Entangling power and local invariants of two-qubit gates

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We show a simple relation connecting entangling power and local invariants of two-qubit gates. From the relation, a general condition under which gates have the same entangling power is derived. The relation also helps in finding the lower bound of entangling power for perfect entanglers, from which the classification of gates as perfect and nonperfect entanglers is obtained in terms of local invariants.

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Entanglement, a nonlocal property of a quantum state, is regarded as a resource for realizing various fascinating features such as teleportation, quantum cryptography, and quantum computation [1,2]. On one side, much work has been carried out to understand and exploit the entanglement for various information processing. On the other side, attention has been given to quantum operations (gates) as they are responsible for creating entanglement when acting on a state.

Since two-qubit gates are capable of producing entanglement, it is of vital importance to understand their entangling characterization. One such useful tool is the entangling power of an operator $e_p(U)$, which quantifies the average entanglement produced [3]. Another tool to characterize the nonlocal attributes of a two-qubit gate is local invariants, namely, G_1 and G_2 (first introduced in Ref. [4]), such that gates differing only by local operations possess the same invariants. Furthermore, nonlocal two-qubit gates form an irreducible geometry of tetrahedron known as a Weyl chamber. Of all the gates, exactly half of them are perfect entanglers (operators capable of producing a maximally entangled state from some input product state), and they form a polyhedron within the Weyl chamber [5].

It is known that gates differing only by local operations possess the same entangling power. Similarly, gates which are inverse to each other possess the same entangling power. For instance, SWAP^{α} and $\text{SWAP}^{-\alpha}$ gates assume the same entangling power as they are inverse to each other. In our earlier study on the geometrical edges of two-qubit gates [6], it was found that gates which do not belong to the preceding category also possess the same entangling power. For example, the entangling power of the gates lying in the polyhedron edges QP, MN, and PN are identical [6]. Motivated by this fact, here we investigate the entangling power of two-qubit gates in detail. In this Brief Report, we establish a simple relation between the entangling power and local invariants. It is shown that if the $|G_1|$ of two gates is the same, they possess the same entangling power. The relation also facilitates in showing that the minimum entangling power of perfect entanglers is possessed by the three edges of the polyhedron mentioned earlier. Furthermore, we find the conditions for the perfect entanglers in terms of local invariants, which are useful for the classification of two-qubit gates as perfect and nonperfect entanglers.

Let us consider a general two-qubit gate U [7]:

$$U = \begin{pmatrix} e^{-\frac{ic_3}{2}}c^- & 0 & 0 & -ie^{-\frac{ic_3}{2}}s^- \\ 0 & e^{\frac{ic_3}{2}}c^+ & -ie^{\frac{ic_3}{2}}s^+ & 0 \\ 0 & -ie^{\frac{ic_3}{2}}s^+ & e^{\frac{ic_3}{2}}c^+ & 0 \\ -ie^{-\frac{ic_3}{2}}s^- & 0 & 0 & e^{-\frac{ic_3}{2}}c^- \end{pmatrix}, \quad (1)$$

where $c^{\pm} = \cos[(c_1 \pm c_2)/2]$, $s^{\pm} = \sin[(c_1 \pm c_2/2]$, and $[c_1, c_2, c_2]$ is the geometrical point of a two-qubit gate [4,5]. We note that the geometrical representation of two-qubit gates (Weyl chamber) is described by $c_1 \geqslant c_2 \geqslant c_3 \geqslant 0$. The entangling capability of a unitary quantum gate can be quantified by the entangling power, which is defined as [3,8]

$$e_p(U) = \overline{[E(U|\Psi_1\rangle \otimes |\Psi_2\rangle)]}_{|\Psi_1\rangle\otimes|\Psi_2\rangle}, \quad (2)$$

where the overbar denotes the average over all product states distributed uniformly in the state space. In the preceding formula, E is the linear entropy of entanglement measure, defined as

$$E(|\Psi\rangle_{AB}) = 1 - \text{tr}\left[\rho^2_{A(B)}\right], \quad (3)$$

where $\rho_{A(B)} = \text{tr}_{B(A)}(|\Psi\rangle_{AB}\langle\Psi|)$ is the reduced density matrix of system $A(B)$.

The expression to calculate the entangling power of a two-qubit gate U is [3,9]

$$e_p(U) = \tfrac{5}{9} - \tfrac{1}{36}\{\langle U^{\otimes 2}, T_{1,3}U^{\otimes 2}T_{1,3}\rangle + \langle(\text{SWAP} \times U)^{\otimes 2}, T_{1,3}(\text{SWAP} \times U)^{\otimes 2}T_{1,3}\rangle\}, \quad (4)$$

where $\langle A,B\rangle = \text{tr}(A^{\dagger}B)$ is referred to as the Hilbert-Schmidt scalar product and $T_{1,3}$ is the transposition operator, defined as $T_{1,3}|a,b,c,d\rangle = |c,b,a,d\rangle$ on a four-qubit system. In what follows, we use the definitions $A = U^{\otimes 2}$, $S = \text{SWAP}^{\otimes 2}$, $B = (\text{SWAP} \times U)^{\otimes 2}$, and $T = T_{1,3}$. Exploiting the property of tensor products [10], $(A_1A_2) \otimes (B_1B_2) = (A_1 \otimes B_1)(A_2 \otimes B_2)$, we can write $B = SA$. With this, we have $\langle B,\text{TBT}\rangle = \text{tr}(A^{\dagger}S^{\dagger}\text{TSAT})$, and hence the entangling power can be rewritten as

$$e_p(U) = \tfrac{5}{9} - \tfrac{1}{36}[\text{tr}(A^{\dagger}\text{TAT}) + \text{tr}(A^{\dagger}S^{\dagger}\text{TSAT})]. \quad (5)$$

Using the fact that $\text{tr}(A) + \text{tr}(B) = \text{tr}(A + B)$, we write the entangling power as

$$e_p(U) = \tfrac{5}{9} - \tfrac{1}{36}[\text{tr}(A^{\dagger}\text{RAT})], \quad (6)$$

where $R = T + S^\dagger T S$. Substituting Eq. (1) in the preceding expression, after some simplifications, the entangling power can be rewritten as

$$e_p(U) = \tfrac{2}{9}[1 - |G_1|], \tag{7}$$

where

$$|G_1| = \cos^2 c_1 \cos^2 c_2 \cos^2 c_3 + \sin^2 c_1 \sin^2 c_2 \sin^2 c_3. \tag{8}$$

Thus we obtain a simple relation between the entangling power e_p and the local invariant G_1 of a two-qubit gate. The relation also implies that gates having the same $|G_1|$ must necessarily possess the same e_p. Since the invariant G_1 for a gate and its inverse are complex conjugate to each other, both gates will have same e_p. Since $0 \leqslant |G_1| \leqslant 1$, it is evident that $0 \leqslant e_p \leqslant 2/9$. Here we note that Eq. (7) can also be rewritten as [7]

$$e_p(U) = \tfrac{1}{18}[3 - (\cos 2c_1 \cos 2c_2 + \cos 2c_2 \cos 2c_3 + \cos 2c_3 \cos 2c_1)]. \tag{9}$$

In our earlier study on the geometrical edges of a polyhedron, it was shown that $e_p = 1/6$ for the edges *QP*, *MN,* and *PN* [6]. In terms of Eq. (7), this result is understandable as $|G_1| = 1/4$ for all these edges. We also note that the identical parameter dependence of e_p for the other edges of the polyhedron, *LQ*, *LN,* and A_2P, is also reflected through their $|G_1|$ [6]. Furthermore, Eq. (7) is also useful in identifying the gates with maximum and minimum e_p. If $e_p = 2/9$, $|G_1| = 0$, which is possible only for $[\pi/2,\varphi,0]$, where $0 \leqslant \varphi \leqslant \pi/2$. These gates correspond to the well-known family of special perfect entanglers (SPE) [7]. If $e_p = 0$, $|G_1| = 1$, which is possible only for (i) [0,0,0], a local gate, and (ii) $[\pi/2,\pi/2,\pi/2]$, a SWAP gate.

A two-qubit gate is called a perfect entangler (PE) if it produces a maximally entangled state for some input product state. Considering the symmetry in the maximal entanglement production by the gates, we confine our attention to one half of the Weyl chamber: $\pi/2 \geqslant c_1 \geqslant c_2 \geqslant c_3 \geqslant 0$. If the geometrical points are such that

$$\text{(A)} \quad c_1 + c_2 \geqslant \pi/2 \quad \text{and} \quad \text{(B)} \quad c_2 + c_3 \leqslant \pi/2, \tag{10}$$

then the corresponding gate is a perfect entangler [11].

Having known that SPE possess the maximum e_p, here we exploit Eq. (7) to identify PEs which possess minimum e_p. In other words, we find PEs which possess maximum $|G_1|$. Let us rewrite the first term of Eq. (8) as

$$\cos^2 c_1 \cos^2 c_2 \cos^2 c_3 = \{[\cos(c_1 + c_2) + \cos(c_1 - c_2)]^2 \cos^2 c_3\}/4. \tag{11}$$

Imposing the condition (A) implies that $-1 \leqslant \cos(c_1 + c_2) \leqslant 0$ and $0 \leqslant \cos(c_1 - c_2) \leqslant 1$. Then $|G_1|$ has the maximum value of 1/4 only for $c_1 = c_2 = \pi/4$, for which the condition (B) becomes $0 \leqslant c_3 \leqslant \pi/4$. In other words, the edge *QP* $[\pi/4,\pi/4,\eta]$ with $0 \leqslant \eta \leqslant \pi/4$ is such that $|G_1| = 1/4$ and hence $e_p = 1/6$ [6]. It is worth recollecting that if $[c_1,c_2,c_3]$ is a perfect entangler, then $[\pi - c_1,c_2,c_3]$ is also a perfect entangler. Since the edge *QP* $[\pi/4,\pi/4,\eta]$ is a PE, the edge *MN* $[3\pi/4,\pi/4,\eta]$ is also a PE with $e_p = 1/6$. In a similar way, the second term of Eq. (8) is rewritten as

$$\sin^2 c_1 \sin^2 c_2 \sin^2 c_3 = \{\sin^2 c_1 [\cos(c_2 - c_3) - \cos(c_2 + c_3)]^2\}/4. \tag{12}$$

Imposing the condition (B) implies that $0 \leqslant \cos(c_2 + c_3) \leqslant 1$ and $0 \leqslant \cos(c_2 - c_3) \leqslant 1$. Then $|G_1|$ has the maximum value of 1/4 only for $c_2 = c_3 = \pi/4$, for which the condition (A) becomes $\pi/4 \leqslant c_1 \leqslant \pi/2$. In other words, the gates $[\pi/4 + \theta,\pi/4,\pi/4]$ with $0 \leqslant \theta \leqslant \pi/4$ are such that $|G_1| = 1/4$ and $e_p = 1/6$. Since the gates $[\pi/4 + \theta,\pi/4,\pi/4]$ are PEs, the gates $[3\pi/4 - \theta,\pi/4,\pi/4]$ are also PEs with $e_p = 1/6$. Alternatively, the edge *PN* $[\pi/4 + \eta,\pi/4,\pi/4]$ with $0 \leqslant \eta \leqslant \pi/2$ is a PE having $e_p = 1/6$. From the preceding analysis, it is clear for the PE that $1/6 \leqslant e_p \leqslant 2/9$. In this range, SPE possess the maximum and the polyhedron edges *QP*, *MN,* and *PN* possess the minimum. In terms of local invariant G_1, the preceding inequality reads as $0 \leqslant |G_1| \leqslant 1/4$.

Having found the range of G_1, the following theorem identifies the range of local invariant G_2 for PEs.

Theorem 1. PEs are such that $-1 \leqslant G_2 \leqslant 1$.

Proof. The expression for G_2 is as given in Ref. [5],

$$G_2 = 4\cos^2 c_1 \cos^2 c_2 \cos^2 c_3 - 4\sin^2 c_1 \sin^2 c_2 \sin^2 c_3 - \cos 2c_1 \cos 2c_2 \cos 2c_3, \tag{13}$$

or [7],

$$G_2 = \cos 2c_1 + \cos 2c_2 + \cos 2c_3. \tag{14}$$

The preceding expression is rewritten as

$$G_2 = 2\cos(c_1 + c_2)\cos(c_1 - c_2) + \cos 2c_3. \tag{15}$$

On imposing the condition (A), we have $-1 \leqslant \cos(c_1 + c_2) \leqslant 0$ and $0 \leqslant \cos(c_1 - c_2) \leqslant 1$. Then G_2 has the maximum value of 1 for $c_1 = \pi/2 - \theta$, $c_2 = \theta$, and $c_3 = 0$. In other words, the edge *LQ* $[\pi/2 - \theta,\theta,0]$ of a Weyl chamber with $0 \leqslant \theta \leqslant \pi/4$ is such that $G_2 = 1$ [6]. Similarly, G_2 takes the minimum value of -1 for $c_1 = c_2 = \pi/2$ and $c_3 = 0$, which corresponds to the double controlled-NOT operation (DCNOT). Hence the proof is completed.

From the earlier analysis on G_1, we observe that all the PEs lie within the range $0 \leqslant |G_1| \leqslant 1/4$. It is worth mentioning that non-PEs are also found within this range, for example, some controlled unitary gates [6]. In order to classify the gates based on the local invariants, we prove the following theorem.

Theorem 2. Non-PEs lie within the range $0 \leqslant |G_1| \leqslant 1/4$ do not satisfy $-1 \leqslant G_2 \leqslant 1$.

Proof. Consider all the non-PEs within the range $0 \leqslant |G_1| \leqslant 1/4$, for which the condition (A) or (B) must be violated. Violation of both (A) and (B) amounts to violation of the Weyl chamber condition: $\pi/2 \geqslant c_1 \geqslant c_2 \geqslant c_3 \geqslant 0$. First, let us assume that (A) is violated, that is, $c_1 + c_2 < \pi/2$. This implies that $0 < \cos(c_1 + c_2) \leqslant 1$ and $0 < \cos(c_1 - c_2) \leqslant 1$, and G_2 takes the maximum value of 3 for $c_1 = c_2 = c_3 = 0$ (local gate). In order to find the minimum value of G_2, we take $c_1 + c_2 = \pi/2 - \epsilon$ and $c_1 - c_2 = \pi/2 - \delta$, where $0 < \epsilon < \delta \ll 1$. Then, the minimum value of G_2 is $1 + 3\epsilon\delta$ for $c_1 = (\pi - \epsilon - \delta)/2$, $c_2 = c_3 = (\delta - \epsilon)/2$. Second, let us consider that (B) is violated, that is, $c_2 + c_3 > \pi/2$. Rewriting Eq. (14) as

$$G_2 = \cos 2c_1 + 2\cos(c_2 + c_3)\cos(c_2 - c_3), \qquad (16)$$

we have $-1 \leqslant \cos(c_2 + c_3) < 0$ and $0 < \cos(c_2 - c_3) \leqslant 1$. Then the minimum value of G_2 is -3 for $c_1 = c_2 = c_3 = \pi/2$ (SWAP gate). For the maximum value of G_2, we take $c_2 + c_3 = \pi/2 + \epsilon$ and $c_2 - c_3 = \pi/2 - \delta$. With this, the maximum value of G_2 is $-(1 + 3\epsilon\delta)$ for $c_1 = c_2 = (\pi + \epsilon - \delta)/2$, $c_3 = (\epsilon + \delta)/2$. Hence non-PEs which violate (A) or (B) do not fall in the range $-1 \leqslant G_2 \leqslant 1$, and the proof is completed.

From theorem 2, we conclude that PEs satisfy

$$\text{(C)} \quad 0 \leqslant |G_1| \leqslant 1/4 \quad \text{and} \quad \text{(D)} \quad -1 \leqslant G_2 \leqslant 1, \qquad (17)$$

and the gates that do not satisfy both these conditions are non-PEs. It is easy to recognize that Eq. (17) and Eq. (10) are equivalent. Thus the local invariants associated with a gate are found to be useful for the classification as perfect and nonperfect entanglers. It is worth emphasizing that Eq. (17) involves two parameters, namely, $|G_1|$ and G_2, while Eq. (10) involves three geometrical parameters.

In this work, we have shown a simple relation between entangling power e_p and local invariant G_1 of a two-qubit gate. The relation implies that the gates with the same $|G_1|$ possess the same entangling power. Thus the local invariant G_1 of a gate also signifies the average entanglement produced. Gates differing only by local operations have same G_1 and hence e_p. Since the invariant G_1 for a gate and its inverse are complex conjugate to each other, both gates will have same e_p. It is identified that three geometrical edges of the polyhedron, namely, *QP*, *MN,* and *PN,* are such that $|G_1| = 1/4$, and hence $e_p = 1/6$. It is shown for perfect entanglers that $1/6 \leqslant e_p \leqslant 2/9$ or $0 \leqslant |G_1| \leqslant 1/4$ such that $e_p = 1/6$ for the preceding three edges and $e_p = 2/9$ for special perfect entanglers.

Furthermore, the local invariant G_2 for the perfect entanglers are such that $-1 \leqslant G_2 \leqslant 1$. From the obtained range of local invariants G_1 and G_2, it is shown that the invariants are also useful in classifying the two-qubit gates as perfect and nonperfect entanglers. It is worth noting that the obtained classification based on local invariants does not require the geometrical point of a gate.

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