Schrödinger uncertainty relation with Wigner-Yanase skew information

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We shall give an alternative Schrödinger-type uncertainty relation for a quantity representing a quantum uncertainty, introduced by Luo [Phys. Rev. A **72**, 042110 (2005)]. Our result improves the Heisenberg-type uncertainty relation shown in Luo's paper for a mixed state.

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I. INTRODUCTION

In quantum mechanical systems, the expectation value of an observable (self-adjoint operator) H in a quantum state (density operator) ρ is expressed by $\mathrm{Tr}(\rho H)$. Also, the variance for a quantum state ρ and an observable H is defined by $V_\rho(H) \equiv \mathrm{Tr}\{\rho[H - \mathrm{Tr}(\rho H)I]^2\} = \mathrm{Tr}(\rho H^2) - \mathrm{Tr}(\rho H)^2$. The Heisenberg uncertainty relation is well known [1]:

$$V_\rho(A)V_\rho(B) \geqslant \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2 \tag{1}$$

for a quantum state ρ and two observables A and B. The further strong result was given by Schrödinger [2]:

$$V_\rho(A)V_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2 \geqslant \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2, \tag{2}$$

where the covariance is defined by $\mathrm{Cov}_\rho(A,B) \equiv \mathrm{Tr}(\rho[A - \mathrm{Tr}(\rho A)I][B - \mathrm{Tr}(\rho B)I])$.

On the other hand, as a degree for noncommutativity between a quantum state ρ and an observable H, the Wigner-Yanase skew information $I_\rho(H)$ was defined in Ref. [3] (see Definition 1 in Sec. II). It is well known that the convexity of the Wigner-Yanase-Dyson skew information $I_{\rho,\alpha}(H) \equiv \frac{1}{2}\mathrm{Tr}\{(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])\}$, $\alpha \in [0,1]$, which is a one-parameter extension of the Wigner-Yanase skew information $I_\rho(H)$, with respect to ρ was successfully proven by Lieb in Ref. [4]. We have the relation between $I_\rho(H)$ and $V_\rho(H)$ such that $0 \leqslant I_\rho(H) \leqslant V_\rho(H)$ so it is quite natural to consider that we have the further sharpened uncertainty relation for the Wigner-Yanase skew information:

$$I_\rho(A)I_\rho(B) \geqslant \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2.$$

However, the above relation failed (see Refs. [5–7]). Luo then introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture:

$$U_\rho(H) \equiv \sqrt{V_\rho(H)^2 - [V_\rho(H) - I_\rho(H)]^2}, \tag{3}$$

and then he successfully showed a new the Heisenberg-type uncertainty relation on $U_\rho(H)$ in Ref. [8]:

$$U_\rho(A)U_\rho(B) \geqslant \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2. \tag{4}$$

As stated in Ref. [8], the physical meaning of the quantity $U_\rho(H)$ can be interpreted as follows. For a mixed state ρ, the variance $V_\rho(H)$ has both classical mixture and quantum uncertainty. Also, the Wigner-Yanase skew information $I_\rho(H)$ represents a kind of quantum uncertainty [9,10]. Thus, the difference $V_\rho(H) - I_\rho(H)$ has a classical mixture so we can consider that the quantity $U_\rho(H)$ has a quantum uncertainty excluding a classical mixture. Therefore it is meaningful and suitable to study an uncertainty relation for a mixed state by use of the quantity $U_\rho(H)$.

Recently, Yanagi gave a one-parameter extension of the inequality (4) in Ref. [11], using the Wigner-Yanase-Dyson skew information $I_{\rho,\alpha}(H)$. Note that we have the following ordering among three quantities:

$$0 \leqslant I_\rho(H) \leqslant U_\rho(H) \leqslant V_\rho(H). \tag{5}$$

The inequality (4) is a refinement of the original Heisenberg's uncertainty relation (1) in the sense of the above ordering (5).

In this Brief Report, we show the further strong inequality (Schrödinger-type uncertainty relation) for the quantity $U_\rho(H)$ representing a quantum uncertainty.

II. MAIN RESULTS

To show our main theorem, we prepare the definition for a few quantities and a lemma representing properties on their quantities.

Definition 1. For a quantum state ρ and an observable H, we define the following quantities.

(i) The Wigner-Yanase skew information:

$$I_\rho(H) \equiv \frac{1}{2}\mathrm{Tr}\{(i[\rho^{1/2}, H_0])^2\} = \mathrm{Tr}\left(\rho H_0^2\right) - \mathrm{Tr}(\rho^{1/2}H_0\rho^{1/2}H_0),$$

where $H_0 \equiv H - \mathrm{Tr}(\rho H)I$ and $[X,Y] \equiv XY - YX$ is a commutator.

(ii) The quantity associated to the Wigner-Yanase skew information:

$$J_\rho(H) \equiv \frac{1}{2}\mathrm{Tr}[(\{\rho^{1/2}, H_0\})^2] = \mathrm{Tr}\left(\rho H_0^2\right) + \mathrm{Tr}(\rho^{1/2}H_0\rho^{1/2}H_0),$$

where $\{X,Y\} \equiv XY + YX$ is an anticommutator.

(iii) The quantity representing a quantum uncertainty:

$$U_\rho(H) \equiv \sqrt{V_\rho(H)^2 - [V_\rho(H) - I_\rho(H)]^2}.$$

For two quantities $I_\rho(H)$ and $J_\rho(H)$, by simple calculations, we have

$$I_\rho(H) = \mathrm{Tr}(\rho H^2) - \mathrm{Tr}(\rho^{1/2}H\rho^{1/2}H)$$

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and

$$J_\rho(H) = \mathrm{Tr}(\rho H^2) + \mathrm{Tr}(\rho^{1/2} H \rho^{1/2} H) - 2[\mathrm{Tr}(\rho H)]^2 = 2V_\rho(H) - I_\rho(H), \quad (6)$$

which implies $I_\rho(H) \leqslant J_\rho(H)$. In addition, we have the following relations.

Lemma 1. (i) For a quantum state ρ and an observable H, we have the following relation among $I_\rho(H)$, $J_\rho(H)$, and $U_\rho(H)$:

$$U_\rho(H) = \sqrt{I_\rho(H) J_\rho(H)}.$$

(ii) For a spectral decomposition of $\rho = \sum_{j=1}^{\infty} \lambda_j |\phi_j\rangle\langle\phi_j|$, putting $h_{ij} \equiv \langle\phi_i|H_0|\phi_j\rangle$, we have

$$I_\rho(H) = \sum_{i<j} (\sqrt{\lambda_i} - \sqrt{\lambda_j})^2 |h_{ij}|^2.$$

(iii) For a spectral decomposition of $\rho = \sum_{j=1}^{\infty} \lambda_j |\phi_j\rangle\langle\phi_j|$, putting $h_{ij} \equiv \langle\phi_i|H_0|\phi_j\rangle$, we have

$$J_\rho(H) \geqslant \sum_{i<j} (\sqrt{\lambda_i} + \sqrt{\lambda_j})^2 |h_{ij}|^2.$$

The relation (i) immediately follows from Eq. (6). See Ref. [11] for the proofs of (ii) and (iii).

Theorem 1. For a quantum state (density operator) ρ and two observables (self-adjoint operators) A and B, we have

$$U_\rho(A)U_\rho(B) - |\mathrm{Re}\{\mathrm{Corr}_\rho(A,B)\}|^2 \geqslant \tfrac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2, \quad (7)$$

where the correlation measure is defined by

$$\mathrm{Corr}_\rho(X,Y) \equiv \mathrm{Tr}(\rho X^* Y) - \mathrm{Tr}(\rho^{1/2} X^* \rho^{1/2} Y)$$

for any operators X and Y.

Proof. We take a spectral decomposition $\rho = \sum_{j=1}^{\infty} \lambda_j |\phi_j\rangle\langle\phi_j|$. If we put $a_{ij} = \langle\phi_i|A_0|\phi_j\rangle$ and $b_{ji} = \langle\phi_j|B_0|\phi_i\rangle$, where $A_0 = A - \mathrm{Tr}(\rho A)I$ and $B_0 = B - \mathrm{Tr}(\rho B)I$, then we have

$$\begin{aligned}\mathrm{Corr}_\rho(A,B) &= \mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho^{1/2} A \rho^{1/2} B) \\ &= \mathrm{Tr}[\rho A_0 B_0] - \mathrm{Tr}(\rho^{1/2} A_0 \rho^{1/2} B_0) \\ &= \sum_{i,j=1}^{\infty} \left(\lambda_i - \lambda_i^{1/2}\lambda_j^{1/2}\right) a_{ij} b_{ji} \\ &= \sum_{i \neq j} \left(\lambda_i - \lambda_i^{1/2}\lambda_j^{1/2}\right) a_{ij} b_{ji} \\ &= \sum_{i<j} \left[\left(\lambda_i - \lambda_i^{1/2}\lambda_j^{1/2}\right) a_{ij} b_{ji} \right. \\ &\quad \left. + \left(\lambda_j - \lambda_j^{1/2}\lambda_i^{1/2}\right) a_{ji} b_{ij}\right].\end{aligned}$$

Thus we have

$$\begin{aligned}|\mathrm{Corr}_\rho(A,B)| \leqslant \sum_{i<j} &\left(\left|\lambda_i - \lambda_i^{1/2}\lambda_j^{1/2}\right| |a_{ij}||b_{ji}| \right. \\ &\left. + \left|\lambda_j - \lambda_j^{1/2}\lambda_i^{1/2}\right| |a_{ji}||b_{ij}|\right).\end{aligned}$$

Since $|a_{ij}| = |a_{ji}|$ and $|b_{ij}| = |b_{ji}|$, taking a square of both sides and then using Schwarz inequality for a scalar and Lemma 1, we have

$$\begin{aligned}&|\mathrm{Corr}_\rho(A,B)|^2 \\ &\leqslant \left(\sum_{i<j} \left(\left|\lambda_i - \lambda_i^{1/2}\lambda_j^{1/2}\right| + \left|\lambda_j - \lambda_j^{1/2}\lambda_i^{1/2}\right|\right) |a_{ij}||b_{ji}|\right)^2 \\ &= \left(\sum_{i<j} \left(\lambda_i^{1/2} + \lambda_j^{1/2}\right) \left|\lambda_i^{1/2} - \lambda_j^{1/2}\right| |a_{ij}||b_{ji}|\right)^2 \\ &\leqslant \left(\sum_{i<j} (\sqrt{\lambda_i} - \sqrt{\lambda_j})^2 |a_{ij}|^2\right) \left(\sum_{i<j} (\sqrt{\lambda_i} + \sqrt{\lambda_j})^2 |b_{ij}|^2\right) \\ &\leqslant I_\rho(A) J_\rho(B).\end{aligned}$$

In a similar way, we also have

$$|\mathrm{Corr}_\rho(A,B)|^2 \leqslant I_\rho(B) J_\rho(A).$$

Thus we have

$$|\mathrm{Corr}_\rho(A,B)|^2 \leqslant U_\rho(A) U_\rho(B),$$

which is equivalent to the inequality

$$U_\rho(A)U_\rho(B) - |\mathrm{Re}\{\mathrm{Corr}_\rho(A,B)\}|^2 \geqslant \tfrac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2,$$

since we have

$$|\mathrm{Im}\{\mathrm{Corr}_\rho(A,B)\}|^2 = \tfrac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2.$$

■

Theorem 1 improves the uncertainty relation (4) shown in Ref. [8], in the sense that the upper bound of the right-hand side of our inequality (7) is tighter than that of Luo's (4).

Remark 1. For a pure state $\rho = |\varphi\rangle\langle\varphi|$, we have $I_\rho(H) = V_\rho(H)$, which implies $U_\rho(H) = V_\rho(H)$ for an observable H and $\mathrm{Corr}_\rho(A,B) = \mathrm{Cov}_\rho(A,B)$ for two observables A and B. Therefore our Theorem 1 coincides with the Schrödinger uncertainty relation (2) for a particular case that a given quantum state is a pure state, $\rho = |\varphi\rangle\langle\varphi|$.

Remark 2. As a similar problem, we may consider the following uncertainty relation:

$$U_\rho(A)U_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2 \geqslant \tfrac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2.$$

However, the above inequality does not hold in general, since we have a counterexample as follows. We take

$$\rho = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and then we have

$$U_\rho(A)U_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2 - \tfrac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2 = -\tfrac{3}{4}.$$

Remark 3. From Theorem 1 and Remark 2, we may expect that the following inequality holds:

$$|\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2 \geqslant |\mathrm{Re}\{\mathrm{Corr}_\rho(A,B)\}|^2. \quad (8)$$

However, the above inequality does not hold in general, since we have a counterexample as follows. We take

$$\rho = \frac{1}{10}\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}, \quad A = \begin{pmatrix} 4 & 4 \\ 4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix},$$

and then we have

$$|\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2 - |\mathrm{Re}\{\mathrm{Corr}_\rho(A,B)\}|^2 \simeq -0.1539.$$

Actually, from Theorem 1, the example in Remark 2, and the above example, we find that there is no ordering between $|\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2$ and $|\mathrm{Re}\{\mathrm{Corr}_\rho(A,B)\}|^2$.

Remark 4. The example given in Remark 2 shows

$$V_\rho(A)V_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2 - \{U_\rho(A)U_\rho(B) - |\mathrm{Re}\{\mathrm{Corr}_\rho(A,B)\}|^2\} \simeq -0.232\,051.$$

The example given in Remark 3 also shows

$$V_\rho(A)V_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2 - \{U_\rho(A)U_\rho(B) - |\mathrm{Re}\{\mathrm{Corr}_\rho(A,B)\}|^2\} \simeq 13.7862.$$

Therefore there is no ordering between $V_\rho(A)V_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A,B)\}|^2$ and $U_\rho(A)U_\rho(B) - |\mathrm{Re}\{\mathrm{Corr}_\rho(A,B)\}|^2$ so we can conclude that neither the inequality (2) nor the inequality (7) is uniformly better than the other.

III. CONCLUSION

As we have seen, we proved an alternative Schrödinger-type uncertainty relation for a quantum state (generally a mixed state). Our result coincides with the original Schrödinger uncertainty relation for a particular case that a quantum state is a pure state. In addition, our result improves the uncertainty relation shown in Ref. [8] as well as the original Heisenberg uncertainty relation. Moreover, it is impossible to conclude that our result is always better than the original Schrödinger uncertainty relation for a mixed state, from the viewpoint of finding a tight upper bound for $\frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2$, where $\mathrm{Tr}(\rho[A,B])$ can be regarded as an average of the commutator $[A,B]$ for two observables A and B in a quantum state ρ. However, in other words, it is also impossible to conclude that our result is a trivial one, since there is no ordering between the left-hand side of the inequality (2) and that of (7).

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