varepsilon(u_{n+1} + u_{n-1} - 2u_n) - |u_n|^2u_n$. In that context and either based on modulational instability [18], through empirical numerical testing [17], or, more rigorously, via Lyapunov-Schmidt perturbative expansions around the so-called *anticontinuum* (AC) limit of zero coupling ($\varepsilon = 0$) [20], it is known that lobes which are phase separated by more than $\pi/2$ are stable next to each other, while those separated by less than $\pi/2$ are unstable. A simple intuitive argument for this situation is that the effective potential which out-of-phase neighboring nodes exert on one another through the focusing nonlinearity is repulsive, and hence, the nodes remain localized in their respective separate wells. In contrast,

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if the neighbors are in phase, then the effective neighboring potentials are attractive and hence the solution is unstable to remaining localized in separate wells. The possible relative phases interpolate between these cases, with $\pi/2$ being exactly in the middle. A similar discussion is used in [21] to justify (upon suitable phase variation) the existence of soliton clusters in bulk media. This leads to stability of the more *highly* charged vortices for contours of larger numbers of nodes (see also [9]). We briefly review the Lyapunov-Schmidt argument. In the limit $\varepsilon \to 0$ one can construct exact solutions of the form $u_j = \sqrt{\mu} e^{[-i(\beta z - \theta_j)]}$ for any arbitrary $\theta_j \in [0,2\pi)$ [20]. The case we are considering is that of $\theta_j = jS\pi/n$. We linearize around the solution for $\varepsilon = 0$, and the condition for existence of solutions with $\varepsilon > 0$ reduces to the vanishing of a vector function $\mathbf{g}(\boldsymbol{\theta})$ of the phase vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_N)$, where

$$g_j \equiv \sin(\theta_{j-1} - \theta_j) + \sin(\theta_{j+1} - \theta_j), \tag{2}$$

subject to periodic boundary conditions. This includes the preceding discrete reduction of the vortex solutions for $0 \leqslant S \leqslant n$. The fundamental contours M will have length $|M| = 2n$, and $|\phi_{j+1} - \phi_j| = \Delta\phi = \pi S/n$ is constant for all $j \in M$, $|\theta_1 - \theta_{|M|}| = \Delta\theta$, and $\Delta\theta|M| = 0 \bmod 2\pi$.

For the contour M, there are $|M|$ eigenvalues γ_j of the $|M| \times |M|$ Jacobian $\mathcal{M}_{jk} = \partial g_j / \partial \theta_k$ of the diffeomorphism given in Eq. (2). The eigenvalues of this matrix γ_j can be mapped to eigenvalues of the full linearization. In particular, eigenvalues of the linearization, denoted λ_j, are given to leading order by the relation [20] $\lambda_j = \pm\sqrt{2\gamma_j \varepsilon}$. Thus, solutions are stable to leading order if $\gamma_j < 0$ (so $\lambda_j \in i\mathbb{R}$) and unstable if $\gamma_j > 0$ (so $\lambda_j \in \mathbb{R}$). We have $\gamma_j = 4\cos(\Delta\phi)\sin^2(\frac{\pi j}{|M|})$ and so these cases correspond exactly to $\Delta\phi > \pi/2$ (or $S > n/2$) and $\Delta\phi < \pi/2$ (or $S < n/2$). In the boundary case of $\Delta\phi = \pi/2$, one needs to expand to the next order in the Lyapunov-Schmidt reduction. We note that a so-called staggering transformation along the contour, $u_j^d = (-1)^j u_j^f$, allows the foregoing conclusions for the focusing problem to be mapped immediately to the defocusing problem (with a change in the sign of the nonlinearity). We do not consider the defocusing case further here. The preceding considerations illustrate the expectation that $S = 3$ vortices may be stable in the $N = 5$ and $N = 8$ cases, and the $S = 4$ vortex may be stable in the $N = 5$ case.

We now turn to numerical computations. We also explore the evolution of some radial Gaussian beams. First, we confirm the expectation of stability of the $S = 4$ vortex for saturable and cubic nonlinearities, over continuations in the semi-infinite gap (see Figs. 1 and 2, respectively). The profiles and phases are depicted in Figs. 1(a), 1(a.i), 2(a), and 2(a.i); linear spectra, in Figs. 1(b) and 2(b); Fourier spectra in Figs. 1(b.ii) and 2(b.i); and continuations of the power $P = \int |U|^2 d\mathbf{x}$, as a function of the propagation constant β, in Figs. 1(c) and 2(c). The power of the solution branches differs substantially between nonlinearities, and the power of the branch of saturable solutions approaches some resonant frequency at which $dP/d\beta \to \infty$ and $P \to \infty$ [see Fig. 1(c)]. The lattice is depicted in Fig. 1(d), while Fig. 2(d) shows the maximal perturbation growth rate, or $\max_\lambda[\text{Re}(\lambda)]$, corresponding to the branches in Fig. 2(c).

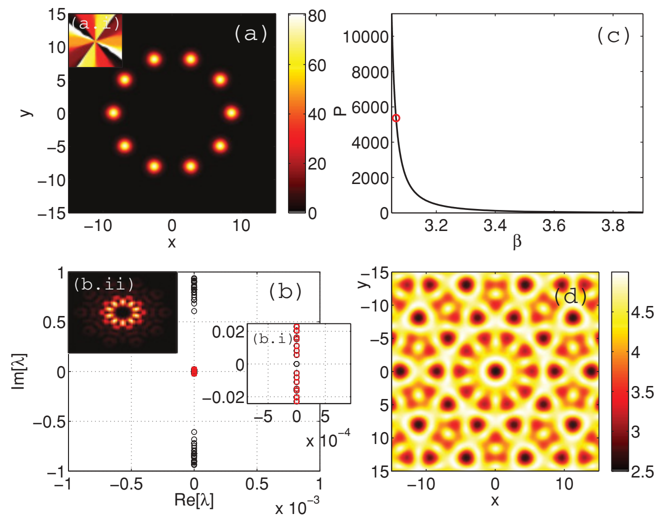

FIG. 1. (Color online) Stable $S = 4$ vortex in a quasicrystal lattice of $N = 5$ and with a saturable nonlinearity. (a), (a.i) Profile and phase; (b) linear spectrum [(b.i) closeup]; (b.ii) Fourier spectrum; (c) continuation of the power $P = \int |U|^2 d\mathbf{x}$ as a function of the propagation constant β; (d) $N = 5$ lattice.

For the structures we consider, there is one pair of eigenvalues at the origin accounting for the U(1) (phase) invariance and the other $2n - 1$ eigenvalue pairs associated with the excited lobes all have negative energy, hence being candidates for instability [22], and are all either purely imaginary or purely real. If real, the instability is immediate, while if imaginary, instability may still arise due to their collision with the phonon band, resulting in a Hamiltonian-Hopf bifurcation and eigenvalue quartets. The spectral plane, with the negative-energy modes indicated by red squares, for the saturable and cubic cases are given in Figs. 1(b) and 2(b),

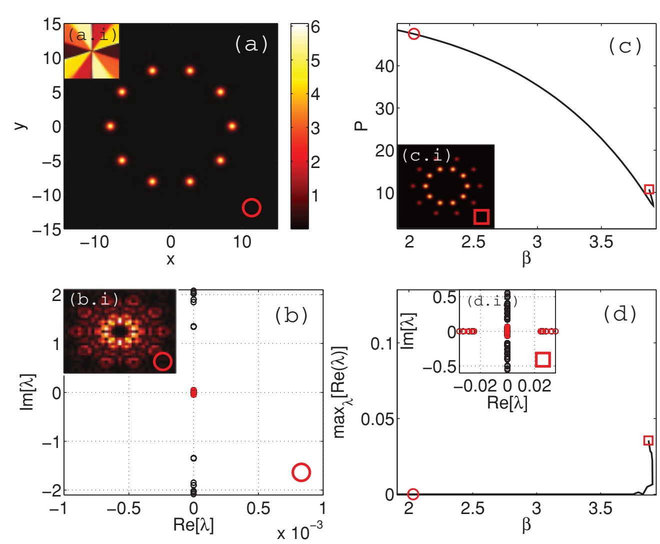

FIG. 2. (Color online) (a)–(c) The same as in Fig. 1 except for a cubic nonlinearity. (d) Growth rate, or $\max_\lambda[\text{Re}(\lambda)]$. (c.i) Profile and (d.i) linear spectra of the highly unstable solution indicated by the (red) square on the branches in (c) and (d), which collides with the main branch and disappears in a saddle-node bifurcation close to the phonon band edge.

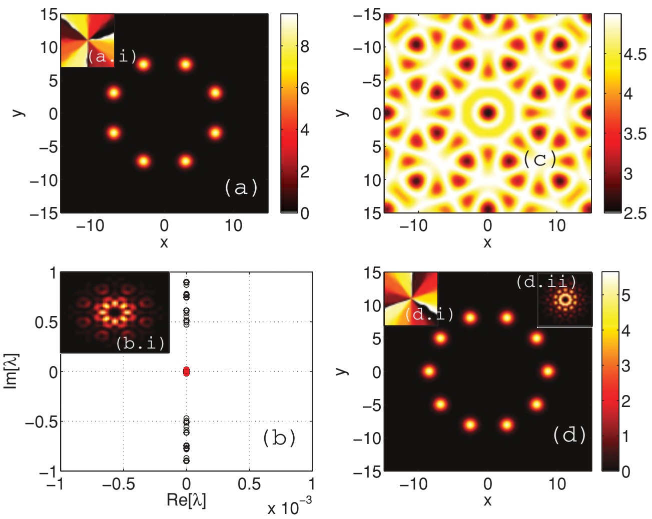

FIG. 3. (Color online) (a), (b) The same as in Figs. 1 and 2 for the stable $S = 3$ vortex in the $N = 8$ quasicrystal lattice (c). (d) $S = 3$ vortex for $N = 5$. (d.i) Phase and (d.ii) Fourier spectra of this solution. For both solutions, $\beta = 3.4$.

respectively. Figure 1(b.ii) is a closeup of the origin showing the nine negative-energy pairs close to the origin and the one pair at the origin. The potential instability arising from these negative-energy modes is prevented by their proximity to the origin and distance from the phonon band. The expected saddle-node bifurcation [23,24] occurs close to the band edge (which we computed as ≈3.9) in which the main solution collides (and disappears) with an unstable solution branch of a configuration with additional populated sites external (and in phase) to the original contour.

Next we present results of the $S = 3$ vortex for saturable nonlinearity in both the $N = 8$ [Figs. 3(a) and 3(b)] and the $N = 5$ [Fig. 3(d)] cases for $\beta = 3.4$. Figure 3(c) depicts the $N = 8$ lattice, and Figs. 3(d.i) and 3(d.ii) are the phase and Fourier spectrum, respectively, of the solution in Fig. 3(d). These solutions are both stable, and again, there is a resonance in the semi-infinite gap (not shown) similar to what is shown in Fig. 1. The vortices for $S < 3$ are unstable (not shown).

To examine the potential experimental realizability of the preceding wave forms, we consider a radial Gaussian beam with topological charge $S = 4$ of the form $e^{iS\theta-(r-R)^2/(2b^2)}$, with (r,θ) denoting polar coordinates, $R = 8.5$, approximately the radius of the contour, and $b = 1$, as an initial condition of the system with saturable nonlinearity. In order to prevent the inevitable radiation from scattering back from the boundaries of the computational domain, we employ an extra term $-i\Gamma$ on the right-hand side of Eq. (1) with $\Gamma = 1 - \tanh(D - \mathbf{r})$. With $D = 16$, this layer initially absorbs the shed radiation and, subsequently, affects the intensity distribution very little. However, the phase dynamics may be sensitive to the presence of such a layer. Imposing a wider dissipation layer, with $D = 24$, the solution actually never settles into one of constant charge; this topological instability effect has been analyzed, for example, in [25–28]. Specifically, vortices may nucleate in the very low-amplitude region and pass in and out of the main configuration (without affecting its intensity). Note that the above suggests that such effects could be avoided

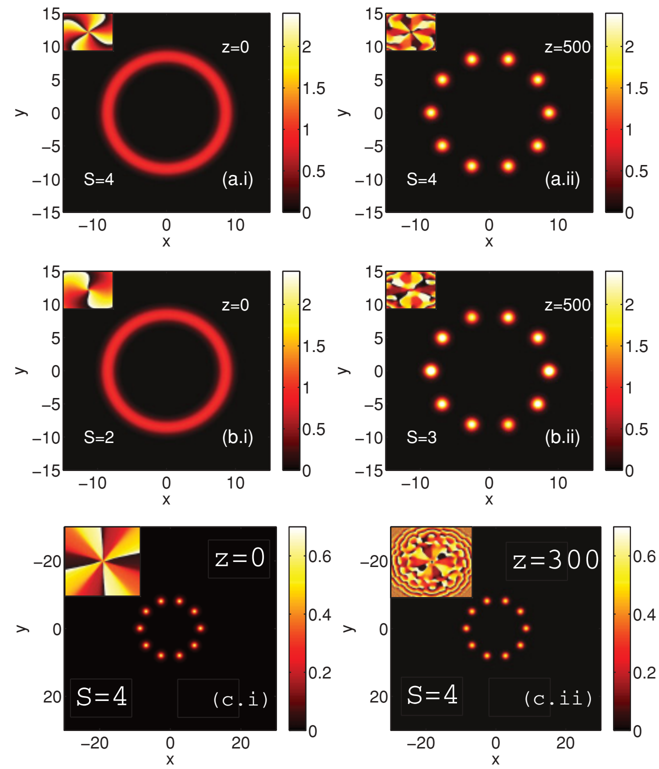

FIG. 4. (Color online) (a.i), (b.i) Initial conditions and (a.ii), (b.ii) profiles at a later time of the $S = 4$ and $S = 2$ radial Gaussian initial conditions for a saturable nonlinearity, with a "tight dissipation layer" (see text), $D = 16$. (c.i), (c.ii) Similar to the above plots but for $D = 24$ and a modulated initial condition.

experimentally if some form of dissipation is imposed. For comparison, we launch a similar initial condition with $S = 2$ and note that it never settles into a stable configuration of fixed charge *independently of the dissipation layer size* (and despite its seemingly robust intensity distribution).

Figures 4(a.i) and 4(b.i) present the initial conditions and profiles for a long evolution Figs. 4(a.ii) and 4(b.ii) of the $S = 4$ and $S = 2$ initial conditions, respectively, for saturable nonlinearity and $D = 16$. The charge of each fluctuates, as power is shed and vortices nucleate in the surrounding low-amplitude regions and enter and leave the contour as the solution traces a stationary state. However, for the $S = 4$ initial condition, the field settles into a solution of constant charge 4 for $D = 16$, while for the $S = 2$ initial condition, the phase continues to fluctuate throughout the numerical experiment. These results are typical in this setting. For D large, the charge may never settle (topological phase instability). However, this does not contradict the linear stability results (which we have confirmed separately for near-stationary configurations). The intial condition $e^{iS\theta-(r-R)^2/(2b^2)}\cos^2(5\theta)$ is far from a stationary configuration and does not prevent contamination of the resulting state by radiation. Although, for example, $\sum_{k=1}^{10} e^{ikS\pi/5-(x-c_{xk})^2-(y-c_{yk})^2}$, with (c_{xk},c_{yk}) the center of one of the wells, is sufficiently localized, and with the latter initial condition, we observe topological stability (indeed, without the initial turbulent fluctuating regime) for $S = 4$ [see Fig. 4(c)] but not for $S = 2$. Finally, Fig. 5 shows the evolution of unstable ($S = 2$) vortices in the presence of a cubic nonlinearity.

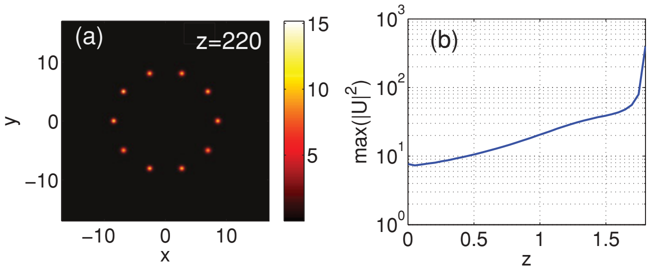

FIG. 5. (Color online) Dynamics of the unstable $S = 2$ vortex in the case of a cubic nonlinearity. Evolution of the same solution with the same perturbation of random noise with 5% of the initial maximum amplitude of the field can lead to (a) robust structures that persist for long distances or (b) almost immediately collapse in different trials.

The evolution depends sensitively on the particular initial condition. Using the initial condition $u = U(1 + X)$ with U the stationary solution and $X \sim 0.05\ \max_{\mathbf{x}}[U(\mathbf{x})]$ Uniform(0,1), two different particular instances can lead to significantly different dynamics. Either the phase merely reshapes, as for the saturable nonlinearity, but the structure persists [see Fig. 5(a)], or the solution collapses almost immediately, as shown by the maximum amplitude of the field in Fig. 5(b). For larger additive noise, collapse seems more likely from several sample trials. The relevant mechanism involves one of the solution lobes exceeding a minimum collapse threshold, leading to an "in-lobe" collapse.

We have demonstrated numerically stable vortices of topological charge $S = 3$ in quasicrystals with $n = 4$ and $n = 5$ directions of symmetry and $S = 4$ with $n = 5$, in the cases of both cubic and saturable focusing nonlinearities. The negative-energy modes for these configurations remain close to the origin in the spectral plane, preventing collision with the phonon band. Hence, the configurations can be experimentally realizable in photonic quasicrystals in a photorefractive (or a Kerr) medium. This has additionally been demonstrated by simulation of the evolution of a radial Gaussian beam into such robust vortex states. This is a prime prospect for an immediate future experimental direction related to the present work.

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