Fermionic Casimir energy in a three-dimensional box

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In this paper we calculate the Casimir energy for a massless fermionic field confined inside a three-dimensional rectangular box. We use the MIT bag model boundary condition for the confinement. We use the direct mode summation method along with the Abel-Plana summation formula to compute the Casimir energy, without any use of regularization or analytic continuation techniques. We obtain a negative Casimir energy, as opposed to the previously reported result for the interior of a three-dimensional sphere.

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I. INTRODUCTION

Casimir effects, first discovered in 1948 [1], are manifestations of the zero-point energies of the quantum fields and have played an important role in a variety of fields of physics. Casimir energy is defined as the difference between the vacuum energies in the presence and the absence of any external boundary conditions or background fields. Both of these energies are in general infinite. However, the difference between the two has almost invariably been calculated to be finite. The external boundary conditions could be produced for example by the presence of material plates. To calculate the Casimir energy one usually needs to utilize various regularization or analytic continuation schemes. Casimir effects have been calculated for a variety of fields, geometries, number of spatial dimensions, and boundary conditions [2–4].

The first experimental attempt to verify the existence of the Casimir effect for two parallel metallic plates was made by Sparnaay [5] ten years after Casimir's theoretical prediction. Four decades passed until new experiments were made with high-conductivity metallic plates. In 1997, using a torsion pendulum, Lamoreaux [6] initiated a new era of experiments concerning Casimir effects. He measured the Casimir force between a plate and a spherical lens [7]. One year later, using an atomic force microscope, Mohideen and Roy [8] measured the Casimir force between a plate and a sphere with a better accuracy and established an agreement between experimental data and theoretical predictions. These two precision experiments have been followed by many others. Most of the theoretical investigations in relation with Casimir effects are for various fields, geometries, boundary conditions, and different dimensions. The Casimir energy for scalar fields confined by different boundaries has been calculated in many studies. We have sorted some of them according to their geometries: between two large perfectly conducting parallel plates [9], inside cylindrical boundaries [10,11], in a spherical geometry [12–14], for a rectangular cavity [15,16], and in higher-dimensional spaces between two parallel plates [17,18]. Some of the studies containing the calculations of the Casimir energy for the electromagnetic field in different geometries are between two parallel plates [19], between two eccentric cylinders [20], in a rectangular waveguide [21], for an infinite cylinder [22], on a dielectric cylinder [23], in rectangular cavities [24,25], for a solid ball [22,26], for a sphere [3,4,27–34], and for a D-dimensional sphere [35]. Nowadays it is well known that the zero-point energies of configurations depend on the nature of the particular quantum field (i.e., scalar, spinor, etc.), the type of space-time geometry and its dimensionality, and the specific boundary condition imposed on the quantum field on the surfaces. These factors cause a rather complex pattern and could even lead to a switching between attractive and repulsive forces [36,37].

Now we concentrate on the Casimir energy for the Dirac fields to lowest order in perturbation theory. In 1975 Johnson evaluated the Casimir energy per unit area for a massless fermionic field, subject to the MIT bag model boundary condition, between two parallel plates [38] and Mamayev and Trunov were the first ones who computed the Casimir energy for a massive fermionic field [39]. In this regard more studies have been done, of which we mention just some of them: calculating the Casimir energy for fermions in one dimension [40,41], between two parallel plates using various methods in three-dimensional space [3,4,38,41–45] and in $(d+1)$-dimensional space-time [46], and for a spherical geometry for a massive fermionic field [47] and a massless fermionic field [3,4]. The perturbation of the vacuum energy of a massless fermionic field in the presence of a three-dimensional box with antiperiodic boundary condition has been evaluated in [43].

It is worth mentioning that in the phenomenological studies of hadrons, bag models have played an important role. The simplest such model is the MIT bag model. In such models "the bag constant" is an input parameter to the theory. As is well known, "the bag constant," B, is added to the Lagrangian density in order to balance the outward pressure of the quarks by the inward vacuum pressure B on the surface of the bag. There have been several studies relating this constant to the Casimir energy (e.g., in [4,48]). Hence the study of the Casimir energy for fermions in closed geometries could have direct phenomenological implications. However, the bag constant—in hadron physics—is even more so influenced by the presence of the gauge fields (the gluons); see, for example, [49]. The bag model is only compatible with the large-N_c expansion of QCD under this premise. In this paper we present the calculation of Casimir energy for a massless fermionic field in a three-dimensional cube with the MIT bag model boundary condition

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imposed on each side, which guaranties complete confinement of the fermionic field inside the box. In Sec. II we present the solution to the Dirac equation subject to the MIT bag model boundary condition on all of the surfaces. Then we compute the Casimir energy by performing a direct sum over all modes of the field using the Abel-Plana summation formula. As we shall show, there will be no need for any analytic continuation techniques in this case.

II. THE DIRAC FIELD CONFINED IN A THREE-DIMENSIONAL CUBE

In this section we construct the eigenfunctions of the Hamiltonian for this system, satisfying the appropriate confinement boundary conditions, that is, the MIT bag model boundary condition. The wave vectors and the resulting energy spectrum will be discrete. We can then calculate the Casimir energy by summing over the eigenenergies.

The most general eigenfunction can be written as follows:

$$\Psi(x_1,x_2,x_3,t) = \psi(x_1,x_2,x_3)\exp(-iEt), \tag{1}$$

where

$$\begin{aligned}
&\psi(x_1,x_2,x_3)\\
&= f\begin{pmatrix}\alpha\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\alpha\end{pmatrix}e^{i(p_1x_1+p_2x_2+p_3x_3)} + g\begin{pmatrix}\beta\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\beta\end{pmatrix}\\
&\quad\times e^{i(-p_1x_1+p_2x_2+p_3x_3)} + h\begin{pmatrix}\eta\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\eta\end{pmatrix}e^{i(p_1x_1-p_2x_2+p_3x_3)}\\
&\quad+ j\begin{pmatrix}\mu\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\mu\end{pmatrix}e^{i(p_1x_1+p_2x_2-p_3x_3)} + k\begin{pmatrix}\nu\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\nu\end{pmatrix}\\
&\quad\times e^{-i(p_1x_1+p_2x_2+p_3x_3)} + l\begin{pmatrix}\tau\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\tau\end{pmatrix}e^{-i(-p_1x_1+p_2x_2+p_3x_3)}\\
&\quad+ q\begin{pmatrix}\chi\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi\end{pmatrix}e^{-i(p_1x_1-p_2x_2+p_3x_3)} + r\begin{pmatrix}\rho\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\rho\end{pmatrix}\\
&\quad\times e^{-i(p_1x_1+p_2x_2-p_3x_3)}.
\end{aligned} \tag{2}$$

Here $\vec{p}$ denotes momentum operator and α, β, η, μ, ν, τ, χ, and ρ are general two-component spinors. Coefficients f through r can be determined using the boundary conditions. Here we use the usual Dirac-Pauli representation:

$$\gamma^0 = \beta = \begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix}, \quad \gamma^k = \beta\alpha_k = \begin{pmatrix} 0 & \sigma_k\\ -\sigma_k & 0\end{pmatrix},\quad k = 1,2,3.$$

The MIT bag model boundary condition is usually said to imply that there is no flux of fermions through the boundary; that is, if j denotes the current of the Dirac field and n is the normal unit vector to the boundary, then $n_\mu j^\mu = 0$. However, it implies an even stronger condition, which is the absolute confinement of the fermionic field. This model was first considered by Bogolioubov [50] and later developed as the MIT bag model by Chodos *et al.* [51] for hadrons. The prevalent form of the MIT bag model boundary condition is as follows:

$$[1 + i(\hat{n}\cdot\vec{\gamma})]\psi(x)\Big|_{\text{Boundary}} = 0. \tag{3}$$

This boundary condition for our special case becomes

$$(1 \mp i\gamma^k)\psi(x_1,x_2,x_3)\Big|_{x_k=\mp a_k/2} = 0, \quad k = 1,2,3, \tag{4}$$

where a_1, a_2, and a_3 denote the lengths of the sides of the box. Substituting Eq. (2) into Eq. (4) we obtain, for example, the following two equations for $x_1 = \pm a_1/2$ surfaces:

$$\begin{aligned}
&\underbrace{((E+m)+i(p_1+\sigma_1\sigma_2p_2+\sigma_1\sigma_3p_3))}_{A}f\alpha e^{ip_1\frac{a_1}{2}}\\
&\quad+\underbrace{((E+m)+i(-p_1+\sigma_1\sigma_2p_2+\sigma_1\sigma_3p_3))}_{B}g\beta e^{-ip_1\frac{a_1}{2}} = 0,
\end{aligned} \tag{5}$$

$$\begin{aligned}
&\underbrace{((E+m)-i(p_1+\sigma_1\sigma_2p_2+\sigma_1\sigma_3p_3))}_{C}f\alpha e^{-ip_1\frac{a_1}{2}}\\
&\quad+\underbrace{((E+m)-i(-p_1+\sigma_1\sigma_2p_2+\sigma_1\sigma_3p_3))}_{D}g\beta e^{ip_1\frac{a_1}{2}} = 0.
\end{aligned} \tag{6}$$

Multiplying Eq. (5) from left by A^{-1} and from right by $(g\beta)^{-1}$, and likewise multiplying Eq. (6) from left by C^{-1} and from right by $(g\beta)^{-1}$, yields

$$\left.\begin{aligned}
(f\alpha)(g\beta)^{-1} &= -e^{-ip_1a_1}A^{-1}B,\\
(f\alpha)(g\beta)^{-1} &= -e^{ip_1a_1}C^{-1}D,
\end{aligned}\right\}$$
$$\Rightarrow e^{-ip_1a_1}A^{-1}B = e^{ip_1a_1}C^{-1}D. \tag{7}$$

Expanding Eq. (7), one can easily show that the equation holds if and only if

$$\begin{aligned}
&\text{Im}\{(\cos p_1a - i\sin p_1a)(m - ip_1)\} = 0\\
&\Longrightarrow p_1\cot p_1a = -m.
\end{aligned} \tag{8}$$

Analogous conditions can be obtained for the components of momenta in the other two directions. By setting $m = 0$ for a massless Dirac field, the quantization condition Eq. (8) yields

$$p_k = \left(n_k + \frac{1}{2}\right)\frac{\pi}{a_k}, \quad k = 1,2,3. \tag{9}$$

From this point on we concentrate on the massless case with $a_1 = a_2 = a_3 = a$, for simplicity. By using the second quantized form of the Dirac field, the vacuum expectation value of the free Hamiltonian [44,52] can be expressed in the form

$$E_{\text{FV}} = -a^3\sum_s\int_{-\infty}^{+\infty}\frac{d^3p}{(2\pi)^3}E_p, \tag{10}$$

where the summation index s runs over the spin states and the subscript FV stands for free vacuum. Obviously, in the massless case $E_p = \sqrt{p_1^2 + p_2^2 + p_3^2}$, where the components of the momenta can take on any real value.

In the presence of the boundaries, all of the components of the momentum are subject to quantization condition Eq. (9). Therefore the integrals turn into summations:

$$E_{\rm BV} = -\sum_{s}\sum_{n_1,n_2,n_3=0}^{+\infty}\left(\frac{\pi}{a}\right)\times\sqrt{\left(n_1+\frac{1}{2}\right)^2+\left(n_2+\frac{1}{2}\right)^2+\left(n_3+\frac{1}{2}\right)^2}, \qquad (11)$$

where $E_{\rm BV}$ denotes the vacuum energy in the presence of the boundaries. Obviously, in both situations the vacuum energy is divergent. However, the Casimir energy, which is the difference between these two quantities, is usually expected to be finite. One usually needs to utilize a regularization prescription to give a physical meaning to such a difference. In this paper we choose a modified form of the Abel-Plana formula, which is useful for the summation over half-integer numbers (see, e.g., [53,54]),

$$\sum_{n=0}^{\infty} F\left(n+\frac{1}{2}\right) = \int_0^\infty dt\, F(t) - i\int_0^\infty \frac{dt}{e^{2\pi t}+1}[F(it)-F(-it)], \qquad (12)$$

where $F(z)$ is assumed to be an analytic function in the right half-plane. The first term is the main term for turning a sum into an integral. The second term is called the branch-cut term. Since we have a triple sum over the wave numbers n_i for $E_{\rm BV}$ as given in Eq. (11), we need to apply the Abel-Plana formula [Eq. (12)] three times. The details are given in the Appendix. The final result is

$$E_{\rm BV} = -2\left(\frac{\pi}{a}\right)\Bigg\{\int_0^\infty\int_0^\infty\int_0^\infty du\,dk\,dt\,\sqrt{u^2+k^2+t^2} + 2\sum_{n_2,n_3=0}^{\infty}\int_{\sqrt{(n_2+\frac{1}{2})^2+(n_3+\frac{1}{2})^2}}^\infty dt\,\frac{\sqrt{t^2-\left(n_2+\frac{1}{2}\right)^2-\left(n_3+\frac{1}{2}\right)^2}}{e^{2\pi t}+1}$$
$$+2\sum_{n_3=0}^{\infty}\int_0^\infty dt\int_{\sqrt{t^2+(n_3+\frac{1}{2})^2}}^\infty dk\,\frac{\sqrt{k^2-t^2-\left(n_3+\frac{1}{2}\right)^2}}{e^{2\pi k}+1} + 2\int_0^\infty dk\int_0^\infty dt\int_{\sqrt{t^2+k^2}}^\infty du\,\frac{\sqrt{u^2-k^2-t^2}}{e^{2\pi u}+1}\Bigg\}. \qquad (13)$$

It is extremely important to note that the only divergent quantity in Eq. (13) is the first term, which is precisely the free vacuum energy $E_{\rm FV}$ and is supposed to be subtracted from $E_{\rm BV}$ in order to obtain the Casimir energy. In order to compute the three branch-cut terms, we evaluated all the integrals analytically. However, in the final step of the calculations we have evaluated the sums numerically. It is worth mentioning that the numerical method used for the final sums yields extremely accurate values (see the Appendix) The final result is

$$E^{\rm cube}_{\rm Casimir} := E_{\rm BV} - E_{\rm FV} = -\frac{0.048\,927\,541}{a}. \qquad (14)$$

As a check on our procedure we have computed the Casimir energy for a fermionic field between two parallel plates in three-dimensional space (per unit area of the plates), separated by a distance a, and obtained a result that agrees extremely well with the analytic result obtained in [38,41,43–45]:

$$\mathcal{E} = -\frac{7\pi^2}{2880a^3} \simeq -\frac{0.023\,988\,621\,8}{a^3}. \qquad (15)$$

III. CONCLUSION

In this paper we have computed the Casimir energy for a massless Dirac field with the MIT bag model boundary condition in a three-dimensional cube. We have used the direct mode summation method in order to compute the Casimir energy for this field. In this regard, we have made repeated use of the Abel-Plana summation formula. It is important to note that the divergent part is automatically canceled once we subtract the free vacuum energy. Therefore, we do not need to resort to any regularization or analytic continuation techniques. We have also used the same method to compute the Casimir energy for the two parallel plates geometry in order to check our procedure, and our result matches the analytic result obtained earlier. We like to mention that the Casimir energy for a massless fermionic field confined inside a three-dimensional sphere with the MIT bag model boundary condition has been computed analytically and the results for a sphere with the same volume as our cube, along with our result for the cube, are

$$E^{\rm cube}_{\rm Casimir} = -\frac{0.048\,927\,541}{a}, \quad E^{\rm sphere}_{\rm Casimir} = +\frac{0.032\,900\,755}{a}. \qquad (16)$$

Note that the Casimir energy for the sphere is of the same order of magnitude as the one for the cubical geometry. However, they have opposite signs [3,4,54]. For a discussion of the sign of the Casimir energy as a function of the type of fields under consideration, the dimensionality of space-time, the boundary conditions imposed, and the geometry see [36].

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APPENDIX: USE OF THE ABEL-PLANA FORMULA IN CALCULATING THE CASIMIR ENERGY FOR A THREE-DIMENSIONAL CUBE AND CALCULATION OF THE BRANCH-CUT TERMS

In this Appendix we present the details of the calculations leading to our main expression for the Casimir energy of a massless fermionic field confined inside a cube via the MIT bag model boundary condition [Eq. (13), Eq. (14)]. In order to apply the Abel-Plana formula [Eq. (12)] to the triple sum in Eq. (11), we first define

$$F\left(n_1+\frac{1}{2}\right) = -\frac{2\pi}{a}\sum_{n_2,n_3=0}^{\infty}\sqrt{\left(n_1+\frac{1}{2}\right)^2+\left(n_2+\frac{1}{2}\right)^2+\left(n_3+\frac{1}{2}\right)^2}, \tag{A1}$$

The factor 2 is associated with the spin multiplicity. The branch-cut term can be calculated using the following:

$$F(\pm it) = -\frac{2\pi}{a}\sum_{n_2,n_3=0}^{\infty}\sqrt{(\pm it)^2+\beta^2} = -\frac{2\pi}{a}\sum_{n_2,n_3=0}^{\infty}\begin{cases}(\pm i)\sqrt{t^2-\beta^2} & \text{for} \quad |t|>\beta,\\ \sqrt{-t^2+\beta^2} & \text{for} \quad |t|<\beta,\end{cases} \tag{A2}$$

where here $\beta = \sqrt{(n_2+\frac{1}{2})^2+(n_3+\frac{1}{2})^2}$. By using Eq. (12) and Eq. (A2), Eq. (11) turns into

$$E_{\rm BV} = -2\left(\frac{\pi}{a}\right)\left\{\sum_{n_2,n_3=0}^{\infty}\int_0^{\infty}dt\sqrt{t^2+\left(n_2+\frac{1}{2}\right)^2+\left(n_3+\frac{1}{2}\right)^2}+2\sum_{n_2,n_3=0}^{\infty}\int_{\sqrt{(n_2+\frac{1}{2})^2+(n_3+\frac{1}{2})^2}}^{\infty}dt\,\frac{\sqrt{t^2-(n_2+\frac{1}{2})^2-(n_3+\frac{1}{2})^2}}{e^{2\pi t}+1}\right\}. \tag{A3}$$

The first term is infinite and we have to use the Abel-Plana formula one more time only for the first term. We obtain

$$\begin{aligned}E_{\rm BV} = &-2\left(\frac{\pi}{a}\right)\left\{\sum_{n_3=0}^{\infty}\int_0^{\infty}\int_0^{\infty}dt\,dk\sqrt{t^2+k^2+\left(n_3+\frac{1}{2}\right)^2}+2\sum_{n_3=0}^{\infty}\int_0^{\infty}dt\int_{\sqrt{t^2+(n_3+\frac{1}{2})^2}}^{\infty}dk\,\frac{\sqrt{k^2-t^2-\left(n_3+\frac{1}{2}\right)^2}}{e^{2\pi k}+1}\right.\\ &\left.+2\sum_{n_2,n_3=0}^{\infty}\int_{\sqrt{(n_2+\frac{1}{2})^2+\left(n_3+\frac{1}{2}\right)^2}}^{\infty}dt\,\frac{\sqrt{t^2-\left(n_2+\frac{1}{2}\right)^2-\left(n_3+\frac{1}{2}\right)^2}}{e^{2\pi t}+1}\right\}.\end{aligned} \tag{A4}$$

Again the first term is infinite and we must apply the Abel-Plana formula one more time to obtain

$$\begin{aligned}E_{\rm BV} = &-2\left(\frac{\pi}{a}\right)\left\{\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}dt\,dk\,du\sqrt{t^2+k^2+u^2}+2\int_0^{\infty}dk\int_0^{\infty}dt\int_{\sqrt{t^2+k^2}}^{\infty}du\,\frac{\sqrt{u^2-k^2-t^2}}{e^{2\pi u}+1}\right.\\ &\left.+2\sum_{n_3=0}^{\infty}\int_0^{\infty}dt\int_{\sqrt{t^2+(n_3+\frac{1}{2})^2}}^{\infty}dk\,\frac{\sqrt{k^2-t^2-\left(n_3+\frac{1}{2}\right)^2}}{e^{2\pi k}+1}+2\sum_{n_2,n_3=0}^{\infty}\int_{\sqrt{(n_2+\frac{1}{2})^2+\left(n_3+\frac{1}{2}\right)^2}}^{\infty}dt\,\frac{\sqrt{t^2-\left(n_2+\frac{1}{2}\right)^2-\left(n_3+\frac{1}{2}\right)^2}}{e^{2\pi t}+1}\right\}.\end{aligned} \tag{A5}$$

Note that all of the branch-cut terms are finite and only the first term is infinite. On the other hand the free vacuum energy is

$$E_{\rm FV} = -2a^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dp_1dp_2dp_3}{(2\pi)^3}\sqrt{p_1^2+p_2^2+p_3^2}. \tag{A6}$$

Making appropriate changes of variables one obtains

$$E_{\rm FV} = \frac{-2\pi}{a}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}dt\,dk\,du\sqrt{t^2+k^2+u^2}, \tag{A7}$$

It is extremely important to note that this term is infinite and precisely equals the infinite term which appears in $E_{\rm BV}$. Therefore when we compute the Casimir energy these two terms precisely cancel each other. That is,

$$\begin{aligned}E_{\rm Casimir} := E_{\rm BV}-E_{\rm FV} = -4\left(\frac{\pi}{a}\right)&\left\{\int_0^{\infty}dk\int_0^{\infty}dt\int_{\sqrt{t^2+k^2}}^{\infty}du\,\frac{\sqrt{u^2-k^2-t^2}}{e^{2\pi u}+1}+\sum_{n_3=0}^{\infty}\int_0^{\infty}dt\int_{\sqrt{t^2+(n_3+\frac{1}{2})^2}}^{\infty}dk\right.\\ &\left.\times\frac{\sqrt{k^2-t^2-\left(n_3+\frac{1}{2}\right)^2}}{e^{2\pi k}+1}+\sum_{n_2,n_3=0}^{\infty}\int_{\sqrt{(n_2+\frac{1}{2})^2+(n_3+\frac{1}{2})^2}}^{\infty}dt\,\frac{\sqrt{t^2-\left(n_2+\frac{1}{2}\right)^2-\left(n_3+\frac{1}{2}\right)^2}}{e^{2\pi t}+1}\right\}.\end{aligned} \tag{A8}$$

Here we explain the details of the calculation of the last term and then outline the calculations for the remaining terms. We expand the denominator as follows:

$$\frac{1}{e^{2\pi t}+1}=\sum_{j=0}^{\infty}(-1)^j e^{-2\pi t(j+1)}. \tag{A9}$$

The last term turns into

$$-\frac{4\pi}{a}\sum_{n_2,n_3,j=0}^{\infty}(-1)^j\int_{\sqrt{(n_2+\frac{1}{2})^2+(n_3+\frac{1}{2})^2}}^{\infty}dt \times e^{-2\pi t(j+1)}\sqrt{t^2-\left(n_2+\frac{1}{2}\right)^2-\left(n_3+\frac{1}{2}\right)^2}. \tag{A10}$$

By using the identity

$$\int_a^{\infty}du\,(u^2-a^2)^{\nu-1}e^{-\mu u} = \frac{1}{\sqrt{\pi}}\left(\frac{2a}{\mu}\right)^{\nu-\frac{1}{2}}\Gamma(\nu)K_{\text{Bessel}}\left(\nu-\frac{1}{2},a\mu\right), \tag{A11}$$

which holds when

$$a>0,\quad \text{Re}(\nu)>0\quad \text{and}\quad \mu>0,$$

the last term becomes

$$\sum_{j=0}^{\infty}\left[-\frac{4\pi}{a}\sum_{n_2,n_3=0}^{\infty}(-1)^j\sqrt{\left(n_2+\frac{1}{2}\right)^2+\left(n_3+\frac{1}{2}\right)^2}\,\frac{K_{\text{Bessel}}\big(1,2\pi\sqrt{\left(n_2+\frac{1}{2}\right)^2+\left(n_3+\frac{1}{2}\right)^2}(j+1)\big)}{2\pi(j+1)}\right]=-0.010\,762\,308\,1. \tag{A12}$$

The numerical value reported is obtained by the following procedure. For a given j, due to the symmetry between n_2 and n_3, we choose a common cutoff denoted by N for the sums over n_2 and n_3. Then we increase the cutoff N until the value of the double sum converges to a finite final value (e.g., see Fig. 1 for $j=1$).

Then we repeat the procedure for the next value of j, and so on. What we observe is that the values of the cutoffs depend on j. However, the cutoff $N(j)$ is a monotonically decreasing function of j (see Fig. 2).

Therefore in order to compute the sum over j it suffices to choose the largest value of $N(j)$, which is $N(0)$. Once the cutoff for the sums over n_2 and n_3 are determined we can compute the sum over j by an analogous procedure.

That is, we compute the value of the sum over j for various cutoffs J and search for an asymptote. We can find the value of the asymptote by fitting a series in the inverse power of J. This is how the numerical value reported in Eq. (A12) has been obtained (see Fig. 3). It is interesting to note that the sums converge extremely rapidly in both cases.

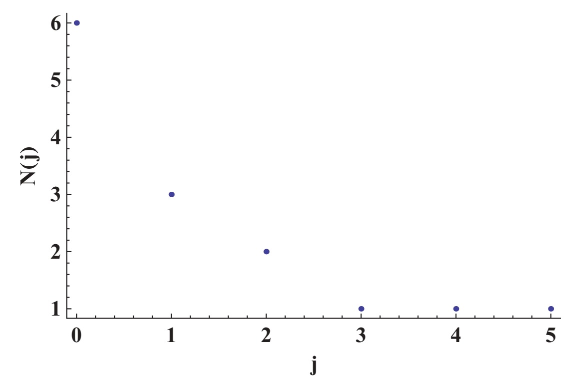

FIG. 2. The plot of the cutoff $N(j)$. Due to the downward trend of $N(j)$, $N(0)$ is the most appropriate cutoff for summations over n_2 and n_3.

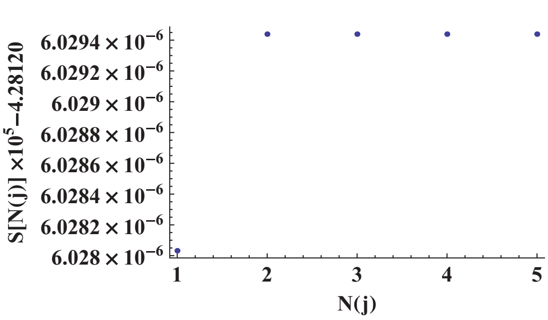

FIG. 1. For $j=1$ we have computed the double sum in Eq. (A12), $S(N(j))$, using a common cutoff (N). By increasing N the double sum converges extremely rapidly to a finite value 0.000 042 812 060 294. The values on the vertical axes up to 14 digits after the decimal point are the same. The only way to present the differences between the values is to define a new origin as we did for labeling the vertical axes. For this case it is sufficient to choose $N(1)=3$.

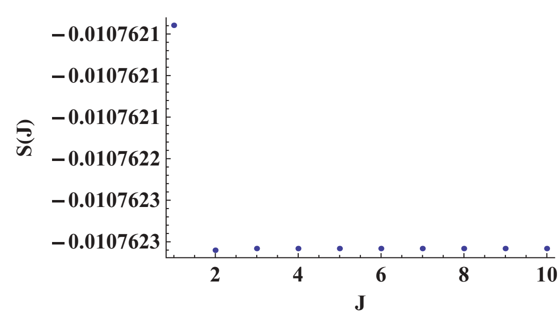

FIG. 3. As illustrated in Fig. 1 and Fig. 2 we obtain the appropriate cutoff for the sums over n_2 and n_3. In order to complete the computation of Eq. (A12) we evaluate the summation over j for various cutoffs J. It can be seen that the whole summation, $S(J)$, reaches its asymptotic value very quickly. We find the precise value of the asymptote (−0.010 762 308 1) by fitting a series in the inverse power of J.

In order to compute the second term of Eq. (A8) we first interchange the order of integrations to obtain

$$-\frac{4\pi}{a}\sum_{n_3=0}^{\infty}\int_{n_3+\frac{1}{2}}^{\infty}\frac{dk}{e^{2\pi k}+1}\int_0^{Y}dt\,\sqrt{Y^2-t^2}, \qquad \text{(A13)}$$

where $Y=\sqrt{k^2-(n_3+\frac{1}{2})^2}$. The innermost integral is easily computed and yields $\frac{\pi}{4}[k^2-(n_3+\frac{1}{2})^2]$. Now using Eq. (A9) and Eq. (A11) we obtain

$$\begin{aligned}&-\frac{\pi^2}{a}\sum_{j,n_3=0}^{\infty}(-1)^j\int_{n_3+\frac{1}{2}}^{\infty}dk\,e^{-2\pi k(j+1)}\left[k^2-\left(n_3+\frac{1}{2}\right)^2\right]\\&=-\frac{1}{a}\sum_{j}^{\infty}\frac{(-1)^j}{(j+1)^{\frac{3}{2}}}\sum_{n_3=0}^{\infty}\left(n_3+\frac{1}{2}\right)^{\frac{3}{2}}K_{\text{Bessel}}\left[\frac{3}{2},2\pi\left(n_3+\frac{1}{2}\right)(j+1)\right]=-0.014\,176\,611\,3. \qquad \text{(A14)}\end{aligned}$$

The numerical value reported at the end of Eq. (A14) is obtained by a procedure analogous to the one explained previously. Going through this same procedure, we can compute the first branch-cut term in Eq. (A8) as follows:

$$-\frac{1}{a}\sum_{j=0}^{\infty}\frac{(-1)^j}{(j+1)^{\frac{3}{2}}}\int_0^{\infty}dt\,t^{\frac{3}{2}}K_{\text{Bessel}}\left[\frac{3}{2},2\pi t(j+1)\right]=-0.023\,988\,621\,9. \qquad \text{(A15)}$$

Finally, we arrive at

$$\begin{aligned}E_{\text{Casimir}}=&-\frac{1}{a}\sum_{j=0}^{\infty}\frac{(-1)^j}{(j+1)}\Bigg\{2\sum_{n_2,n_3=0}^{\infty}\sqrt{\left(n_2+\frac{1}{2}\right)^2+\left(n_3+\frac{1}{2}\right)^2}\,K_{\text{Bessel}}\left[1,2\pi\sqrt{\left(n_2+\frac{1}{2}\right)^2+\left(n_3+\frac{1}{2}\right)^2}(j+1)\right]\\&+\frac{1}{(j+1)^{\frac{1}{2}}}\sum_{n_3=0}^{\infty}\left(n_3+\frac{1}{2}\right)^{\frac{3}{2}}K_{\text{Bessel}}\left[\frac{3}{2},2\pi\left(n_3+\frac{1}{2}\right)(j+1)\right]+\frac{1}{(j+1)^{\frac{1}{2}}}\int_0^{\infty}dt\,t^{\frac{3}{2}}K_{\text{Bessel}}\left[\frac{3}{2},2\pi t(j+1)\right]\Bigg\}\\&=-\frac{0.048\,927\,541}{a}. \qquad \text{(A16)}\end{aligned}$$

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