Quantum search by partial adiabatic evolution

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A quantum search algorithm based on the partial adiabatic evolution [Phys. Rev. A **80**, 052328 (2009)] is provided. We calculate its time complexity and find that the algorithm improves the time complexity, which is $O(\sqrt{N/M})$, of the local adiabatic search algorithm [Phys. Rev. A **65**, 042308 (2002)] to $O(\sqrt{N}/M)$.

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I. INTRODUCTION

Quantum adiabatic computation has attracted a lot of attention in the past decades, such as Refs. [1–9], since it was proposed by Farhi *et al.* [10]. In Ref. [4], an adiabatic algorithm was proposed to solve the Deutsch-Jozsa problem. The algorithm took an exponential time, which provided only a quadratic speed up over the best classical algorithm for the problem. A modified algorithm proposed by Wei and Ying [11] improved the performance to constant time. In Ref. [5], quantum adiabatic computation was proved to be polynomially equivalent to the quantum circuit model. The proof showed that adiabatic quantum computation using Hamiltonians with long-range five- or three-body interactions, or nearest-neighbor two-body interactions with six-state particles, could efficiently simulate the circuit model. This results were soon modified to qubits with two-body interactions [12]. A simpler proof of the equivalence was presented in Ref. [7]. In Ref. [13], quantum adiabatic computation was applied to solve random instances of *NP*-complete problems. A research outline of its application to solve *NP*-complete problems can also be found in the paper.

A typical quantum adiabatic algorithm starts with the ground state of the initial Hamiltonian H_i, and evolves slowly to the ground state of the final Hamiltonian H_f. The system that implements the algorithm uses the time-dependent Hamiltonian,

$$H(s(t)) = [1 - s(t)]H_i + s(t)H_f. \tag{1}$$

The running time (evolution time) is essentially determined by [1]

$$T \geqslant \Theta\left(\frac{1}{g_{\min}^2}\right), \tag{2}$$

where

$$g_{\min} = \min[E(1,s) - E(0,s)] \quad (0 \leqslant s \leqslant 1). \tag{3}$$

$E(0,s)$ and $E(1,s)$ are the two lowest eigenvalues of $H(s)$. We can see that the evolution time defined by Eq. (2) is determined only by the minimum energy gap. Note that, in general, the evolution time of a quantum adiabatic process cannot be determined only by the minimum energy gap. But in quantum adiabatic computation, the system Hamiltonian is usually chosen to be real and nonoscillating [6]. This means that the commonly used condition [14], which is reduced to Eq. (2) by Tulsi in Ref. [1], is sufficient to guarantee the adiabaticity of the system process [6,15,16].

Roland and Cerf [17] considered the unstructured search problem [18], and designed a quantum search algorithm based on local adiabatic evolution. In the algorithm, H_i and H_f are specified as

$$H_i = 1 - |\Psi\rangle\langle\Psi|, \tag{4}$$

and

$$H_f = 1 - |\beta\rangle\langle\beta|, \tag{5}$$

where

$$|\Psi\rangle = \frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}|i\rangle, \tag{6}$$

and $|\beta\rangle$ is an equal superposition of all marked states. The algorithm can find one marked item in a running time of order $\sqrt{N/M}$.

In Ref. [1], Tulsi proposed a partial adiabatic evolution with H_f a one-dimensional projector Hamiltonian. It was also checked in the paper that Roland and Cerf's results (for the case that there is only one marked state) can be obtained as a special case of the partial adiabatic evolution. In this Brief Report, we give a specified quantum search algorithm based on the partial adiabatic evolution, and show that the algorithm provides a better time complexity, which is $O(\sqrt{N}/M)$, than that, which is $O(\sqrt{N/M})$, of the local adiabatic search algorithm.

This Brief Report is organized as follows. In Sec. II, we specify the partial adiabatic search algorithm. In Sec. III, we calculate its time complexity. We conclude the Brief Report in Sec. IV.

II. PARTIAL ADIABATIC SEARCH ALGORITHM

In this section, we specify the partial adiabatic evolution [1] as a search algorithm. For convenience, we use different notations. The search algorithm executes the four steps as follows:

(1) The initial state is prepared to be $|\Psi\rangle$.

(2) At $t = 0$, the Hamiltonian is suddenly changed to $H(s^-)$ without disturbing state $|\Psi\rangle$.

(3) The Hamiltonian evolves from $H(s^-)$ to $H(s^+)$ linearly in time over duration T'.

(4) Measure the state of the system.

Repeat these four steps until we find a marked state.

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Note that the sudden change in step (2) is not an adiabatic process [the adiabatic evolution only occurs in step (3)], and s^- and s^+ are used to define the time interval in which the algorithm runs adiabatically, and are similar to the notations μ^- and μ^+ in Ref. [1]. For the algorithm, they are specified as $s^- = \frac{1}{2} - \frac{1}{2\sqrt{N}}$ and $s^+ = \frac{1}{2} + \frac{1}{2\sqrt{N}}$. Because the algorithm evolves adiabatically only within a small time interval $[s^-,s^+]$, it is called a partial adiabatic search algorithm. After step (2), the system that implements the algorithm will still be in state $|\Psi\rangle$. That is, the system state will be the ground state of $H(s^-)$ with probability $|\langle\Psi|E(0,s^-)\rangle|^2$. The adiabatic theorem [14] guarantees that it will be the ground state of $H(s^+)$ with probability $|\langle\Psi|E(0,s^-)\rangle|^2$ after step (3). Measuring the state of the system will give the ground state of $H_f = 1 - |\beta\rangle\langle\beta|$ with probability $P = |\langle\Psi|E(0,s^-)\rangle|^2 \times |\langle\beta|E(0,s^+)\rangle|^2$. We call P one-round success probability, and accordingly, T' one-round evolution time. The overall time complexity (evolution time) of the algorithm is $T = T'/P$. Here, $T' = \omega/g_{\min}^2$, where $\omega = s^+ - s^- = \frac{1}{\sqrt{N}}$.

III. TIME COMPLEXITY

As we know, the minimum energy gap is $g_{\min} = \sqrt{M/N}$ [17]. This gives rise to the one-round evolution time $T' = \sqrt{N}/M$. We turn to calculate the one-round success probability in the following. To achieve this, we follow a good method usually used to analyze the quantum search. That is, we work in a two-dimensional Hilbert space spanned by the marked and unmarked states. Let

$$|\alpha\rangle = \frac{1}{\sqrt{N-M}} \sum_{x\notin S} |x\rangle, \tag{7}$$

$$|\beta\rangle = \frac{1}{\sqrt{M}} \sum_{x\in S} |x\rangle, \tag{8}$$

where S is the set of the marked states, and M is the number of the marked states. Throughout this Brief Report, we suppose S is not empty. This means that $1 \leqslant M \leqslant N$. State $|\Psi\rangle$ can be rewritten as [19]

$$|\Psi\rangle = \sqrt{\frac{N-M}{N}}|\alpha\rangle + \sqrt{\frac{M}{N}}|\beta\rangle. \tag{9}$$

Let the eigenspectrum of $H(s)$ be

$$H(s)|E(k,s)\rangle = E(k,s)|E(k,s)\rangle, \tag{10}$$

where $E(k,s)$ and $|E(k,s)\rangle$ are the k-level eigenvalue and eigenstate of $H(s)$, respectively. Throughout this Brief Report, we only consider the two lowest eigenstates and eigenvalues (i.e., $k = 0,1$). Left multiplying $\langle\alpha|$ to Eq. (10), we get

$$\langle\alpha|H(s)|E(k,s)\rangle = E(k,s)\langle\alpha|E(k,s)\rangle. \tag{11}$$

Substituting Eq. (1) into the left side of Eq. (11), we also get

$$\langle\alpha|H(s)|E(k,s)\rangle = \langle\alpha|E(k,s)\rangle - (1-s)\langle\alpha|\Psi\rangle\langle\Psi|E(k,s)\rangle, \tag{12}$$

with $\langle\alpha|\beta\rangle = 0$. Combining Eqs. (11) with (12), we obtain

$$\langle\alpha|E(k,s)\rangle = \frac{(1-s)\langle\alpha|\Psi\rangle\langle\Psi|E(k,s)\rangle}{1-E(k,s)}, \tag{13}$$

when $1 - E(k,s) \neq 0$. Similarly,

$$\langle\beta|E(k,s)\rangle = \frac{(1-s)\langle\beta|\Psi\rangle\langle\Psi|E(k,s)\rangle}{1-s-E(k,s)}, \tag{14}$$

when $1 - s - E(k,s) \neq 0$.

Substituting Eqs. (13) and (14) into $|\langle\alpha|E(k,s)\rangle|^2 + |\langle\beta|E(k,s)\rangle|^2 = 1 (k = 0,1)$, we have

$$\left\{\frac{(1-s)^2A}{[1-E(k,s)]^2} + \frac{(1-s)^2B}{[1-s-E(k,s)]^2}\right\}|\langle\Psi|E(k,s)\rangle|^2 = 1, \tag{15}$$

when $1 - E(k,s) \neq 0$ and $1 - s - E(k,s) \neq 0$, where $A = |\langle\Psi|\alpha\rangle|^2 = \frac{N-M}{N}$ and $B = |\langle\Psi|\beta\rangle|^2 = \frac{M}{N}$. If $1 - s \neq 0$, this immediately gives

$$|\langle\Psi|E(k,s)\rangle|^2 = \frac{1}{(1-s)^2\left\{\frac{A}{[1-E(k,s)]^2} + \frac{B}{[1-s-E(k,s)]^2}\right\}}. \tag{16}$$

Substituting Eq. (16) into Eq. (14), we get

$$|\langle\beta|E(k,s)\rangle|^2 = \frac{B}{\left\{\frac{[1-s-E(k,s)]}{1-E(k,s)}\right\}^2 A + B}. \tag{17}$$

Because

$$(1-s^-)^2 = \frac{1}{4}\left(1 + \frac{1}{\sqrt{N}}\right)^2 < 1, \tag{18}$$

$$\frac{A}{[1-E(0,s^-)]^2} < 4, \tag{19}$$

and

$$\frac{B}{[1-s^- - E(0,s^-)]^2} < 4, \tag{20}$$

we obtain

$$|\langle\Psi|E(0,s^-)\rangle|^2 > \tfrac{1}{8}. \tag{21}$$

using Eq. (16).

In addition,

$$\frac{1-s^+ - E(0,s^+)}{1-E(0,s^+)} < \sqrt{\frac{2M}{N}} = \sqrt{2B}, \tag{22}$$

we obtain

$$|\langle\beta|E(0,s^+)\rangle|^2 > \tfrac{1}{3}, \tag{23}$$

using Eq. (17). We finally get a lower bound of the one-round success probability,

$$P = \left|\left\langle\Psi\middle|E\left(0,\frac{1}{2} - \frac{1}{2\sqrt{N}}\right)\right\rangle\right|^2 \left|\left\langle\beta\middle|E\left(0,\frac{1}{2} + \frac{1}{2\sqrt{N}}\right)\right\rangle\right|^2 > \frac{1}{24}, \tag{24}$$

with Eqs. (21) and (23). As a result, the overall time complexity of the partial adiabatic search algorithm is $T = T'/P = O(\sqrt{N}/M)$.

IV. CONCLUSION

We have provided a quantum search algorithm based on the partial adiabatic evolution. As we have seen, the minimum energy gap along with the one-round success probability determines the overall time complexity of the algorithm. It is proved that the one-round success probability is bounded from below by a constant, and hence, the overall time complexity is $O(\sqrt{N}/M)$, which provides a speed up of $\sqrt{M}$ over the local adiabatic search algorithm.

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