Solitonic Bloch oscillations in two-dimensional optical lattices

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A theoretical description for nonlinear beam propagation in a two-dimensional optical lattice in the presence of a refractive-index gradient has been developed. This problem is associated with nonlinear Bloch oscillations; it has been reduced to a nonlinear Schrödinger equation with a varying dispersion coefficient. It is shown that, if the periodicity of longitudinal modulation coincides with the transverse gradient of the refractive index, a stationary oscillatory picture emerges in the nonlinear regime.

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I. INTRODUCTION

Bloch oscillations [1] of electrons in a perfect lattice in a static electric field (see, e.g. [2]) have stimulated a number of investigations in completely different physical systems: Optical waves in waveguide arrays [3–7], ultracold atoms in optical lattices [8–11], acoustic waves in supersonic superlattices [12], and elastic mechanical systems [13] are just recent examples. The influence of additional nonlinear terms in the wave equations that are generated, for example, by noninear response of the medium (light) or by a mean-field treatment of quantum many-body interactions in Bose-Einstein condensates (BECs), has been analyzed. These terms generally lead to decoherence and deterioration of the Bloch oscillations, which was observed on time scales of a few oscillation periods in ordinary [14–18] and disordered lattices [19–21]. In the optical context this means that Bloch oscillations for stationary beams can be observed only in the case of low intensities, while for strong intensities a transition to chaotic light distribution is observed.

Very recently, some new scenarios have been suggested for observation of persistent Bloch oscillations in the nonlinear limit. In Ref. [22] it was suggested to make the nonlinear coefficient spatially periodic, while in Refs. [23,24] the authors considered a time-dependent nonlinear term in the BEC context. In the present paper I consider optical spatial soliton propagation in a film of Kerr medium for which the nonlinear coefficient is constant, while the linear refractive index is modulated along both spatial directions. In the case of refractive-index modulation along only one spatial (transverse) direction n, noninear Bloch oscillations do not survive, as has been shown in previous studies. I show here that, if one has a certain periodicity of refractive-index modulations (coinciding with the Bloch oscillation period) along the longitudinal direction as well, it is possible to see persistent Bloch oscillations even in the nonlinear regime, as displayed in the main plot of Fig. 1.

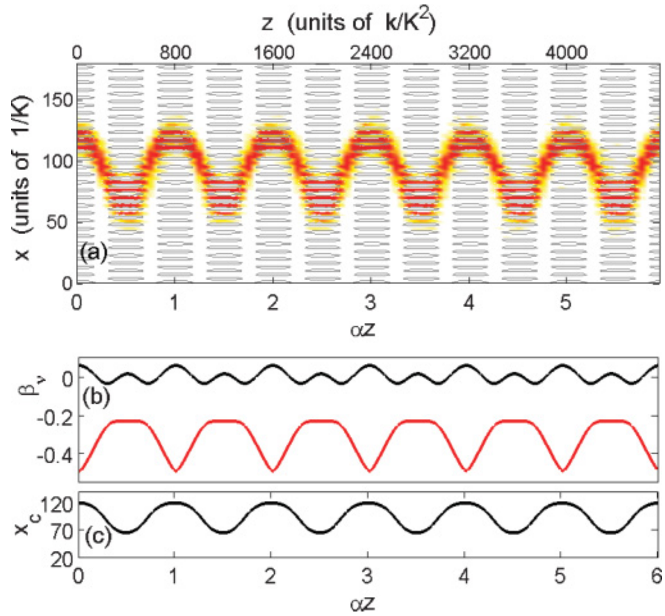

FIG. 1. (Color online) (a) Results of numerical simulations (with only the first Bloch mode excited) of Eq. (1) showing persistent Bloch oscillations in the case of refractive-index modulation along both spatial dimensions; they are displayed as a gray contour plot (light areas correspond to the larger-refractive-index regions). A refractive-index gradient $\alpha = 1/800$ is applied along the x axis; the nonlinearity parameter $\chi = 0.016$; the wave envelope is initially normalized to unity $|\Psi(z=0)|_{\max} = 1$; and modulation along z has the form $w(z) = 0.25\cos(2\pi z/\alpha)$. (b) The two first Bloch bands in this two-dimensional lattice. (c) The calculated trajectory from the simplified model according to the formula (12), assuming the initial excitation of the first Bloch band.

II. NONLINEAR EVOLUTION OF INTERACTING BLOCH MODES

Let us start by writing the nonlinear Schrödinger equation (NLS) describing the stationary distribution of a linearly polarized light envelope in an optical film in the paraxial approximation:

$$i\frac{\partial\Psi}{\partial z} + \frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} + [\alpha x - w(z)\cos(x)]\Psi + \chi|\Psi|^2\Psi = 0, \quad (1)$$

where I assume light propagation along the z direction, the complex wave envelope is normalized as $|\Psi(x,z=0)|_{\max} = 1, \chi$ is a nonlinearity coefficient, the potential term (including harmonic and gradient parts) represents the linear refractive-index variation

$$\alpha x - w(z)\cos(x) = k^2(n-n_0)/n_0K^2, \quad (2)$$

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the spatial variables x and z are scaled in units of $1/K$ and k/K^2, respectively, K is the inverse spacing of the harmonic transverse modulations, and k is the carrier wave number, defined as $k = n_0\omega/c$ with ω being the laser beam frequency and n_0 the averaged refractive index; pinned boundary conditions $\Psi(0,z) = \Psi(L,z) = 0$ are considered.

The solution of (1) is sought via the following expansion over extended Bloch waves:

$$\Psi(x,z) = \sum_{n=-\infty}^{\infty} e^{i(\alpha z+n)x+\lambda(z)} A_n(x,z), \tag{3}$$

and by putting this into Eq. (1) one finds the following set of equations:

$$i\frac{\partial A_n}{\partial z} - \left(\beta + \frac{(\kappa+n)^2}{2}\right) A_n - \frac{w}{2}(A_{n-1} + A_{n+1}) + i(\kappa+n) \times \frac{\partial A_n}{\partial x} + \frac{1}{2}\frac{\partial^2 A_n}{\partial x^2} + \chi \sum_{n_1,n_2} A_{n_1} A^*_{n_2} A_{n-n_1+n_2} = 0, \tag{4}$$

where the z-dependent propagation constant $\beta(z) = d\lambda/dz$ and the wave vector $\kappa(z) = \alpha z$ have been introduced. Now we turn to the weakly nonlinear approach, considering small amplitudes and slowly varying variables; in other words, let us consider the following representation:

$$A_n(x,z) = \sum_{\ell=1}^{\infty} \varepsilon^{\ell} \varphi_n^{(\ell)}(\xi,\tau), \quad \xi = \varepsilon\,[x - \theta(z)]\,, \quad \tau = \varepsilon^2 z. \tag{5}$$

This has to be substituted into (4), building perturbation theory for various orders of ε. In particular, in the linear limit one gets the following algebraic set of equations written in the matrix representation:

$$\sum_{n'} Q_{nn'}\varphi_{n'}^{(1)} = -\beta\varphi_n^{(1)}, \quad Q_{nn} = \frac{(\kappa+n)^2}{2} \quad Q_{nn\pm1} = \frac{w}{2}.$$

This is an eigenvalue problem for given z, and one can define the spectrum of the propagation constant $\beta_\nu\,[\kappa(z), w(z)]$ and the corresponding orthonormalized real eigenvectors $R_n^\nu(z)$. By this we define a complete set of Bloch eigenmodes as a function of z and write the function $\varphi_n^{(1)}$ as an expansion over these modes:

$$\varphi_n^{(1)} = \sum_\nu R_n^\nu(z)\mathcal{E}_\nu(\xi,\tau), \qquad \sum_{n'} Q_{nn'} R_{n'}^\nu = -\beta_\nu R_n^\nu, \tag{6}$$

where $\mathcal{E}_\nu(\xi,\tau)$ are arbitrary functions to be defined in the further approximations. Now let us make the assumption that for given z one has only a single mode ν; in other words, all amplitudes $\mathcal{E}_\mu$ with $\mu \neq \nu$ are zero. Then, in a second approximation over ε, choosing $\beta \equiv \beta_\nu$ from (4), we get the following set of equations:

$$\sum_{n'} (\beta_\nu \delta_{nn'} + Q_{nn'})\,\varphi_{n'}^{(2)} = i\,[(\kappa+n) - v]\,\frac{\partial \mathcal{E}_\nu}{\partial \xi} R_n^\nu, \tag{7}$$

where $v(z) = d\theta/dz$. Multiplying Eq. (7) by R_n^ν and summing over n, we get an expression for the velocity $v_\nu(z)$ associated with the νth mode:

$$v_\nu(z) = \sum_n (\kappa+n)\left(R_n^\nu\right)^2 \equiv \frac{\partial\beta_\nu(\kappa,w)}{\partial\kappa}. \tag{8}$$

Proceeding in a similar way to the third approximation over ε, we finally get from (4)

$$i\frac{\partial\mathcal{E}_\nu}{\partial\tau} - \frac{1}{2}\frac{\partial^2\beta_\nu(\kappa,w)}{\partial\kappa^2}\frac{\partial^2\mathcal{E}_\nu}{\partial\xi^2} + \Delta_\nu|\mathcal{E}_\nu|^2\mathcal{E}_\nu = 0 \tag{9}$$

where the modified nonlinearity parameter is defined as follows:

$$\Delta_\nu = \chi \sum_{n,n_1,n_2} R_n^\nu R_{n_1}^\nu R_{n_2}^\nu R_{n-n_1+n_2}^\nu. \tag{10}$$

Now, rescaling the slow variable $\tau \to \int d\tau\,\Delta_\nu(\tau)$, we finally get the NLS equation with varying dispersion coefficient:

$$i\frac{\partial\mathcal{E}_\nu}{\partial\tau} + \frac{\Omega_\nu(\tau)}{2}\frac{\partial^2\mathcal{E}_\nu}{\partial\xi^2} + |\mathcal{E}_\nu|^2\mathcal{E}_\nu = 0, \qquad \Omega_\nu = -\frac{\partial^2\beta_\nu(\kappa,w)}{\partial\kappa^2}\Big/\Delta_\nu, \tag{11}$$

where it is assumed that as long as the refractive-index gradient is small, $\alpha \sim \varepsilon^2$, the dispersion coefficient varies slowly along z and is a function of the variable $\tau = \varepsilon^2 z$. As we will see below, Eq. (11) can support a solitonic solution, and then according to definitions (5) the center of the localized stationary distribution should follow the trajectory

$$x = \theta(z) = \int v_\nu\,dz = \int dz\frac{\partial\beta_\nu(\kappa,w)}{\partial\kappa}. \tag{12}$$

III. ANALYSIS OF REDUCED NLS MODEL WITH VARYING DISPERSION COEFFICIENT

It is clear that, if there is no longitudinal modulation of the refractive index [$w(z)$ is a constant] and the localized solution is stable, then the trajectory $x = \beta_\nu(z)/\alpha$, and it simply follows the corresponding Bloch band curve. Such a scenario occurs in the case of only transverse modulations of the refractive index and for low beam intensities (linear regime). This is displayed in Figs. 2(a) and 2(c) and it can

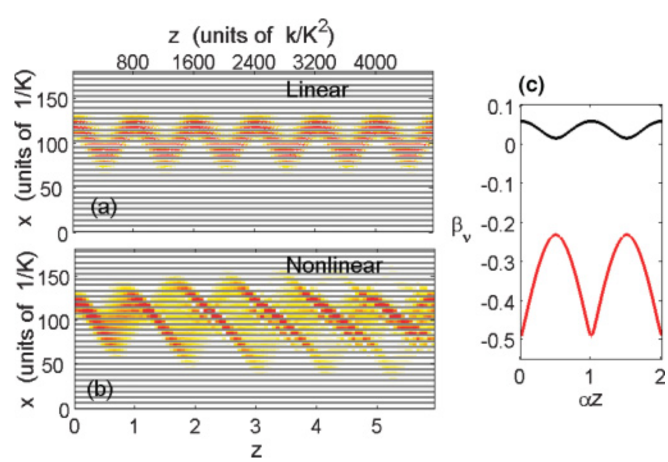

FIG. 2. (Color online) (a) and (b) Results of numerical simulations (stationary spatial light intensity distribution with first Bloch mode excited) on the initial model (1) with only transverse modulation and a gradient of the refractive index in the linear and nonlinear regimes, respectively. (c) Dependence of the propagation constant on z for the first two Bloch modes.

be seen that the beam follows the trajectory of the first Bloch band. When the beam intensity is increased, the decoherence of Bloch oscillations occurs, as shown in Fig. 2(b). This happens because of variation of the effective dispersion coefficient Ω_1, which changes sign periodically, thus switching from the focusing to the defocusing regime of the NLS equation (11). This behavior is displayed in the inset of Fig. 3(a), where in the main plot I show the evolution of the initially localized wave packet according to the simplified model (11). In full accordance with the numerical simulations on the initial model (1) with only transverse modulations, the reduced NLS model also shows the delocalization of the initial wave packet, and this means deterioration of Bloch oscillations in the nonlinear case.

As mentioned above, the coherent oscillatory regime can reemerge if one introduces longitudinal modulation of the nonlinear parameter χ, as was done in Ref. [25], or one can consider time-dependent nonlinearity in the BEC context [24]. Then if the periodicity of this modulation coincides with the Bloch oscillation period $1/\alpha$, the effective dispersion coefficient Ω_1 does not depend on z and (11) is always in the focusing regime, maintaining the coherent Bloch oscillation regime.

A similar consideration applies in the case of refractive-index modulation along both spatial dimensions but with a constant nonlinear coefficient. In this latter case, I take the refractive-index modulation in the form $\cos(2\pi z/\alpha)\cos(x)$, thus taking the longitudinal periodicity equal to the Bloch oscillation period $1/\alpha$. Then the dependence of the effective dispersion coefficient Ω_1 on z is displayed in the inset of Fig. 3(b). It shows that in some regions Ω_1 is still negative (defocusing regime), but this interval is considerably smaller than in the case of only transverse modulations [compare with the inset of Fig. 3(a)]. Thus the system "spends" less in defocusing regime and the localized solution does survive [see the main plot in Fig. 3(b)]. This fully explains our findings using the initial model (1) with a two-dimensional lattice, which are displayed in Fig. 1 and show persistent nonlinear Bloch oscillations. Please note that the trajectory profile derived from (12) [see Fig. 1(c)] is fully consistent with the results of numerical simulations [Fig. 1(a)], although it does not follow the first Bloch band curve [Fig. 1(b)] as long as $w(z)$ depends on z.

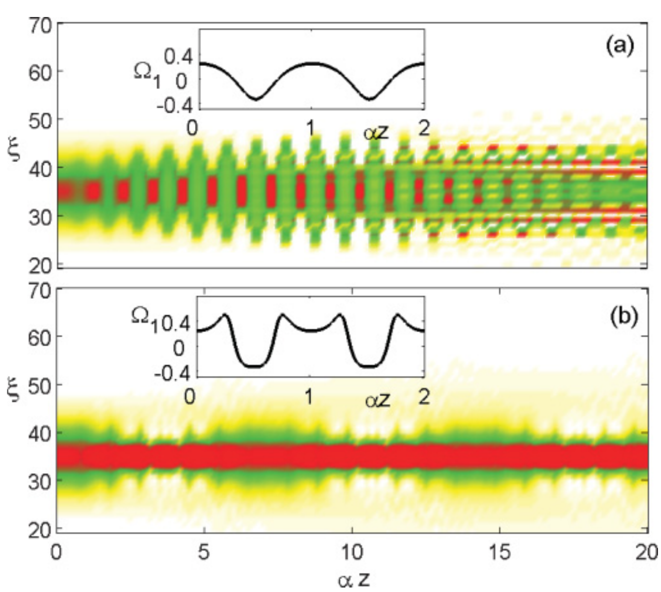

FIG. 3. (Color online) (a) and (b) Results of numerical simulations of the simplified model NLS (11) with varying dispersion coefficients Ω_1 (see the insets). (a) corresponds to the case of only transverse modulation of the refractive index in the initial Eq. (1), while (b) represents the case of a two-dimensional optical lattice when the longitudinal modulation and Bloch periodicities coincide with each other.

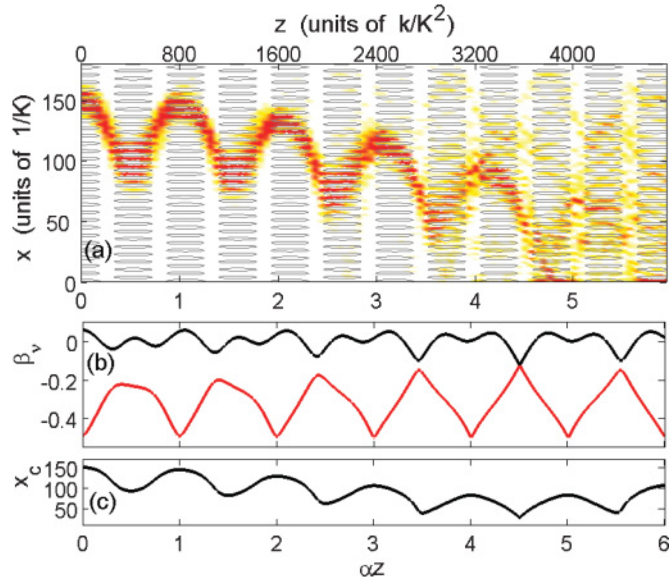

FIG. 4. (Color online) (a) Typical result of numerical simulations of the initial model (1) in the nonlinear regime when the periodicity of the longitudinal modulation and the Bloch period do not coincide. (b) z dependence of the propagation constant of the two first Bloch modes. (c) Calculated trajectory (12) from the simplified NLS model (11).

IV. DISCUSSIONS AND CONCLUSIONS

Next let us consider what happens if the longitudinal modulation periodicity does not coincide with the Bloch period. This scenario is displayed in Fig. 4. As in previous simulations, the Bloch period is fixed as $T = 1/\alpha = 800$, while I now take the longitudinal modulation period $T_1 = 847$. The evolution of the propagation constant of the first and second Bloch bands is displayed in Fig. 4(b) and, as is seen, after several oscillation periods these two bands go close to each other and then separate again. This happens after m Bloch periods. This is calculated by taking the integer part of the following relation:

$$(2T - T_1)/[4(T_1 - T)]. \qquad (13)$$

As a result, first a drift of the oscillatory regime takes place, followed by Landau-Zener tunneling in the range of z where the two band curves are close to each other. Naturally this leads to the destruction of coherent Bloch oscillations, and that is what we see in Fig. 4(a). In Fig. 4(c) the trajectory calculated from the formula (12) is presented;

it is in excellent agreement with numerical simulations. Moreover, the destruction of Bloch oscillations takes place around $m = 4$ Bloch periods, which coincides with the results from formula (13), substituting there the values $T = 800$ and $T_1 = 847$.

In conclusion, the stationary solitonic oscillatory regime in two-dimensional optical lattices under a field gradient has been investigated. The results are directly applicable for BECs in a harmonic optical potential and under an applied field. There, time modulations of the optical lattice depth will be necessary to see nonlinear Bloch oscillations. Another interesting issue would be to consider interaction of the oscillatory solitons from different bands, as was done in considering different types of solitons in Ref. [26]. Longitudinal modulation of the refractive index in an experimental setup with waveguides can be achieved by photolithography, as, for example, in Ref. [15], while in the BEC context the time modulation of the power of counterpropagating optical beams creating optical lattice could be considered.

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