Analytical results for a monochromatically driven two-level system

Qiongtao Xie (谢琼涛) and Wenhua Hai (海文华)
Department of Physics and Key Laboratory of Low-Dimensional Quantum Structure and Quantum Control of Ministry of Education, Hunan Normal University, Changsha 410081, China
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We show that the frequently encountered physical model of a monochromatically driven two-level system is exactly solvable. We present an analytical exact solution to this driven system in terms of two known special functions. Our analytical solution is valid for all parameter regimes and may find applications in current solid-state experiments.

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I. INTRODUCTION

The periodically driven two-level model has been an important paradigm for understanding many fundamental phenomena in diverse branches of physics [1]. One of the simplest nontrivial models is the interaction of a two-level system with a monochromatic driving field. Recently, such a monochromatically driven two-level system has received revived interest due to artificial two-level systems in superconducting Josephson devices [2–7]. In these driven artificial two-level systems, all relevant system parameters are tunable, and thus different dynamical regimes can be reached.

Although widely studied over the past few decades [8], up to now an analytical exact solution to the simple driven two-level system is still lacking. At present, there are various methods for approximate solutions, such as the rotating-wave approximation, the time-averaging method, and perturbation theory [8–17]. The widely used rotating-wave approximation works well under the conditions of near resonance and weak coupling [9–11]. The averaging method is applicable for the weak-coupling region or the high-frequency region [12]. The perturbation theory is generally useful for weak-coupling, strong-coupling, high-frequency, and low-frequency cases with a suitably chosen small parameter in the perturbation expansions [13–17]. However, the usual perturbation series give rise to secular terms [18], which grow linearly as a function of the time variable. In the strong-coupling and high-frequency regimes, two complicated methods have been used to eliminate secular terms [15,16]. In the strong-coupling case, an efficient iterating approach has been suggested to be valid for a wide frequency range [17].

In this paper, we go a step further to present an analytical exact solution of the two-level system under a monochromatic driving field with the help of two known special functions. It is shown that, for a purely oscillating driving field, the solution is given in terms of the Heun confluent function, and, for the oscillating driving field with a nonzero static component, the solution is given in terms of the Heun double confluent function. Therefore, this widely studied driven two-level system is an exactly solvable model. Our analytical solutions work for all parameter regimes and may find applications in current solid-state experiments.

II. ANALYTICAL EXACT SOLUTIONS OF A PERIODICALLY DRIVEN TWO-LEVEL SYSTEM

We consider a two-level system $|1\rangle$ and $|2\rangle$ described by the following Hamiltonian ($\hbar = 1$) [8–17]:

$$H(t) = \frac{f(t)}{2}(|1\rangle\langle 1| - |2\rangle\langle 2|) + \frac{\nu}{2}(|1\rangle\langle 2| + |2\rangle\langle 1|). \quad (1)$$

Here $f(t)$ denotes the oscillating energy bias between the two levels induced by the external periodic force, and in our study we assume a general time dependence of the form $f(t) = f_0 + f_1 \sin(\omega t)$ with oscillating amplitude f_1, oscillating frequency ω, and nonzero static part f_0; ν is the coupling constant between the two levels. The parameters ν, f_1, f_0, and ω have the same units. Depending on the ratios between these parameters, this system can be in different dynamical regimes. The periodically driven two-level model occurs in many physical situations, such as the semiclassical description of the interaction of a two-level atom with a single-mode field [19,20], the tunneling of a single particle in a periodically driven double-well potential [21], and the light propagation in two periodically modulated coupled waveguides [22–24].

Expanding the solution $|\psi(t)\rangle$ of the time-dependent Schrödinger equation for the Hamiltonian (1) in the set $\{|1\rangle, |2\rangle\}$, $|\psi(t)\rangle = a(t)|1\rangle + b(t)|2\rangle$, we have

$$i\frac{da}{d\tau} = \frac{\nu}{2\omega}b + \left(\frac{f_0}{2\omega} + \frac{f_1 \sin(\tau)}{2\omega}\right)a, \quad (2)$$

$$i\frac{db}{d\tau} = \frac{\nu}{2\omega}a - \left(\frac{f_0}{2\omega} + \frac{f_1 \sin(\tau)}{2\omega}\right)b. \quad (3)$$

Since the ratios between these parameters are essential, we have used the scale transformation $\tau = \omega t$. It is straightforward to show that the probability amplitudes $a(\tau)$ and $b(\tau)$ obey the following equations:

$$\frac{d^2a}{d\tau^2} + \left(i\frac{f_1\cos\tau}{2\omega} + \frac{(f_0 + f_1\sin\tau)^2}{4\omega^2} + \frac{\nu^2}{4\omega^2}\right)a = 0, \quad (4)$$

$$\frac{d^2b}{d\tau^2} + \left(-i\frac{f_1\cos\tau}{2\omega} + \frac{(f_0 + f_1\sin\tau)^2}{4\omega^2} + \frac{\nu^2}{4\omega^2}\right)b = 0, \quad (5)$$

which are similar to the Schrödinger equation for a particle in a periodic potential. Therefore, our later analytical results may find applications for ultracold atoms in optical lattices [25,26].

We first consider the simple case of a purely sinusoidal driving field (i.e., $f_0 = 0$). On making the transformations $z(\tau) = \sin^2(\tau/2)$ and $a(z) = \exp[if_1 z/\omega]\phi(z)$, we find that $\phi(z)$ obeys the Heun confluent equation [27,28]

$$\frac{d^2\phi}{dz^2} + \left(\alpha + \frac{\beta+1}{z} + \frac{\gamma+1}{z-1}\right)\frac{d\phi}{dz} + \frac{qz+p}{z(z-1)}\phi = 0, \quad (6)$$

where $q = \delta + \alpha(\beta+\gamma+2)/2$ and $p = \eta + \beta/2 + (\gamma-\alpha)(\beta+1)/2$. Here the parameters α, β, γ, δ, and η are used in the widespread computer package MAPLE, and they are given by $\alpha = i2f_1/\omega$, $\beta = \gamma = -1/2$, $\delta = if_1/\omega$, and $\eta = -if_1/2\omega + 3/8 - \nu^2/4\omega^2$. This equation has two regular singularities at $z = 0$ and 1 and an irregular singularity at $z = \infty$. The local solution of Eq. (6) around $z = 0$ is the confluent Heun function, as a standard power-series expansion around the origin, $\phi_1(z) = \mathrm{HC}(\alpha,\beta,\gamma,\delta,\eta,z) = \sum_{n=0}^{\infty} h_n z^n$ [27,28], where the coefficients h_n are determined by the three-term recurrence relation $A_n h_n = B_n h_{n-1} + C_n h_{n-2}$ ($n \geqslant 1$) with the initial conditions $h_0 = 1$ and $h_{-1} = 0$. Here $A_n = 1 + \beta/n$, $B_n = 1 + (\beta+\gamma-\alpha-1)/n + [\eta - \beta/2 + (\gamma-\alpha)(\beta+1)/2]/n^2$, and $C_n = [\delta + \alpha(\beta+\gamma)/2 + \alpha(n-1)]/n^2$. Another linearly independent solution of Eq. (6) around $z = 0$ can be expressed as $\phi_2(z) = z^{-\beta}\mathrm{HC}(\alpha, -\beta, \gamma,\delta,\eta,z)$ [27,28]. With the two solutions, we obtain the two linearly independent particular solutions for $a(\tau)$:

$$\psi_1(\tau) = e^{i\frac{f_1}{\omega}\sin^2(\tau/2)}\mathrm{HC}(\alpha,\beta,\gamma,\delta,\eta,\sin^2(\tau/2)), \quad (7)$$

$$\psi_2(\tau) = e^{i\frac{f_1}{\omega}\sin^2(\tau/2)}\sin(\tau/2)\mathrm{HC}(\alpha,-\beta,\gamma,\delta,\eta,\sin^2(\tau/2)). \quad (8)$$

It follows from Eqs. (4) and (5) that $\psi_1^*(\tau)$ and $\psi_2^*(\tau)$ are also two linearly independent solutions of Eq. (5). Thus, one can construct the general solutions for a and b: $a(\tau) = c_1\psi_1(\tau) + c_2\psi_2(\tau)$ and $b(\tau) = d_1\psi_1^*(\tau) + d_2\psi_2^*(\tau)$, where the constants $c_{1,2}$ and $d_{1,2}$ are determined by the conditions $a(\tau_0)$, $a'(\tau_0)$, $b(\tau_0)$, and $b'(\tau_0)$ at the initial time $\tau = \tau_0$. Here the prime denotes the derivative with respect to τ. After a straightforward calculation, we finally get

$$a(\tau) = U_{11}^1(\tau,\tau_0)a(\tau_0) + U_{12}^1(\tau,\tau_0)b(\tau_0), \quad (9)$$

$$b(\tau) = U_{21}^1(\tau,\tau_0)a(\tau_0) + U_{22}^1(\tau,\tau_0)b(\tau_0), \quad (10)$$

where $U_{11}^1(\tau,\tau_0) = \Delta_1(\tau,\tau_0)/\Delta_1(\tau_0,\tau_0) + if_1\sin(\tau_0)\Delta_2(\tau,\tau_0)/2\omega\Delta_1(\tau_0,\tau_0)$, $U_{12}^1(\tau,\tau_0) = i\nu\Delta_2(\tau,\tau_0)/2\omega\Delta_1(\tau_0,\tau_0)$, $U_{21}^1(\tau,\tau_0) = -U_{12}^{1*}(\tau,\tau_0)$, and $U_{22}^1(\tau,\tau_0) = U_{11}^{1*}(\tau,\tau_0)$ with $\Delta_1(\tau,\tau_0) = \psi_1(\tau)\psi_2'(\tau_0) - \psi_2(\tau)\psi_1'(\tau_0)$ and $\Delta_2(\tau,\tau_0) = \psi_1(\tau)\psi_2(\tau_0) - \psi_2(\tau)\psi_1(\tau_0)$. This leads to the time-evolution operator

$$U_1(\tau,\tau_0) = \begin{pmatrix} U_{11}^1(\tau,\tau_0) & U_{12}^1(\tau,\tau_0) \\ -U_{12}^{1*}(\tau,\tau_0) & U_{11}^{1*}(\tau,\tau_0) \end{pmatrix}. \quad (11)$$

However, the Heun confluent function $\mathrm{HC}(\alpha,\beta,\gamma,\delta,\eta,z)$ is convergent within the circle $|z| < 1$ [27,28]. At $z = 1$, it is divergent, since $z = 1$ is the singularity of Eq. (6). Therefore, because of $0 \leqslant z(\tau) \leqslant 1$, the exact solutions (9) and (10) are not valid at $z = 1$. To find the solutions at $z = 1$, we need to consider the local solution relative to $z = 1$. After the substitution of $x(\tau) = 1 - z(\tau)$ into Eq. (6), we have

$$\frac{d^2\phi}{dx^2} + \left(-\alpha + \frac{\gamma+1}{x} + \frac{\beta+1}{x-1}\right)\frac{d\phi}{dx} + \frac{\widetilde{q}x + \widetilde{p}}{x(x-1)}\phi = 0, \quad (12)$$

where $\widetilde{q} = -\delta - \alpha(\beta+\gamma+2)/2$ and $\widetilde{p} = \eta + \delta + \gamma/2 + (\gamma+\alpha)(\gamma+1)/2$. Equation (12) is also a Heun confluent equation with different parameters. Near $z = 1$, we obtain two linearly independent particular solutions for $a(\tau)$:

$$\psi_3(\tau) = e^{-i\frac{f_1}{\omega}\cos^2(\tau/2)}\mathrm{HC}(-\alpha,\gamma,\beta,-\delta,\eta+\delta,\cos^2(\tau/2)), \quad (13)$$

$$\psi_4(\tau) = e^{-i\frac{f_1}{\omega}\cos^2(\tau/2)}\cos(\tau/2) \times \mathrm{HC}(-\alpha,-\gamma,\beta,-\delta,\eta+\delta,\cos^2(\tau/2)). \quad (14)$$

The general solutions for a and b are thus given by $a(\tau) = c_3\psi_3(\tau) + c_4\psi_4(\tau)$ and $b(\tau) = d_3\psi_3^*(\tau) + d_4\psi_4^*(\tau)$, where $c_{3,4}$ and $d_{3,4}$ are constants determined by the conditions $a(\tau_1)$, $a'(\tau_1)$, $b(\tau_1)$, and $b'(\tau_1)$ at time $\tau = \tau_1$. Similarly, we can get

$$a(\tau) = U_{11}^2(\tau,\tau_1)a(\tau_1) + U_{12}^2(\tau,\tau_1)b(\tau_1), \quad (15)$$

$$b(\tau) = U_{21}^2(\tau,\tau_1)a(\tau_1) + U_{22}^2(\tau,\tau_1)b(\tau_1), \quad (16)$$

where $U_{11}^2(\tau,\tau_1) = \Delta_3(\tau,\tau_1)/\Delta_3(\tau_1,\tau_1) + if_1\sin(\tau_1)\Delta_4(\tau,\tau_1)/2\omega\Delta_3(\tau_1,\tau_1)$, $U_{12}^2(\tau,\tau_1) = i\nu\Delta_4(\tau,\tau_1)/2\omega\Delta_3(\tau_1,\tau_1)$, $U_{21}^2(\tau,\tau_1) = -U_{12}^{2*}(\tau,\tau_1)$, and $U_{22}^2(\tau,\tau_1) = U_{11}^{2*}(\tau,\tau_1)$ with $\Delta_3(\tau,\tau_1) = \psi_3(\tau)\psi_4'(\tau_1) - \psi_4(\tau)\psi_3'(\tau_1)$ and $\Delta_4(\tau,\tau_1) = \psi_3(\tau)\psi_4(\tau_1) - \psi_4(\tau)\psi_3(\tau_1)$. The corresponding time-evolution operator is given by

$$U_2(\tau,\tau_1) = \begin{pmatrix} U_{11}^2(\tau,\tau_1) & U_{12}^2(\tau,\tau_1) \\ -U_{12}^{2*}(\tau,\tau_1) & U_{11}^{2*}(\tau,\tau_1) \end{pmatrix}. \quad (17)$$

These analytic exact solutions relative to $z = 0$ and $z = 1$ can be connected at the common regions of $0 < z(\tau) < 1$ and $0 < x(\tau) = 1 - z(\tau) < 1$. With Eqs. (11) and (17), we can construct the total time-evolution operator by dividing the evolution time into different time intervals. For each time interval, we have the corresponding time-evolution operator. For example, the well-known Floquet operator $F(T,0)$, defined as the time-evolution operator over one period $T = 2\pi$, can be written as

$$F(2\pi,0) = U_1(2\pi,\tau_2)U_2(\tau_2,\tau_1)U_1(\tau_1,0), \quad (18)$$

where τ_n must be appropriately chosen to avoid the cases of $z = 1$ and $x = 1$. In Fig. 1, we use the computer package MAPLE to show our analytical and numerical results for three set of parameters corresponding to the high-frequency, resonance, and low-frequency cases, respectively. It is clearly seen that they agree very well. To show our analytical solutions, we have taken $\tau_n = 3n$ to connect solutions in different time intervals. In particular, for the case of $f_1/\omega = 2.404$ and $\nu/\omega = 6$, the system initially in the level $|1\rangle$ remains almost in the same level. This effect is known as coherent destruction of tunneling and was found originally by Grossmann *et al.* for a particle in a driven double-well potential [21].

Now we consider the general case in the presence of the static part f_0. On substituting the transforms $z_1(\tau) = -i\tan(\tau/2)$ and $a(\tau) = \exp[-f_1\sin(\tau)/2\omega]\phi(z_1)$ into Eq. (4),

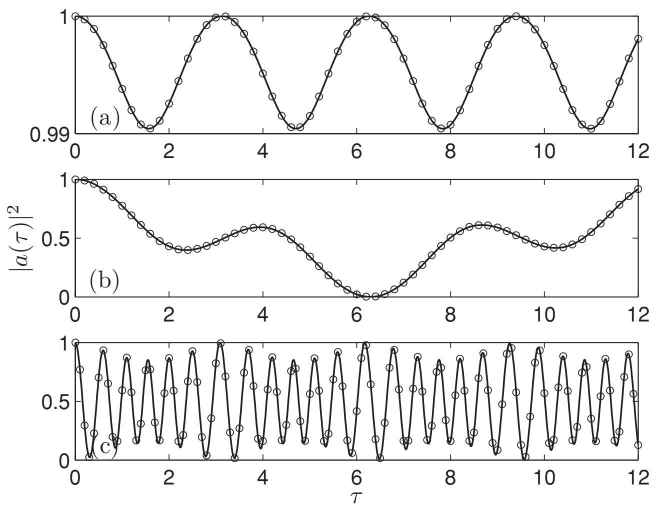

FIG. 1. $|a(\tau)|^2$ vs the dimensionless time τ with $a(0) = 1$ and $b(0) = 0$ for (a) $\nu/\omega = 1/6$ and $f_1/\omega = 2.404$, (b) $\nu/\omega = 1$ and $f_1/\omega = 1$, (c) $\nu/\omega = 10$ and $f_1/\omega = 10$. The circles are for the numerical results with Eqs. (2) and (3), and the solid lines are for the analytical exact results. To compute our analytical exact solutions, we have taken $\tau_n = 3n$ to connect solutions in different time intervals.

it follows that $\phi(z)$ obeys the Heun double confluent equation [27,28]

$$\frac{d^2\phi}{dz_1^2} + \frac{2z_1(z_1^2-1)^2 - \alpha(z_1^4-1)}{(z_1^2-1)^3}\frac{d\phi}{dz_1} + \frac{\beta z_1^2 + (\gamma+\alpha)z_1 + \delta}{(z_1^2-1)^2}\phi = 0, \tag{19}$$

where the parameters α, β, γ, and δ are also used in the widespread computer package MAPLE, which reads $\alpha = i2f_1/\omega$, $\beta = -(f_0^2 + f_1^2 + \nu^2)/\omega^2 + i2f_1/\omega$, $\gamma = i4f_0f_1/\omega^2$, and $\delta = (f_0^2 + f_1^2 + \nu^2)/\omega^2 + i2f_1/\omega$. This equation has two irregular singularities, located at $z = -1$ and $z = 1$, and the origin $z = 0$ is a regular point. The Heun double confluent equation has a local solution $\phi_1(z_1) = \mathrm{HD}(\alpha,\beta,\gamma,\delta,z_1)$, which is called the Heun double confluent function. Another independent solution can be given by $\phi_2(z_1) = \exp[-\alpha z_1/(z_1^2-1)]\mathrm{HD}(-\alpha,\beta,\gamma,\delta,z_1)$. With the two solutions, we obtain the two linearly independent particular solutions for $a(\tau)$:

$$\chi_1(\tau) = e^{-\frac{f_1}{2\omega}\sin(\tau)}\mathrm{HD}\big(\alpha,\beta,\gamma,\delta,-i\tan(\tau/2)\big), \tag{20}$$

$$\chi_2(\tau) = e^{\frac{f_1}{2\omega}\sin(\tau)}\mathrm{HD}\big(-\alpha,\beta,\gamma,\delta,-i\tan(\tau/2)\big). \tag{21}$$

Following our above treatment, we immediately obtain the time-evolution operator

$$\widetilde{U}_1(\tau,\tau_0) = \begin{pmatrix} \widetilde{U}_{11}^1(\tau,\tau_0) & \widetilde{U}_{12}^1(\tau,\tau_0) \\ -\widetilde{U}_{12}^{1*}(\tau,\tau_0) & \widetilde{U}_{11}^{1*}(\tau,\tau_0) \end{pmatrix}, \tag{22}$$

where $\widetilde{U}_{11}^1(\tau,\tau_0) = \widetilde{\Delta}_1(\tau,\tau_0)/\widetilde{\Delta}_1(\tau_0,\tau_0) + i[f_0 + f_1\sin(\tau_0)]\,\widetilde{\Delta}_2(\tau,\tau_0)/2\omega\widetilde{\Delta}_1(\tau_0,\tau_0)$ and $\widetilde{U}_{12}^1(\tau,\tau_0) = i\nu\widetilde{\Delta}_2(\tau,\tau_0)/2\omega\widetilde{\Delta}_2(\tau,\tau_0)$. Here $\widetilde{\Delta}_{1,2}$ have the same expression with $\Delta_{1,2}$ by replacing $\psi_{1,2}$ with $\chi_{1,2}$. The latter $\widetilde{\Delta}_{3,4,5,6,7,8}$ also have similar expressions. Because the radius of convergence of the Heun double confluent function is $|z_1| < 1$, together with $z_1 \in i[-\infty,\infty]$, our solutions are valid only in a certain time interval. For simplicity, we only consider the solutions in a period T. Clearly, our time-evolution operator $\widetilde{U}_1(\tau,\tau_0)$ is valid for $0 \leqslant \tau_0 \leqslant \tau < \pi/2$ and $3\pi/2 < \tau_0 \leqslant \tau \leqslant 2\pi$.

To obtain the solutions relative to $z_1 = -i$ at $\tau = \pi/2$, we can make other different transforms $z_2 = -i\tan(\tau/2 - \pi/4)$ and $a(z_2) = \exp[if_1\cos(\tau)/2\omega]\phi(z_2)$, thereby leading to another version of the Heun double confluent equation:

$$\frac{d^2\phi}{dz_2^2} + \frac{2z_2(z_2^2-1)^2 - \widetilde{\alpha}(z_2^4-1)}{(z_2^2-1)^3}\frac{d\phi}{dz_2} + \frac{\widetilde{\beta}z_2^2 + (\widetilde{\gamma}+\widetilde{\alpha})z_2 + \widetilde{\delta}}{(z_2^2-1)^2}\phi = 0, \tag{23}$$

where the parameters are $\widetilde{\alpha} = -2f_1/\omega$, $\widetilde{\beta} = -(f_0^2 + \nu^2 - 2f_0f_1)/\omega^2$, $\widetilde{\gamma} = 4f_1/\omega$, and $\widetilde{\delta} = (f_0^2 + \nu^2 + 2f_0f_1)/\omega^2$. The resulting two linearly independent particular solutions for $a(\tau)$ can be written as

$$\chi_3(\tau) = e^{i\frac{f_1}{2\omega}\cos(\tau)}\mathrm{HD}\big(\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma},\widetilde{\delta},-i\tan(\tau/2-\pi/4)\big), \tag{24}$$

$$\chi_4(\tau) = e^{-i\frac{f_1}{2\omega}\cos(\tau)}\mathrm{HD}\big(-\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma},\widetilde{\delta},-i\tan(\tau/2-\pi/4)\big). \tag{25}$$

The corresponding time-evolution operator is given by

$$\widetilde{U}_2(\tau,\tau_1) = \begin{pmatrix} \widetilde{U}_{11}^2(\tau,\tau_1) & \widetilde{U}_{12}^2(\tau,\tau_1) \\ -\widetilde{U}_{12}^{2*}(\tau,\tau_1) & \widetilde{U}_{11}^{2*}(\tau,\tau_1) \end{pmatrix}, \tag{26}$$

where $\widetilde{U}_{11}^2(\tau,\tau_1) = \widetilde{\Delta}_3(\tau,\tau_1)/\widetilde{\Delta}_3(\tau_1,\tau_1) + i[f_0 + f_1\sin(\tau_1)]\,\widetilde{\Delta}_4(\tau,\tau_1)/2\omega\widetilde{\Delta}_3(\tau_1,\tau_1)$ and $\widetilde{U}_{12}^2(\tau,\tau_1) = i\nu\widetilde{\Delta}_4(\tau,\tau_1)/2\omega\widetilde{\Delta}_3(\tau,\tau_1)$. We can see that the time-evolution operator $\widetilde{U}_2(\tau,\tau_1)$ is valid for $0 < \tau_1 \leqslant \tau < \pi$. Therefore, the two time-evolution operators can be connected at the time $\tau_1 = \pi/2 - \Delta\tau_1$ with $\Delta\tau_1$ being positive small.

To give the solutions at $\tau = \pi$, for which $|z_1| > 1$ and $z_2 = -i$, the substitution of $x_1 = 1/z_1 = i\cot(\tau/2)$ into Eq. (19) can yield another version of itself. The corresponding parameters transform according to $\alpha = -\alpha$, $\beta = -\delta$, and $\gamma = -\gamma$. Therefore, the resulting two linearly independent particular solutions for $a(\tau)$ can be written as

$$\chi_5(\tau) = e^{-\frac{f_1}{2\omega}\sin(\tau)}\mathrm{HD}\big(-\alpha,-\delta,-\gamma,-\beta,i\cot(\tau/2)\big), \tag{27}$$

$$\chi_6(\tau) = e^{\frac{f_1}{2\omega}\sin(\tau)}\mathrm{HD}\big(\alpha,-\delta,-\gamma,-\beta,i\cot(\tau/2)\big). \tag{28}$$

The corresponding time-evolution operator is given by

$$\widetilde{U}_3(\tau,\tau_2) = \begin{pmatrix} \widetilde{U}_{11}^3(\tau,\tau_2) & \widetilde{U}_{12}^3(\tau,\tau_2) \\ -\widetilde{U}_{12}^{3*}(\tau,\tau_1) & \widetilde{U}_{11}^{3*}(\tau,\tau_2) \end{pmatrix}, \tag{29}$$

where $\widetilde{U}_{11}^3(\tau,\tau_2) = \widetilde{\Delta}_5(\tau,\tau_2)/\widetilde{\Delta}_5(\tau_2,\tau_2) + i[f_0 + f_1\sin(\tau_2)]\,\widetilde{\Delta}_6(\tau,\tau_2)/2\omega\widetilde{\Delta}_5(\tau_2,\tau_2)$ and $\widetilde{U}_{12}^3(\tau,\tau_2) = i\nu\widetilde{\Delta}_6(\tau,\tau_2)/2\omega\widetilde{\Delta}_5(\tau,\tau_2)$. The time-evolution operator is valid for $\pi/2 < \tau_2 \leqslant \tau < 3\pi/2$. Therefore, the two time-evolution operators $\widetilde{U}_2(\tau,\tau_1)$ and $\widetilde{U}_3(\tau,\tau_2)$ can be connected at $\tau_2 = \pi - \Delta\tau_2$ with $\Delta\tau_2$ being positive small.

To obtain the solutions at $\tau = 3\pi/2$, on substituting $x_2 = 1/z_2 = i\cot(\tau/2 - \pi/4)$ into Eq. (23), we have another version of it. The corresponding parameters transform according to $\widetilde{\alpha} = -\widetilde{\alpha}$, $\widetilde{\beta} = -\widetilde{\delta}$, and $\widetilde{\gamma} = -\widetilde{\gamma}$. Therefore, the resulting two linearly independent particular solutions for $a(\tau)$ can be written as

$$\chi_7(\tau) = e^{i\frac{f_1}{2\omega}\cos(\tau)}\mathrm{HD}(-\tilde{\alpha}, -\tilde{\delta}, -\tilde{\gamma}, -\tilde{\beta}, i\cot(\tau/2-\pi/4)), \tag{30}$$

$$\chi_8(\tau) = e^{-i\frac{f_1}{2\omega}\cos(\tau)}\mathrm{HD}(\tilde{\alpha}, -\tilde{\delta}, -\tilde{\gamma}, -\tilde{\beta}, i\cot(\tau/2-\pi/4)). \tag{31}$$

The corresponding time-evolution operator is given by

$$\tilde{U}_4(\tau,\tau_3) = \begin{pmatrix} \tilde{U}_{11}^4(\tau,\tau_3) & \tilde{U}_{12}^4(\tau,\tau_3) \\ -\tilde{U}_{12}^{4*}(\tau,\tau_3) & \tilde{U}_{11}^{4*}(\tau,\tau_3) \end{pmatrix}, \tag{32}$$

where $\tilde{U}_{11}^4(\tau,\tau_3) = \tilde{\Delta}_7(\tau,\tau_3)/\tilde{\Delta}_7(\tau_3,\tau_3) + i[f_0 + f_1\sin(\tau_3)]\tilde{\Delta}_8(\tau,\tau_3)/2\omega\tilde{\Delta}_7(\tau_3,\tau_3)$ and $\tilde{U}_{12}^4(\tau,\tau_3) = i\nu\tilde{\Delta}_8(\tau,\tau_3)/2\omega\tilde{\Delta}_7(\tau,\tau_3)$. The time-evolution operator is valid for $\pi < \tau_3 \leqslant \tau < 2\pi$. Therefore, the two time-evolution operators $\tilde{U}_3(\tau,\tau_2)$ and $\tilde{U}_4(\tau,\tau_3)$ can be connected at $\tau_3 = 3\pi/2 - \Delta\tau_3$ with $\Delta\tau_3$ being positive small. In addition, the two time-evolution operators $\tilde{U}_1(\tau,\tau_0)$ and $\tilde{U}_4(\tau,\tau_3)$ can be connected at $\tau_0 = 2\pi - \Delta\tau_4$ with $\Delta\tau_4$ being positive small. With these time-evolution operators, the well-known Floquet operator $\tilde{F}(2\pi,0)$ can be written as

$$\tilde{F} = \tilde{U}_1(2\pi,\tau_4)\tilde{U}_4(\tau_4,\tau_3)\tilde{U}_3(\tau_3,\tau_2)\tilde{U}_2(\tau_2,\tau_1)\tilde{U}_1(\tau_1,0). \tag{33}$$

It is known that in certain parameter regimes some approximations can be used to obtain analytical results for this periodically driven two-level system, such as high-frequency, rotating-wave, and adiabatic approximations. For simplicity, we only focus on the simple case of $f_0 = 0$. We assume that the system is in an initial state with $a(0) = 1$ and $b(0) = 0$. In the high-frequency limit $0 < \nu/\omega \ll 1$, the high-frequency approximation (HFA) gives the following approximate results [12–16]:

$$a_{\mathrm{HFA}}(\tau) = e^{-i\frac{f_1}{\omega}\sin^2(\frac{\tau}{2})}\cos\left[\frac{\nu\tau}{2\omega}J_0(f_1/\omega)\right], \tag{34}$$

$$b_{\mathrm{HFA}}(\tau) = -ie^{-i\frac{f_1}{\omega}\cos^2(\frac{\tau}{2})}\sin\left[\frac{\nu\tau}{2\omega}J_0(f_1/\omega)\right], \tag{35}$$

where $J_0(f_1/\omega)$ is the zero-order Bessel function. The rotating-wave approximation usually works well for near-resonance and weak-coupling cases. For brevity, we consider the exact resonant case of $\nu/\omega = 1$. The weak coupling corresponds to the weak field situation with $0 < f_1/\omega \ll 0$ in the Hamiltonian (1) for the driven system. The rotating-wave approximation (RWA) leads to [9–11]

$$a_{\mathrm{RWA}}(\tau) = \frac{1}{\sqrt{2}}\left[e^{-i\pi/4}\cos\left(\frac{f_1\tau}{4\omega} - \frac{\tau}{2}\right) + e^{i\pi/4}\cos\left(\frac{f_1\tau}{4\omega} + \frac{\tau}{2}\right)\right], \tag{36}$$

$$b_{\mathrm{RWA}}(\tau) = \frac{i}{\sqrt{2}}\left[e^{-i\pi/4}\sin\left(\frac{f_1\tau}{4\omega} - \frac{\tau}{2}\right) + e^{i\pi/4}\sin\left(\frac{f_1\tau}{4\omega} + \frac{\tau}{2}\right)\right]. \tag{37}$$

For the low-frequency (LF) case of $\nu/\omega \gg 1$, it then follows on applying the adiabatic approximation that [13]

$$a_{\mathrm{LF}}(\tau) = \tfrac{1}{2}\left[e^{-i\int_0^\tau E_+(s)ds}\varphi_+(\tau) + e^{-i\int_0^\tau E_-(s)ds}\varphi_-(\tau)\right], \tag{38}$$

$$b_{\mathrm{LF}}(\tau) = \tfrac{1}{2}\left[-e^{-i\int_0^\tau E_+(s)ds}\varphi_-(\tau) + e^{-i\int_0^\tau E_-(s)ds}\varphi_+(\tau)\right], \tag{39}$$

where $E_\pm(\tau) = \pm\sqrt{f_1^2\sin^2(\tau) + \nu^2}/2\omega$ are the instantaneous eigenvalues and $\varphi_\pm = \sqrt{1 \pm f_1^2\sin^2(\tau)/\sqrt{f_1^2\sin^2(\tau)+\nu^2}}$. These approximate results are applicable for the total time. However, the analytical solutions given by Eqs. (9), (10), (15), and (16) are valid in different time intervals, and they are connected at the common region of $0 < \sin^2(\tau/2) < 1$ and $0 < \cos^2(\tau/2) < 1$. For example, with $\tau_0 = 0$ in Eqs. (9) and (10), we have

$$a(\tau) = e^{-i\frac{f_1}{\omega}\sin^2(\frac{\tau}{2})}\mathrm{HC}\left(-i\frac{2f_1}{\omega}, -\frac{1}{2}, -\frac{1}{2}, i\frac{f_1}{\omega}, -i\frac{f_1}{2\omega} + \frac{3}{8} - \frac{\nu^2}{4\omega^2}, \sin^2\left(\frac{\tau}{2}\right)\right), \tag{40}$$

$$b(\tau) = -\frac{i\nu}{\omega}e^{i\frac{f_1}{\omega}\sin^2(\frac{\tau}{2})}\sin\left(\frac{\tau}{2}\right)\mathrm{HC}^*\left(-i\frac{2f_1}{\omega}, \frac{1}{2}, -\frac{1}{2}, i\frac{f_1}{\omega}, -i\frac{f_1}{2\omega} + \frac{3}{8} - \frac{\nu^2}{4\omega^2}, \sin^2\left(\frac{\tau}{2}\right)\right), \tag{41}$$

where we have used the identity $\mathrm{HC}(\alpha,\beta,\gamma,\delta,\eta,z) = \exp(-\alpha z)\mathrm{HC}(-\alpha,\beta,\gamma,\delta,\eta,z)$ for $z < 1$. Although the above analytical exact solution is valid for $0 \leqslant \tau < \pi$, on applying the following relations [29]

$$\cos\left(\frac{u\tau}{2}\right) = \mathrm{HC}\left(0, -\frac{1}{2}, -\frac{1}{2}, 0, \frac{3}{8} - \frac{u^2}{4}, \sin^2\left(\frac{\tau}{2}\right)\right), \tag{42}$$

$$\sin\left(\frac{u\tau}{2}\right) = u\sin\left(\frac{\tau}{2}\right)\mathrm{HC}\left(0, \frac{1}{2}, -\frac{1}{2}, 0, \frac{3}{8} - \frac{u^2}{4}, \sin^2\left(\frac{\tau}{2}\right)\right), \tag{43}$$

and $\mathrm{HC}(-i2f_1/\omega, -1/2, -1/2, if_1/\omega, -if_1/2\omega + 3/8, \sin^2(\tau/2)) = 1$, we obtain the expected results that when $f_1/\omega = 0$, we have $a(\tau) = \cos(\nu\tau/2\omega)$ and $b(\tau) = -i\sin(\nu\tau/2\omega)$, and when $\nu/\omega = 0$, we have $a(\tau) = \exp[-if_1\sin^2(\tau/2)/\omega]$ and $b(\tau) = 0$. However, it is a very difficult task to give the analytical expressions for the asymptotic behavior of the analytical exact solutions in various parameter regimes, since the theory of the Heun functions is not developed enough.

III. CONCLUSION

In conclusion, we have given an analytical exact solution of a two-level system driven by a sinusoidal driving field in terms of two special functions. For a purely sinusoidal driving field, the analytical exact solution is given with the help of the Heun confluent function. For a general sinusoidal driving field with a nonzero static component, the analytical exact solution is given with the help of the Heun double confluent function. Although one of our original aims is to use these analytical exact solutions to test the validity of various approximation methods, there is not enough mathematical knowledge about the asymptotic behavior of the two special functions in different parameter ranges. This needs further study of the two special functions. Additionally, our analytical solutions may find applications in optical lattice systems and current solid-state experiments.

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