Invisibility in non-Hermitian tight-binding lattices

Stefano Longhi*

Dipartimento di Fisica, Politecnico di Milano, Piazza L. da Vinci 32, I-20133 Milano, Italy

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Reflectionless defects in Hermitian tight-binding lattices, synthesized by the intertwining operator technique of supersymmetric quantum mechanics, are generally not invisible and time-of-flight measurements could reveal the existence of the defects. Here it is shown that, in a certain class of non-Hermitian tight-binding lattices with complex hopping amplitudes, defects in the lattice can appear fully invisible to an outside observer. The synthesized non-Hermitian lattices with invisible defects possess a real-valued energy spectrum; however, they lack parity-time ($\mathcal{PT}$) symmetry, which does not play any role in the present work.

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I. INTRODUCTION

In recent years, the subject of invisibility physics has attracted considerable renewed interest, mainly triggered by the publication of a few seminal papers by Pendry and Leonhardt on transformation optics and electromagnetic cloaking [1,2], which has led to the first experimental observation of invisibility at microwave frequencies [3]. Since then, a large body of works inspired by the concepts of transformation optics has been published, and applications to matter wave cloaking have been suggested as well [4]. An invisible object or scatter is, by definition, an object which does not scatter any wave incident upon it; that is, a wave which shines on the object is not reflected or absorbed, but instead it is transmitted in such a way that it appears to the outside observer as if there were no object present. The concepts and methods of invisibility based on the idea of transformation optics apply to two- or three-dimensional objects. In one-dimensional systems, the possibility of achieving an invisible scatter is closely related to the realization of reflectionless potentials. For continuous media, this problem was investigated in a pioneering work by Kay and Moses in 1956 [5], and then studied in great detail in the context of the inverse scattering theory [6,7] and supersymmetric quantum mechanics for Hermitian systems [8]. The potentials obtained by such techniques, though being transparent, are generally not invisible. This is due to the dependence of the phase of the transmitted wave on energy, which is generally responsible for some delay and/or for the distortion of a wave packet transmitted across the potential [9].

The possibility of synthesizing reflectionless potentials has been also investigated for wave scattering on a lattice, in which wave transport occurs due to hopping among adjacent sites of the lattice. In the mathematical literature, this problem is solved by the inverse spectral theory of Jacobi operators (i.e., second-order symmetric difference operators [10]); in this context, Darboux transformations and the intertwining operator technique of supersymmetric quantum mechanics have been successfully extended to the discrete Schrödinger equation, with applications to the synthesis of transparent (i.e., reflectionless) defects in Hermitian tight-binding lattices [11,12]. An optical realization of a special class of these reflectionless potentials on a lattice has been recently proposed for waveguide arrays and coupled-resonator structures with modulated coupling rates [13], suggesting new possibilities for pulse and beam shaping. For Hermitian lattices, such reflectionless potentials are nevertheless not invisible because the bound states of the lattice modify the time of flight of a wave packet and generally also distorts its shape: The existence of defects in the lattice, though being transparent, could be then inferred form simple time-of-flight measurements.

It is the aim of this work to show that ful invisibility of localized defects can be realized in *non-Hermitian* tight-binding lattices, which are synthesized by iterated application of the intertwining operator technique (Darboux transformation) to a defect-free tight-binding Hermitian lattice. The study of non-Hermitian tight-binding lattices has received in recent years a great attention (see, e.g. [14–19] and references therein); such previous studies have been mainly focused to lattices possessing parity-time ($\mathcal{PT}$) symmetry and were framed in the context of non-Hermitian quantum mechanics [17–20]; however, the possibility of realizing invisibility in a non-Hermitian lattice was not investigated in such previous works [21]. It should be noted that the class of non-Hermitian lattices synthesized in the present work by application of the Darboux transformation and showing the property of invisibility are not $\mathcal{PT}$-symmetric. Nevertheless, their energy spectrum is real-valued because they are isospectral to a Hermitian lattice. Therefore, $\mathcal{PT}$ symmetry does not play any role in the realization of invisible defects discussed in this work.

The paper is organized as follows. In Secs. II and III, the intertwining operator technique and its application to the synthesis of tight-binding lattices with reflectionless defects are briefly reviewed. The scattering and invisibility properties of the synthesized lattices are discussed in Sec. IV; in particular, it is shown that, as for any Hermitian lattice, invisibility can never be achieved and time-of-flight measurements can be used to reveal the existence of defects in the lattice, but in non-Hermitian lattices with certain complex hopping rates invisibility can occur. The main conclusions are outlined in Sec. V, whereas some mathematical details and a possible realization of non-Hermitian lattice models based on light propagation in optical waveguide arrays are presented in three appendixes.

*longhi@fisi.polimi.it

II. THE INTERTWINING OPERATOR TECHNIQUE FOR SPECTRAL ENGINEERING OF TIGHT-BINDING LATTICES

The synthesis of reflectionless and invisible defects in a tight-binding lattice discussed in the next sections is based on the discrete analogs of the intertwining operator technique of supersymmetric quantum mechanics [8]. Extensions of the intertwining operator technique to the discrete Schrödinger equation, together with the related issue of inverse scattering for Jacobi operators, have been discussed mainly in the mathematical literature (see, for instance [10–12]); however, they are not so common in physical contexts. In this section we thus provide a brief review of the intertwining operator technique and its application to the problem of spectral engineering of tight-binding lattices.

Let us consider a one-dimensional tight-binding lattice described by the Hamiltonian

$$\mathcal{H} = \sum_n \kappa_n(|n-1\rangle\langle n| + |n\rangle\langle n-1|) + \sum_n V_n|n\rangle\langle n|, \tag{1}$$

where $|n\rangle$ is a Wannier state localized at site n of the lattice, κ_n is the hopping rate between sites $|n-1\rangle$ and $|n\rangle$, and V_n is the energy of Wannier state $|n\rangle$. Note that $\mathcal{H}$ turns out to be Hermitian provided that the hopping amplitudes κ_n and site energies V_n are real-valued parameters. Let us indicate by $\mathcal{H}_1$ the tight-binding Hamiltonian defined by Eq. (1) with hopping amplitudes and site energies given by $\kappa_n^{(1)}$ and $V_n^{(1)}$, respectively, and let us assume that $\kappa_n^{(1)} \to \kappa > 0$ and $V_n^{(1)} \to 0$ as $n \to \pm\infty$, that is. that the lattice is asymptotically homogeneous and free of defects. Let $\sigma^{(1)} = \sigma_c \cup \sigma_p$ be the spectrum of $\mathcal{H}_1$, which comprises the continuous spectrum σ_c (the tight-binding band $-\kappa < E < \kappa$) and the point spectrum σ_p. Our goal is to synthesize a new tight-binding lattice Hamiltonian $\mathcal{H}_2$ of the form of Eq. (1), whose spectrum $\sigma^{(2)}$ is the same as that of $\mathcal{H}_1$, except for the addition of a new real-valued energy level μ_1 in the point spectrum, with $|\mu_1| > 2\kappa$. To this aim, let us indicate by $|\phi^{(1)}\rangle = \sum_n \phi_n^{(1)}|n\rangle$ a solution to the second-order difference equation

$$\kappa_n^{(1)}\phi_{n-1}^{(1)} + \kappa_{n+1}^{(1)}\phi_{n+1}^{(1)} + V_n^{(1)}\phi_n^{(1)} = \mu_1\phi_n^{(1)} \tag{2}$$

with the asymptotic behavior $|\phi_n^{(1)}| \to \infty$ for $n \to \pm\infty$. Note that such a solution does exist because μ_1 does not belong to the point spectrum nor to the continuous spectrum of $\mathcal{H}_1$. More precisely, $\phi_n^{(1)}$ is given by an arbitrary superposition of two linearly independent solutions to Eq. (2), which behave asymptotically as $\phi_n^{(1)} \sim \exp(\pm\omega_1 n)$ at $n \to \pm\infty$ for $\mu_1 > 2\kappa$, or as $\phi_n^{(1)} \sim (-1)^n \exp(\pm\omega_1 n)$ at $n \to \pm\infty$ for $\mu_1 < -2\kappa$, where $\omega_1 > 0$ is the root of the equation $2\kappa \cosh(\omega_1) = |\mu_1|$. It can then be shown by direct calculations that the following factorization for $\mathcal{H}_1$ holds:

$$\mathcal{H}_1 = \mathcal{Q}_1\mathcal{R}_1 + \mu_1, \tag{3}$$

where

$$\mathcal{Q}_1 = \sum_n \left(q_n^{(1)}|n\rangle\langle n| + \bar{q}_{n-1}^{(1)}|n-1\rangle\langle n|\right), \tag{4}$$

$$\mathcal{R}_1 = \sum_n \left(r_n^{(1)}|n\rangle\langle n| + \bar{r}_{n+1}^{(1)}|n+1\rangle\langle n|\right), \tag{5}$$

and

$$r_n^{(1)} = -\sqrt{\frac{\kappa_n^{(1)}\phi_{n-1}^{(1)}}{\phi_n^{(1)}}}, \tag{6}$$

$$\bar{r}_n^{(1)} = -\frac{\kappa_n^{(1)}}{r_n^{(1)}}, \tag{7}$$

$$q_n^{(1)} = -r_n^{(1)}, \tag{8}$$

$$\bar{q}_n^{(1)} = -\bar{r}_{n+1}^{(1)}. \tag{9}$$

Let us then introduce the new Hamiltonian $\mathcal{H}_2$ obtained from $\mathcal{H}_1$ by interchanging the operators $\mathcal{R}_1$ and $\mathcal{Q}_1$; that is, let us set

$$\mathcal{H}_2 = \mathcal{R}_1\mathcal{Q}_1 + \mu_1. \tag{10}$$

$\mathcal{H}_2$ will be referred to as the partner Hamiltonian of $\mathcal{H}_1$. By using Eqs. (4)–(9), from Eq. (10) it can be readily shown that $\mathcal{H}_2$ describes the Hamiltonian of a tight-binding lattice [i.e., it is of the form (1)] with hopping amplitudes and site energies $\{\kappa_n^{(2)}, V_n^{(2)}\}$ given by

$$\kappa_n^{(2)} = \kappa_n^{(1)}\frac{r_{n-1}^{(1)}}{r_n^{(1)}}, \tag{11}$$

$$V_n^{(2)} = V_n^{(1)} + \kappa_{n+1}^{(1)}\frac{\phi_{n+1}^{(1)}}{\phi_n^{(1)}} - \kappa_n^{(1)}\frac{\phi_n^{(1)}}{\phi_{n-1}^{(1)}}. \tag{12}$$

Note that, owing to the asymptotic behavior of $\kappa_n^{(1)}$, $V_n^{(1)}$, and $\phi_n^{(1)}$ at $n \to \pm\infty$, one has $\kappa_n^{(2)} \to \kappa$ and $V_n^{(2)} \to 0$ for $n \to \pm\infty$; that is, the partner lattice described by the Hamiltonian $\mathcal{H}_2$ is still a homogeneous lattice without defects at $n \to \pm\infty$. An interesting property of the Hamiltonian $\mathcal{H}_2$ is that its spectrum $\sigma^{(2)}$ is given by $\sigma^{(2)} = \sigma^{(1)} \cup \{\mu_1\}$; that is, it is the same as that of $\mathcal{H}_1$ plus the additional energy level μ_1 in the point spectrum. In fact, let us indicate by $|\psi_E\rangle = \sum_n \psi_n(E)|n\rangle$ a proper (or improper) eigenfunction of $\mathcal{H}_1$ with energy E. Note that, if E belongs to the point spectrum of $\mathcal{H}_1$, $|\psi_n(E)| \to 0$ as $n \to \pm\infty$, whereas if E belongs to the continuous spectrum of $\mathcal{H}_1$, $|\psi_n(E)|$ remains bounded as $n \to \pm\infty$. Since μ_1 does not belong to the point spectrum of $\mathcal{H}_1$, one has $E \neq \mu_1$. By using the factorization (3) for $\mathcal{H}_1$, the eigenvalue equation $\mathcal{H}_1|\psi_E\rangle = E|\psi_E\rangle$ reads explicitly

$$\mathcal{Q}_1\mathcal{R}_1|\psi_E\rangle = (E-\mu_1)|\psi_E\rangle, \tag{13}$$

from which it follows that $\mathcal{R}_1|\psi_E\rangle \neq 0$ since $E \neq \mu_1$. Applying the operator $\mathcal{R}_1$ to both sides of Eq. (13), one obtains

$$\mathcal{R}_1\mathcal{Q}_1|\tilde{\psi}_E\rangle = (E-\mu_1)|\tilde{\psi}_E\rangle, \tag{14}$$

that is, $\mathcal{H}_2|\tilde{\psi}_E\rangle = E|\tilde{\psi}_E\rangle$, where we have set $|\tilde{\psi}_E\rangle = \mathcal{R}_1|\psi_E\rangle$ or, explicitly [see Eq. (5)],

$$\tilde{\psi}_n(E) = r_n^{(1)}\psi_n(E) + \bar{r}_n^{(1)}\psi_{n-1}(E). \tag{15}$$

Therefore, $|\tilde{\psi}_E\rangle$ is an eigenfunction of $\mathcal{H}_2$ corresponding to the energy E. Also, from Eqs. (6), (7), and (15) and from the assumed asymptotic behavior of $\kappa_n^{(1)}$ and $V_n^{(1)}$ as $n \to \pm\infty$, it follows that $|\tilde{\psi}_E\rangle$ is a proper (improper) eigenfunction of $\mathcal{H}_2$ in the same way as $|\psi_E\rangle$ is a proper (improper) eigenfunction of $\mathcal{H}_1$. In a similar way, one can show that any eigenvalue E of $\mathcal{H}_2$, belonging to its continuous or to its point spectrum, is also an eigenvalue of $\mathcal{H}_1$ provided that $E \neq \mu_1$. Therefore the continuous and point spectra of $\mathcal{H}_1$ and $\mathcal{H}_2$ do coincide, apart from the energy level $E = \mu_1$, which needs a separate analysis. For $E = \mu_1$, the eigenvalue equation $\mathcal{H}_2|\psi\rangle = \mu_1|\psi\rangle$ can be satisfied by taking $\mathcal{Q}_1|\psi\rangle = 0$, which reads explicitly

$$q_n^{(1)}\psi_n + q_n^{(1)}\psi_{n+1} = 0. \tag{16}$$

By using the expressions of $q_n^{(1)}$ and $\bar{q}_n^{(1)}$ given by Eqs. (6)–(9), the difference Eq. (16) for ψ_n can be solved in a closed form, yielding

$$\psi_n = \frac{1}{\sqrt{\kappa_n^{(1)}\phi_n^{(1)}\phi_{n-1}^{(1)}}}. \tag{17}$$

In view of the asymptotic behaviors of $\phi_n^{(1)}$ and κ_n as $n \to \pm\infty$ and by assuming that $\phi_n^{(1)}$ does not vanish for any integer n, it turns out that ψ_n is bounded and $\psi_n \to 0$ as $n \to \pm\infty$; that is, $E = \mu_1$ belongs to the point spectrum of $\mathcal{H}_2$ and its eigenfunction is given by Eq. (17).

It should be noted that the synthesis of the partner Hamiltonian $\mathcal{H}_2$, with spectrum $\sigma_2 = \sigma_1 \cup \{\mu_1\}$, is not unique because of some freedom left in the choice of $\phi_n^{(1)}$ satisfying Eq. (2) once μ_1 has been fixed: Different choices of $\phi_n^{(1)}$ lead in fact to different lattice realizations of $\mathcal{H}_2$ (i.e., different values of hopping amplitudes $\kappa_n^{(2)}$ and site energies $V_n^{(2)}$).

The factorization method can be iterated to synthesize new Hamiltonians $\mathcal{H}_3$, $\mathcal{H}_4$, $\mathcal{H}_5, \ldots$, whose energy spectra differ from that of $\mathcal{H}_1$ owing to the addition of the discrete energy levels $\{\mu_1,\mu_2\}$, $\{\mu_1,\mu_2,\mu_3\}$, $\{\mu_1,\mu_2,\mu_3,\mu_4\}, \ldots$, with $|\mu_k| > 2\kappa$ $(k = 1,2,3,4,\ldots)$.

An interesting property, which is proven in Appendix A, is the following one. Let us assume $V_n^{(1)} = 0$ for the lattice Hamiltonian $\mathcal{H}_1$. Then a partner Hamiltonian $\mathcal{H}_{2N+1}$, obtained from $\mathcal{H}_1$ by adding $2N$ new energy levels $\mu_1,\mu_2,\mu_3,\ldots,\mu_{2N}$ with $\mu_2 = -\mu_1$, $\mu_4 = -\mu_3, \ldots, \mu_{2N} = -\mu_{2N-1}$, can be synthesized in such a way that $V_n^{(2N+1)} = 0$. This means that the partner lattice described by $\mathcal{H}_{2N+1}$ and supporting $2N$ bound states differs from the original one, defined by $\mathcal{H}_1$, because of different hopping rates κ_n between adjacent sites, but not for the site energies V_n.

As a final note, it should be mentioned that the technique of intertwining operators so far described could generate non-Hermitian lattice Hamiltonians with complex-valued hopping rates κ_n or site energies V_n, even though the initial Hamiltonian $\mathcal{H}_1$ is Hermitian. However, in spite of non-Hermiticity, the energy spectrum of such synthesized Hamiltonians remains by construction real-valued. This situation is especially interesting for the synthesis of invisible defects in the lattice, as discussed in Sec. IV.

III. TIGHT-BINDING LATTICES WITH REFLECTIONLESS DEFECTS

The intertwining operator technique presented in the previous section can be applied to the synthesis of lattices with reflectionless defects. Previous works have so far been limited to considering Hermitian lattices (see, for instance [11,13]); conversely, here we do not necessarily require that the partner Hamiltonians $\mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \ldots$, obtained by the iterated application of intertwining operator method, be self-adjoint. Notably, it will be shown in the next section that true invisibility of the defects requires the synthesis of non-Hermitian lattices. In this section, we first discuss the scattering properties of partner lattice Hamiltonians obtained by the intertwining operator technique, and then apply the results to the synthesis of reflectionless defects in the lattices.

A. Scattering properties of partner lattice Hamiltonians

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the Hamiltonians of the two partner tight-binding lattices defined by Eqs. (3) and (10). By construction, the two Hamiltonians have the same energy spectrum, except for an additional energy level μ_1 for $\mathcal{H}_2$. The two lattices are homogeneous (i.e., free of defects) at $n \to \pm\infty$; therefore, asymptotically they admit plane-wave solutions of the form $\sim\exp(\pm iqn)$, where q is the wave number that varies in the interval $0 \leqslant q < \pi$. Such plane waves belong to the common continuous spectrum of the Hamiltonians, with energy $E(q) = 2\kappa\cos(q)$. The reflection $[r_1(q), r_2(q)]$ and transmission $[t_1(q), t_2(q)]$ coefficients of the two lattices are defined by the asymptotic behavior of scattered waves at $n \to \pm\infty$ from a forward-incident plane wave $\sim\exp(-iqn)$ according to the relations [22]

$$\psi_n^{(1)} \sim \begin{cases} \exp(-iqn) + r_1(q)\exp(iqn), & n \to -\infty, \\ t_1(q)\exp(-iqn), & n \to \infty, \end{cases} \tag{18}$$

for $\mathcal{H}_1$, and

$$\psi_n^{(2)} \sim \begin{cases} \exp(-iqn) + r_2(q)\exp(iqn), & n \to -\infty, \\ t_2(q)\exp(-iqn), & n \to \infty, \end{cases} \tag{19}$$

for $\mathcal{H}_2$. Let us indicate by ω_1 the real-valued and positive solution to the equation

$$|\mu_1| = 2\kappa\cosh(\omega_1) \tag{20}$$

and let $\delta_1 = \mu_1/|\mu_1|$ (i.e., $\delta_1 = 1$ for $\mu_1 > 0$ and $\delta_1 = -1$ for $\mu_1 < 0$). It can then be proven that the following relations between transmission and reflection coefficients of the two partner Hamiltonians hold:

$$t_2(q) = t_1(q)\frac{\exp(-\omega_1/2) - \delta_1\exp(\omega_1/2 + iq)}{\exp(\omega_1/2) - \delta_1\exp(-\omega_1/2 + iq)}, \tag{21}$$

$$r_2(q) = r_1(q)\frac{\exp(\omega_1/2) - \delta_1\exp(-\omega_1/2 - iq)}{\exp(\omega_1/2) - \delta_1\exp(-\omega_1/2 + iq)}. \tag{22}$$

The proof of Eqs. (21) and (22) is given in Appendix B. Here we just notice that $|r^{(1)}(q)| = |r^{(2)}(q)|$ and $|t^{(1)}(q)| = |t^{(2)}(q)|$; that is, the transmittance and reflectance coefficients of the two partner lattices are the same. It should be noted that, as $|t^{(1)}(q)|^2 + |r^{(1)}(q)|^2 = 1$ for the Hermitian $\mathcal{H}_1$ lattice, it follows that $|t^{(2)}(q)|^2 + |r^{(2)}(q)|^2 = 1$ as wll, even if the partner Hamiltonian $\mathcal{H}_2$ is non-Hermitian. This result is a nontrivial one because it is known that unitarity of the scattering matrix in a generic non-Hermitian Hamiltonian is usually broken, and the reflection and transmission coefficients can be unbounded (see, for instance [23] and references therein).

By simple iteration, Eqs. (21) and (22) can be readily extended to the case of the partner Hamiltonian $\mathcal{H}_N$ obtained from $\mathcal{H}_1$ by adding the energy levels μ_1, $\mu_2, \ldots, \mu_N$. The reflection $[r_N(q)]$ and transmission $[t_N(q)]$ coefficients of the lattice described by $\mathcal{H}_N$ are given by

$$t_N(q) = t_1(q) \prod_{k=1}^{N} \frac{\exp(-\omega_k/2) - \delta_k \exp(\omega_k/2 + iq)}{\exp(\omega_k/2) - \delta_k \exp(-\omega_k/2 + iq)}, \quad (23)$$

$$r_N(q) = r_1(q) \prod_{k=1}^{N} \frac{\exp(\omega_k/2) - \delta_k \exp(-\omega_k/2 - iq)}{\exp(\omega_k/2) - \delta_k \exp(-\omega_k/2 + iq)}, \quad (24)$$

where ω_k is the positive root of the equation $2\kappa \cosh(\omega_k) = |\mu_k|$ and $\delta_k = \mu_k/|\mu_k|$ $(k = 1,2,3,\ldots,N)$.

B. Lattice with reflectionless defects

Reflectionless lattices containing localized defects are readily synthesized by assuming for $\mathcal{H}_1$ the Hamiltonian of a homogeneous and defect-free lattice ($\kappa_n^{(1)} = 1$, $V_n^{(1)} = 0$), for which $r_1(q) = 0$ and $t_1(q) = 1$. In fact, from Eq. (24) it follows that the reflection coefficient $r_N(q)$ of any partner Hamiltonian $\mathcal{H}_N$ vanishes, and the incident wave is fully transmitted through the lattice. Depending on the choice of the sequences $\phi_n^{(1)}, \phi_n^{(2)}, \phi_n^{(3)}, \ldots,$ the resulting partner Hamiltonian may be or may not be Hermitian.

1. Hermitian lattices

Examples of reflectionless and Hermitian lattices obtained by the application of the intertwining operator technique or by other techniques have been previously presented in [11–13]. The simplest case corresponds to the addition of a single energy level μ_1 outside the tight-binding band $-\kappa < E < \kappa$. If one assumes for instance $\mu_1 > \kappa$, Eq. (2) can be satisfied with the choice

$$\phi_n^{(1)} = \cosh[\omega_1(n-\alpha)], \quad (25)$$

which ensures the Hermiticity of the partner Hamiltonian $\mathcal{H}_2$. In Eq. (25), $\omega_1 = \operatorname{acosh}(\mu_1/2\kappa)$ and α is an arbitrary real parameter. The hopping amplitudes and site energies of the partner lattice read explicitly [see Eqs. (11) and (12)]

$$\kappa_n^{(2)} = \frac{\sqrt{\cosh[\omega_1(n-\alpha-2)]\cosh[\omega_1(n-\alpha)]}}{\cosh[\omega_1(n-\alpha-1)]}, \quad (26)$$

$$V_n^{(2)} = \frac{\cosh[\omega_1(n-\alpha+1)]}{\cosh[\omega_1(n-\alpha)]} - \frac{\cosh[\omega_1(n-\alpha)]}{\cosh[\omega_1(n-\alpha-1)]}. \quad (27)$$

Such a lattice, in spite of the presence of defects, is reflectionless and supports one bound state, given by [see Eq. (17)]

$$\psi_n = \frac{1}{\sqrt{\cosh[\omega_1(n-\alpha)]\cosh[\omega_1(n-\alpha-1)]}}. \quad (28)$$

Another example, which was recently proposed in Ref. [13], is provided by the partner lattice $\mathcal{H}_3$ obtained from the defect-free lattice $\mathcal{H}_1$ by adding the couple of energy levels $\mu_1 > \kappa$ and $\mu_2 = -\mu_1$ [24]. In this case, by assuming again for $\phi_n^{(1)}$ the expression given by Eq. (25), according to the analysis of Sec. II and Appendix A the hopping amplitudes of the Hermitian lattice $\mathcal{H}_3$ read explicitly [see Eq. (A6)]

$$\kappa_n^{(3)} = \sqrt{\frac{\cosh[\omega_1(n-\alpha)]\cosh[\omega_1(n-\alpha-3)]}{\cosh[\omega_1(n-\alpha-1)]\cosh[\omega_1(n-\alpha-2)]}}, \quad (29)$$

whereas $V_n^{(3)} = 0$ for the site energies. The lattice $\mathcal{H}_3$ is, by construction, reflectionless and supports two bound states. With the procedure outlined in the previous section,

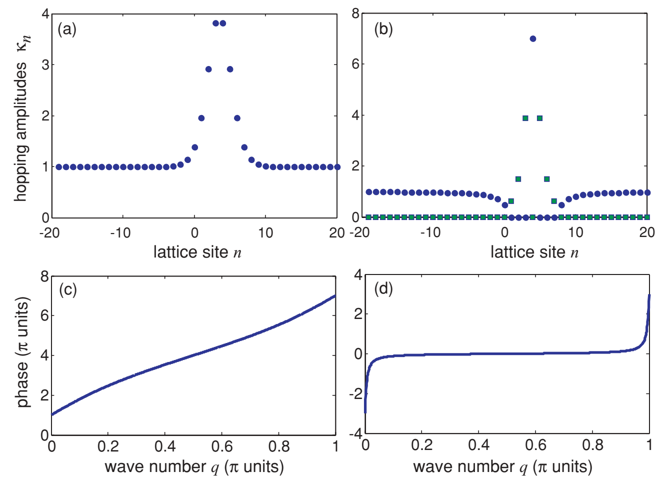

FIG. 1. (Color online) (a) Behavior of the hopping rates κ_n for a Hermitian lattice as predicted by Eq. (30) for parameter values $N = 3$, $\omega_1 = 0.6$, and $\alpha = 0$. (b) Behavior of the hopping rates κ_n for a non-Hermitian lattice as predicted by Eq. (35) for parameter values $N = 3$, $\omega_1 = 0.01$, and $\alpha = 0.5$. In the figure, the dots refer to $\mathrm{Re}(\kappa_n)$, whereas the squares refer to $\mathrm{Im}(\kappa_n)$. In (c) and (d) the behaviors of the phase of the transmission coefficient $t(q)$ of the two lattices are also depicted.

Hermitian lattices supporting an arbitrarily large number of bound states can be constructed in this way. A simple and noteworthy case, which generalizes the previous example, is provided by the lattice Hamiltonian $\mathcal{H}_{2N+1}$ obtained from the defect-free lattice $\mathcal{H}_1$ by adding the $2N$ energy levels $\mu_1 = 2\kappa\cosh(\omega_1)$, $\mu_2 = -\mu_1$, $\mu_3 = 2\kappa\cosh(2\omega_1)$, $\mu_4 = -\mu_3, \ldots, \mu_{2N-1} = 2\kappa\cosh(N\omega_1)$, $\mu_{2N} = -\mu_{2N-1}$. In this case, with the choice (25) for $\phi_n^{(1)}$, one can show that the hopping rates of the lattice $\mathcal{H}_{2N+1}$ take the simple form [11]

$$\kappa_n^{(2N+1)} = \sqrt{\frac{\cosh[\omega_1(n-\alpha)]\cosh[\omega_1(n-\alpha-2N-1)]}{\cosh[\omega_1(n-\alpha-N)]\cosh[\omega_1(n-\alpha-N-1)]}}, \quad (30)$$

which generalizes Eq. (29). An an example, Fig. 1(a) shows the behavior of the hopping rates k_n, as predicted by Eq. (30), for the case $N = 3$ and for $\omega_1 = 0.6$, $\alpha = 0$. As shown in the next section, even though such Hermitian lattices are reflectionless, they are not invisible owing to the energy dependence introduced by the bound states in the phase of the transmission coefficient.

2. Non-Hermitian lattices

A different choice of the sequences $\phi_n^{(1)}, \phi_n^{(2)}, \ldots,$ can be used to synthesize reflectionless non-Hermitian lattices. The simplest case corresponds, as in the previous Hermitian case, to the addition of a single energy level μ_1 outside the tight-binding band $-\kappa < E < \kappa$.

Let us assume, for the sake of definiteness, $\mu_1 > \kappa$ and let us make the choice [which replaces Eq. (25)]

$$\phi_n^{(1)} = \sinh[\omega_1(n-\alpha)], \quad (31)$$

where $\omega_1 = \operatorname{acosh}(\mu_1/2\kappa)$ and α is an arbitrary real (but noninteger) parameter. The expressions of hopping amplitudes and site energies of the partner lattice Hamiltonian $\mathcal{H}_2$ are then given by

$$\kappa_n^{(2)} = \sqrt{\frac{\sinh[\omega_1(n-\alpha-2)]\sinh[\omega_1(n-\alpha)]}{\sinh^2[\omega_1(n-\alpha-1)]}}, \quad (32)$$

$$V_n^{(2)} = \frac{\sinh[\omega_1(n-\alpha+1)]}{\sinh[\omega_1(n-\alpha)]} - \frac{\sinh[\omega_1(n-\alpha)]}{\sinh[\omega_1(n-\alpha-1)]}, \quad (33)$$

which replace Eqs. (26) and (27), respectively. Note that, as the site energies $V_n^{(2)}$ are always real-valued, the hopping amplitudes $\kappa_n^{(2)}$ are not. Specifically, $\kappa_n^{(2)}$ becomes purely imaginary at the two lattice sites n satisfying the condition $\alpha < n < 2+\alpha$. Therefore, the partner Hamiltonian $\mathcal{H}_2$ is not Hermitian, in spite of its spectrum is real-valued by construction.

As a second example, let us synthesize the partner lattice $\mathcal{H}_3$ obtained from the defect-free lattice $\mathcal{H}_1$ by adding the couple of energy levels $\mu_1 > \kappa$ and $\mu_2 = -\mu_1$, assuming again for $\phi_n^{(1)}$ the expression given by Eq. (31). According to the analysis of Sec. II and Appendix A, the hopping amplitudes of the lattice $\mathcal{H}_3$ now read explicitly [compare with Eq. (29)]

$$\kappa_n^{(3)} = \sqrt{\frac{\sinh[\omega_1(n-\alpha)]\sinh[\omega_1(n-\alpha-3)]}{\sinh[\omega_1(n-\alpha-1)]\sinh[\omega_1(n-\alpha-2)]}}, \quad (34)$$

whereas $V_n^{(3)} = 0$ for the site energies. By construction, the lattice Hamiltonian $\mathcal{H}_3$ is reflectionless, has a real-valued energy spectrum, and supports two bound states, corresponding to the energies $E = \pm 2\kappa\cosh(\omega_1)$. However, an inspection of Eq. (34) indicates that $\mathcal{H}_3$ is not Hermitian because the hopping amplitudes $\kappa_n^{(3)}$ take an imaginary value at the two sites n satisfying the condition $\alpha < n < 1+\alpha$ and $2+\alpha < n < 3+\alpha$. More generally, with the choice (31), the non-Hermitian Hamiltonian $\mathcal{H}_{2N+1}$ admitting $2N$ bound states with energies $\mu_1 = 2\kappa\cosh(\omega_1)$, $\mu_2 = -\mu_1$, $\mu_3 = 2\kappa\cosh(2\omega_1)$, $\mu_4 = -\mu_3, \ldots, \mu_{2N-1} = 2\kappa\cosh(N\omega_1)$, $\mu_{2N} = -\mu_{2N-1}$ can be synthesized, corresponding to the hopping amplitudes [compare with Eq. (30)]

$$\kappa_n^{(2N+1)} = \sqrt{\frac{\sinh[\omega_1(n-\alpha)]\sinh[\omega_1(n-\alpha-2N-1)]}{\sinh[\omega_1(n-\alpha-N)]\sinh[\omega_1(n-\alpha-N-1)]}} \quad (35)$$

and site energies $V_n^{(2N+1)} = 0$. Note that the hopping amplitudes are purely imaginary at lattice sites n satisfying the conditions $\alpha < n < \alpha+N$ and $\alpha+N+1 < n < \alpha+2N+1$. As an example, Fig. 1(b) shows the behavior of the real and imaginary parts of the hopping amplitudes κ_n as given by Eq. (35) for $N = 3$, $\omega = 0.01$, and $\alpha = 0.5$.

One could wonder whether non-Hermitian tight-binding lattices with imaginary hopping amplitudes may describe wave transport in some physically realizable systems. Coupled optical waveguide structures with gain and/or loss regions have been recently proposed as experimentally accessible systems to mimic the dynamics of non-Hermitian lattices with complex-valued site energies (see, for instance [17–19,25]); however, the non-Hermitian lattices discussed in the previous examples require imaginary values of the hopping rates at some site energies, an issue which was not considered in such previous works. In Appendix C, it is shown that suitable *longitudinal* modulations of gain or loss and propagation constants in evanescently coupled optical waveguide arrays lead to an effective non-Hermitian lattice with imaginary hopping amplitudes that realizes the models discussed in this section.

IV. INVISIBILITY IN NON-HERMITIAN LATTICES

For a reflectionless lattice synthesized by the intertwining operator technique, the transmission coefficient as a function of the wave number q of the incident wave has the form $t(q) = \exp[i\varphi(q)]$, where according to Eq. (23) the phase $\varphi(q)$ is given by the sum of N contributions associated with each of the N bound states with energies $\mu_1, \mu_2, \ldots, \mu_N$, that is,

$$\varphi(q) = \sum_{k=1}^{N} \varphi_k(q), \quad (36)$$

where

$$\exp[i\varphi_k(q)] = \frac{\exp(-\omega_k/2) - \delta_k\exp(\omega_k/2 + iq)}{\exp(\omega_k/2) - \delta_k\exp(-\omega_k/2 + iq)}, \quad (37)$$

$\mu_k = 2\kappa\delta_k\cosh(\omega_k)$, $\delta_k = \mu_k/|\mu_k|$, and $\omega_k > 0$ ($k = 1,2,3,\ldots,N$). The behavior of $\varphi_k(q)$, for increasing values of ω_k and for $\delta_k = \pm 1$, is shown in Fig. 2. In case $\delta_k = 1$ [i.e., $\mu_k > 0$, see Fig. 2(a)], one has $\varphi_k(q) \simeq \pi + q$ for $\omega_k \gg 1$ and $\varphi_k(q) \to 0$ (mod 2π, $q \neq 0$) for $\omega_k \to 0^+$. Similarly, in case $\delta_k = -1$ [i.e., $\mu_k < 0$, see Fig. 2(b)], one has $\varphi_k(q) \simeq q$ for $\omega_k \gg 1$ and $\varphi_k(q) \to 0$ ($q \neq \pi$) for $\omega_k \to 0^+$. Note that, according to Eq. (36), the behavior of the overall phase $\varphi(q)$ is given by the superposition of the various terms $\varphi_k(q)$ and does not depend on whether the synthesized partner Hamiltonian $\mathcal{H}_N$ is Hermitian or non-Hermitian.

We now ask ourselves whether the defects in the partner lattice, in addition of being reflectionless, are also *invisible* to

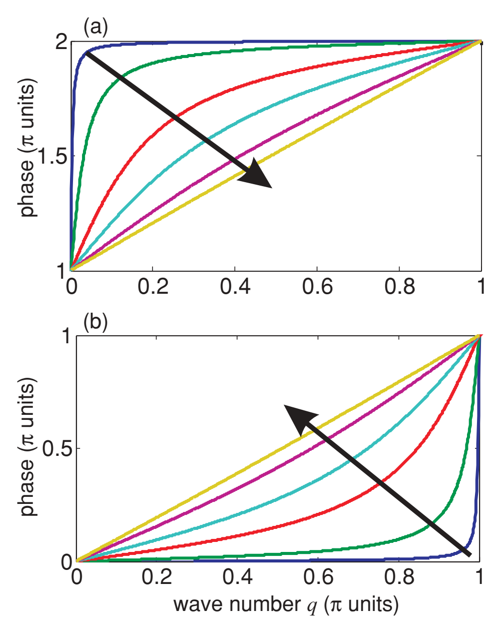

FIG. 2. (Color online) Behavior of the phase $\varphi_k(q)$ versus the wave number q (in units of π), defined by Eq. (37), for (a) $\delta_k = 1$ and (b) $\delta_k = -1$ and for increasing values of ω_k. The curves refer to the values $\omega_k = 0.01$, 0.1, 0.5, 1, 2, and 4. The arrows in the figures show the direction of increasing ω_k.

an outside observer. This condition requires that the phase $\varphi(q)$ of the transmission coefficient be flat, that is, that $(d\varphi/dq) = 0$ almost everywhere. If this condition is not satisfied, the spectral components of a wave packet crossing the defect region of the partner lattice would acquire the additional phase contribution $\varphi(q)$, absent in the defect-free lattice, which would be responsible for a *different* time of flight and for a *different* distortion of the wave packet as compared to the same wave packet propagating in the ideal defect-free lattice. Therefore, an outside observer could detect the existence of defects somewhere in the lattice by, for example, simple time-of-flight measurements. The *advance* in the time of flight experienced by the wave packet propagating in the partner lattice with defects can be readily calculated by standard methods of phase or group-delay time analysis and reads

$$\tau_g = \frac{1}{v_g}\left(\frac{d\varphi}{dq}\right)_{q_0} = \frac{1}{2\kappa \sin(q_0)}\left(\frac{d\varphi}{dq}\right)_{q_0}, \qquad (38)$$

where q_0 is the carrier wave number of the wave packet and $v_g = 2\kappa \sin(q_0) > 0$ is its group velocity. In particular, for a partner lattice synthesized by taking $\omega_k \gg 1$, one has $[d\varphi(q)/dq] \simeq N$ (see Fig. 2), and thus the advancement of the wave packet measured by an outside observer (i.e., far from the defect region) would be $\sim N/v_g$. Hence, comparing the time-of-flight measurements in the two lattices, the observer can estimate the number N of bound states of the partner lattice. From these considerations, it follows that the necessary and sufficient condition for a reflectionless lattice to be also invisible is that $\omega_k \to 0$. For a Hermitian lattice, from Eqs. (26) and (27) [and similarly from Eqs. (29) or (30)] it follows that in this limit the lattice becomes defect-free, that is, $\kappa_n \to 1$ and $V_n \to 0$ regardless of the value of the parameter α. This means that, for the Hermitian lattices synthesized in Sec. III B, the invisibility condition is the absence of defects. Conversely, from Eqs. (32) and (33) [and similarly from Eqs. (34) and (35)] it follows that, in the $\omega_k \to 0$ limit, κ_n and V_n do not tend to the values of the defect-free lattice [see, for instance, Fig. 1(b)]. This means that, in the non-Hermitian lattices synthesized in Sec. III B, invisibility of defects can be achieved. It should be noted that such non-Hermitian lattices with localized defects possessing a real-valued energy spectrum are not $\mathcal{PT}$ invariant; that is, $\mathcal{PT}$ symmetry is not of relevance for the achievement of invisibility of the defects.

We have checked these predictions by direct numerical simulations of wave packet propagation in Hermitian and non-Hermitian tight-binding lattices with zero site energies and with hopping amplitudes defined according to Eqs. (30) and (35), respectively. As an example, Fig. 3(a) shows the propagation of an initial Gaussian-shaped wave packet $|\psi(t=0)\rangle = \sum_n \mathcal{N}\exp[-(n+n_0)^2/w^2]\exp(-iq_0 n)|n\rangle$ in a Hermitian lattice with hopping rates given by Eq. (30) for parameter values $\kappa = 1$, $N = 3$, $\omega_1 = 0.6$, $\alpha = 0$, $n_0 = 70$, $w_0 = 10$, and $q_0 = \pi/2$ (where $\mathcal{N}$ is the normalization constant). The profile of hopping rates for this lattice was shown in Fig. 1(a). For comparison, Fig. 3(b) shows the propagation of the same wave packet in the defect-free lattice. The distribution of site occupation probabilities $P_n(t) = |\langle n|\psi(t)\rangle|^2$ at time $t = 70$ in the two cases is shown in Fig. 3(c). Note that, according to the previous analysis, the wave packet is fully transmitted in both

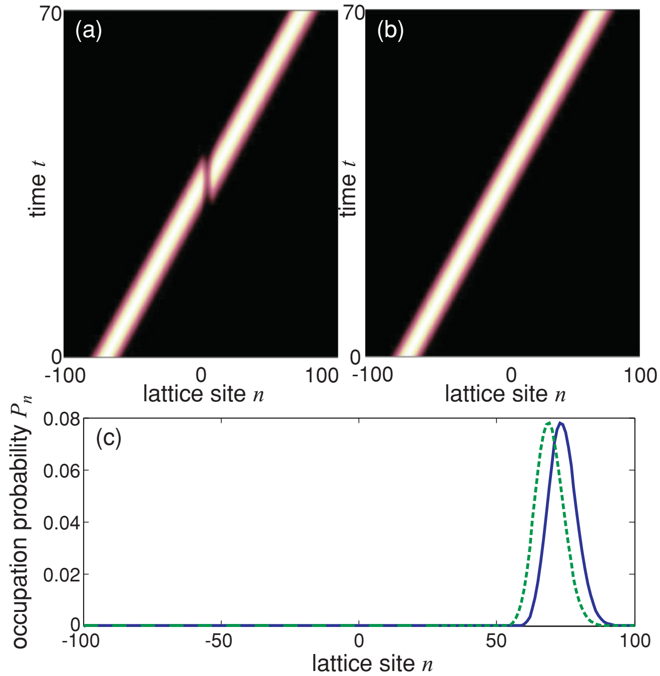

FIG. 3. (Color online) Propagation of an initial Gaussian-shaped wave packet [snapshot of the site occupation probabilities $P_n(t) = |\langle\psi(t)|n\rangle|^2$] (a) in the Hermitian lattice with hopping rates shown in Fig. 1(a) (with parameter values given in the text) and (b) in the defect-free lattice. In (c) the behaviors of site occupation probabilities $P_n(t)$ at time $t = 70$ in the two lattices are depicted. (The solid line refers to the Hermitian lattice with defects; the dashed line to the defect-free lattice.) Note the advancement experienced by the wave packet propagating in the lattice with defects. Such an advancement is basically ascribable to the increase of hopping rates κ_n in the defect region [see Fig. 1(a)].

lattices, and far from the inhomogeneities it propagates with group velocity $v_g = 2\kappa \sin(q_0) = 2$. However, in the lattice with defects the wave packet is advanced, as one can see clearly from an inspection of Fig. 3(c). The behavior of the phase $\varphi(q)$ of the transmission coefficient of the partner lattice corresponding to the simulation of Fig. 3(a) is shown in Fig. 1(c). One might think that, to make the Hermitian lattice invisible, one should reduce the value of ω_1; however, as discussed previously and as shown in Fig. 4, as ω_1 is diminished toward

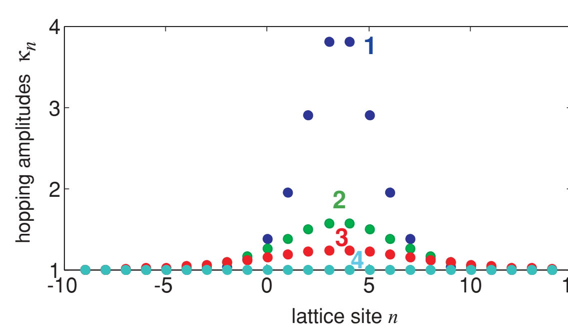

FIG. 4. (Color online) Behavior of the hopping rates κ_n for the Hermitian lattice, as given by Eq. (30), for $N = 3$, $\alpha = 0$ and for decreasing values of ω_1 (curve 1: $\omega_1 = 0.6$; curve 2: $\omega_1 = 0.3$; curve 3: $\omega_1 = 0.2$; curve 4: $\omega_1 = 0.02$).

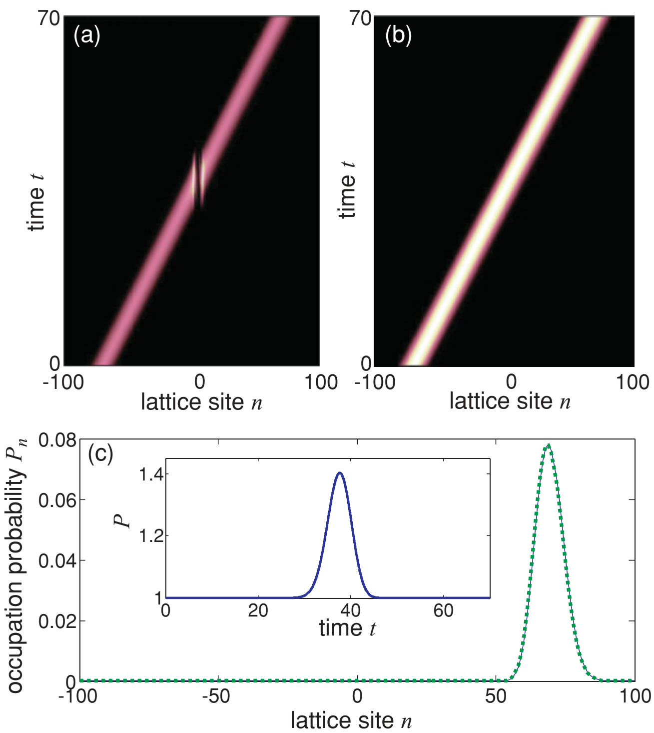

FIG. 5. (Color online) Propagation of an initial Gaussian-shaped wave packet [snapshot of the site occupation probabilities $P_n(t) = |\langle\psi(t)|n\rangle|^2$] (a) in the non-Hermitian lattice with hopping rates shown in Fig. 1(b) (with parameter values given in the text) and (b) in the defect-free lattice. In (c) the behaviors of site occupation probabilities $P_n(t)$ at time $t = 70$ in the two lattices are depicted. (The thin solid line refers to the non-Hermitian lattice with defects; the dashed line, almost overlapped with the solid one, refers to the defect-free lattice.) The inset in (c) shows the behavior of the total occupation probability $P(t) = \sum_n P_n(t)$ versus time in the non-Hermitian lattice of (a).

zero, the defects in the hopping amplitudes vanish and the lattice basically becomes defect-free. Conversely, Fig. 5 shows that a non-Hermitian lattice can be invisible yet present defects in the hopping amplitudes. Figure 5(a) shows the propagation of the same initial Gaussian-shaped wave packet $|\psi(t=0)\rangle = \sum_n \mathcal{N} \exp[-(n+n_0)^2/w^2]\exp(-iq_0 n)|n\rangle$ of Fig. 3, but in the non-Hermitian lattice with hopping rates given by Eq. (35) for parameter values $\kappa = 1$, $N = 3$, $\omega_1 = 0.01$, and $\alpha = 0.5$ [the distribution of hopping rates for this lattice being shown in Fig. 1(b)]. For comparison, Fig. 5(b) shows the propagation of the same wave packet in the defect-free lattice. The distribution of site occupation probabilities $P_n(t) = |\langle n|\psi(t)\rangle|^2$ at time $t = 70$ in the two cases is shown in Fig. 5(c). Note that, owing to the flatness of the phase $\varphi(q)$ for this lattice [see Fig. 1(d)], the wave packet is fully transmitted with no appreciable delay and/or distortion, as one can infer from an inspection of Fig. 5(c). An outside observer thus cannot distinguish whether the transmitted wave packet has been propagated in a defect-free or in an inhomogeneous lattice, and thus the defects in the non-Hermitian lattice are fully invisible. It should be finally noted that the total probability $P(t) = \sum_n P_n(t)$ is transiently not conserved in the non-Hermitian lattice, and it turns out to be amplified during interaction with defects, as shown in the inset of Fig. 5(c). Such an enhancement of the probability, however, is not visible to the outside observer.

V. CONCLUSIONS

In this work we have investigated theoretically the issue of invisibility of reflectionless tight-binding lattices with defects synthesized by the intertwining operator technique of supersymmetric quantum mechanics. As for Hermitian lattices the defects are not invisible and time-of-flight measurements of wave packets crossing the defect region may reveal their existence; in this work it has been shown that, in a certain class of non-Hermitian lattices with complex hopping amplitudes, the defects may appear fully invisible to an outside observer. In spite of non-Hermiticity, such lattices have a real-valued energy spectrum. As discussed in Appendix C, arrays of evanescently coupled optical waveguides with suitable longitudinal modulation of loss or gain coefficients and propagation constants could provide a physically realizable system to test invisibility in non-Hermitian tight-binding lattices.

ACKNOWLEDGMENTS

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APPENDIX A

In this Appendix, the following theorem is proved:

Let $\mathcal{H}_1$ be a tight-binding Hamiltonian with $V_n^{(1)} = 0$, and let $\mathcal{H}_{2N+1}$ be a partner Hamiltonian synthesized from $\mathcal{H}_1$ by adding $2N$ new energy levels $\mu_1, \mu_2, \mu_3, \ldots, \mu_{2N}$, with $\mu_2 = -\mu_1, \mu_4 = -\mu_3, \ldots, \mu_{2N} = -\mu_{2N-1}$. Then $\mathcal{H}_{2N+1}$ can be constructed in such a way that $V_n^{(2N+1)} = 0$.

Let us first prove the theorem for $N = 1$. The partner Hamiltonian $\mathcal{H}_2$ is first constructed following the procedure described in Sec. II, and the corresponding hopping amplitudes $\kappa_n^{(2)}$ and site energies $V_n^{(2)}$ are given by Eqs. (11) and (12), respectively, with $V_n^{(1)} = 0$. To synthesize the Hamiltonian $\mathcal{H}_3$, we need to construct the sequence $\phi_n^{(2)}$ satisfying the difference equation

$$\kappa_n^{(2)}\phi_{n-1}^{(2)} + \kappa_{n+1}^{(2)}\phi_{n+1}^{(2)} + V_n^{(2)}\phi_n^{(2)} = \mu_2\phi_n^{(2)} \tag{A1}$$

and apply again the intertwining operator technique after the factorization $\mathcal{H}_2 = \mathcal{Q}_2\mathcal{R}_2 + \mu_2$. In Eq. (A1), $\mu_2 = -\mu_1$ and the asymptotic behavior $|\phi_n^{(2)}| \to \infty$ for $n \to \pm\infty$ should be satisfied. A possible choice for the sequence $\phi_n^{(2)}$ can be obtained by observing that, since $V_n^{(1)} = 0$, from Eq. (2), it follows that $|\psi\rangle = \sum_n (-1)^n \phi_n^{(1)}|n\rangle$ satisfies the equation $\mathcal{H}_1|\psi\rangle = -\mu_1|\psi\rangle$, and thus $|\phi^{(2)}\rangle = \mathcal{R}_1|\psi\rangle$ satisfies the equation $\mathcal{H}_2|\phi^{(2)}\rangle = -\mu_1|\phi^{(2)}\rangle$, which is precisely Eq. (A1). Using Eqs. (6), (7), and (15) one obtains after some algebra

$$\phi_n^{(2)} = -2(-1)^n\sqrt{\kappa_n^{(1)}\phi_n^{(1)}\phi_{n-1}^{(1)}}. \tag{A2}$$

The hopping rates $\kappa_n^{(3)}$ and site energies of the partner Hamiltonian $\mathcal{H}_3 = \mathcal{R}_2\mathcal{Q}_2 + \mu_2$, obtained from $\mathcal{H}_2$ after changing the order of the operators $\mathcal{Q}_2$ and $\mathcal{R}_2$, are then given by [see Eqs. (11) and (12)]

$$\kappa_n^{(3)} = \kappa_n^{(2)}\frac{r_{n-1}^{(2)}}{r_n^{(2)}}, \tag{A3}$$

$$V_n^{(3)} = V_n^{(2)} + \kappa_{n+1}^{(2)}\frac{\phi_{n+1}^{(2)}}{\phi_n^{(2)}} - \kappa_n^{(2)}\frac{\phi_n^{(2)}}{\phi_{n-1}^{(2)}}, \tag{A4}$$

where

$$r_n^{(2)} = -\sqrt{\frac{\kappa_n^{(2)}\phi_{n-1}^{(2)}}{\phi_n^{(2)}}}. \tag{A5}$$

By using in Eq. (A5) the expressions of $\kappa_n^{(2)}$ defined by Eqs. (6) and (11), and of $\phi_n^{(2)}$ as given by Eq. (A2), substitution of Eq. (A5) into Eqs. (A3) and (A4) finally yields after some straightforward though lengthy algebra

$$\kappa_n^{(3)} = \sqrt{\frac{\kappa_n^{(1)}\kappa_{n-2}^{(1)}\phi_n^{(1)}\phi_{n-3}^{(1)}}{\phi_{n-1}^{(1)}\phi_{n-2}^{(1)}}}, \tag{A6}$$

$$V_n^{(3)} = 0. \tag{A7}$$

Therefore, for the partner Hamiltonian $\mathcal{H}_3$, obtained from $\mathcal{H}_1$ by adding the two energies μ_1 and $\mu_2 = -\mu_1$ with the procedure just described, one has $V_n^{(3)} = 0$. Starting from $\mathcal{H}_3$, one can repeat the procedure to construct a partner Hamiltonian $\mathcal{H}_5$ with $V_n^{(5)} = 0$ by adding to $\mathcal{H}_3$ the couple of eigenvalues μ_3 and $\mu_4 = -\mu_3$. The hopping amplitudes $\kappa_n^{(5)}$of the new Hamiltonian will be given by Eq. (A6), with $\kappa_n^{(1)}$ and $\phi_n^{(1)}$ replaced by $\kappa_n^{(3)}$ and $\phi_n^{(3)}$, respectively. By induction, it follows that a partner Hamiltonian $\mathcal{H}_{2N+1}$, obtained from $\mathcal{H}_1$ by adding N couples of energies $\{\mu_1,\mu_2 = -\mu_1\}$, $\{\mu_3,\mu_4 = -\mu_3\},\ldots,\{\mu_{2N-1},\mu_{2N} = -\mu_{2N-1}\}$, can be always synthesized to have $V_n^{(2N+1)} = 0$, which proves the theorem.

APPENDIX B

In this Appendix we prove Eqs. (21) and (22) given in the text relating the reflection and transmission coefficients of the two partner lattice Hamiltonians $\mathcal{H}_1$ and $\mathcal{H}_2$. To this aim, let us first consider the case $\mu_1 > 2\kappa$, and let us indicate by ω_1 the positive root of the equation $\mu_1 = 2\kappa\cosh(\omega_1)$. As $\kappa_n^{(1)} \to \kappa$ and $V_n^{(1)} \to 0$ at $n \to \pm\infty$, the asymptotic behavior of $\phi_n^{(1)}$, satisfying Eq. (2), is of the form

$$\phi_n^{(1)} \sim \begin{cases} \alpha\exp(\omega_1 n), & n \to +\infty, \\ \beta\exp(-\omega_1 n), & n \to -\infty, \end{cases} \tag{B1}$$

where α and β are two nonvanishing constants. From Eqs. (6), (7), (11), and (12) it then follows that

$$r_n^{(1)} \to -\exp(\mp\omega_1/2) \quad \text{for} \quad n \to \pm\infty, \tag{B2}$$

$$\bar{r}_n^{(1)} \to \exp(\pm\omega_1/2) \quad \text{for} \quad n \to \pm\infty, \tag{B3}$$

$$\kappa_n^{(2)} \to 1 \quad \text{for} \quad n \to \pm\infty, \tag{B4}$$

$$V_n^{(2)} \to 0 \quad \text{for} \quad n \to \pm\infty. \tag{B5}$$

Let us then indicate by $|\psi^{(1)}\rangle = \sum_n \psi_n^{(1)}|n\rangle$ the solution to the equation $\mathcal{H}_1|\psi^{(1)}\rangle = E|\psi^{(1)}\rangle$ corresponding to the scattering of a forward propagating plane wave (coming from $n \to -\infty$) with wave number q and energy $E = 2\kappa\cos(q)$ ($0 \leqslant q < \pi$). The eigenfunction $|\psi^{(1)}\rangle$ has therefore the asymptotic behavior expressed by Eq. (18) given in the text. According to Eq. (15), the function $|\psi^{(2)}\rangle = \sum_n \psi_n^{(2)}|n\rangle$ with

$$\psi_n^{(2)} = r_n^{(1)}\psi_n^{(1)} + \bar{r}_n^{(1)}\psi_{n-1}^{(1)} \tag{B6}$$

satisfies the equation $\mathcal{H}_2|\psi^{(2)}\rangle = E|\psi^{(2)}\rangle$. By using Eqs. (18), (B2), and (B3), it follows that the asymptotic behavior of $\psi_n^{(2)}$ is given by

$$\psi_n^{(2)} \sim \begin{cases} [-\exp(\omega_1/2) + \exp(-\omega_1/2 + iq)]\exp(-iqn) + r_1(q)[\exp(-\omega_1/2 - iq) - \exp(\omega_1/2)]\exp(iqn), & n \to -\infty, \\ t_1(q)[-\exp(\omega_1/2) + \exp(\omega_1/2 + iq)]\exp(-iqn), & n \to \infty; \end{cases} \tag{B7}$$

that is, $|\psi^{(2)}\rangle$ describes the scattering, in the lattice $\mathcal{H}_2$, of a plane wave with wave number q coming from $n \to -\infty$ and with amplitude $[-\exp(\omega_1/2) + \exp(-\omega_1/2 + iq)]$. From Eq. (B7), the transmission (t_2) and reflection (r_2) coefficients of the partner lattice $\mathcal{H}_2$ are readily calculated, yielding the expressions (21) and (22) given in the text with $\delta_1 = 1$.

Let us now consider the case $\mu_1 < -2\kappa$, and let us indicate again by ω_1 the positive root of the equation $2\kappa\cosh(\omega_1) = -\mu_1$. The asymptotic behavior of $\phi_n^{(1)}$, satisfying Eq. (2), is now of the form

$$\phi_n^{(1)} \sim \begin{cases} \alpha(-1)^n\exp(\omega_1 n), & n \to +\infty, \\ \beta(-1)^n\exp(-\omega_1 n), & n \to -\infty, \end{cases} \tag{B8}$$

where α and β are again two nonvanishing constants. In this case, the asymptotic behavior of $r_n^{(1)}$ and $\bar{r}_n^{(1)}$, as obtained from Eqs. (6), (7), and (B8), is given by

$$r_n^{(1)} \to -i\exp(\mp\omega_1/2) \quad \text{for} \quad n \to \pm\infty, \tag{B9}$$

$$\bar{r}_n^{(1)} \to -i\exp(\pm\omega_1/2) \quad \text{for} \quad n \to \pm\infty. \tag{B10}$$

As compared to the previous case $\mu_1 > 0$, from Eqs. (B6), (B9), and (B10) it follows that the asymptotic behavior of $\psi_n^{(2)}$ is now given by the equation

$$\psi_n^{(2)} \sim \begin{cases} -i[\exp(\omega_1/2) + \exp(-\omega_1/2 + iq)]\exp(-iqn) - ir_1(q)[\exp(-\omega_1/2 - iq) + \exp(\omega_1/2)]\exp(iqn), & n \to -\infty, \\ -it_1(q)[\exp(-\omega_1/2) + \exp(\omega_1/2 + iq)]\exp(-iqn), & n \to \infty, \end{cases} \tag{B11}$$

which replaces Eq. (B7). The transmission and reflection coefficients t_2 and r_2 of the lattice $\mathcal{H}_2$ are readily calculated from Eq. (B11), and their expressions are given by Eqs. (21) and (22) with $\delta_1 = -1$.

APPENDIX C

In this Appendix we briefly discuss a possible physical realization of non-Hermitian tight-binding lattices with complex hopping rates, such as those discussed in Secs. III B and IV. In the optical context, it is known that Hermitian lattices can be implemented by considering light propagation in arrays of evanescently coupled optical waveguides, the propagation direction z of light playing the role of time t in the quantum-mechanical problem (see, for instance [13,26]). The evolution along z of the modal amplitudes c_n of light trapped in the various waveguides of the array is governed by the tight-binding Hamiltonian (1), in which the site energies V_n and hopping amplitudes κ_n can be engineered by a suitable design of waveguide channel widths, index changes of the guiding cores, and distances between adjacent waveguides in the array. In ordinary arrays, that is, without loss or gain regions, V_n and κ_n turn out to be real-valued, and thus the Hamiltonian $\mathcal{H}$ is Hermitian. Non-Hermitian lattices with complex site energies can be mimicked by considering arrays of evanescently coupled waveguides in which light propagation in each waveguide is either absorbed or amplified by some loss or gain mechanism (see, for instance [19,25]), where the z-invariant gain or loss coefficients in the various waveguides determine the imaginary parts of the site energies V_n. Such non-Hermitian lattices have been intensively investigated in the past few years, especially in connection with $\mathcal{PT}$-symmetric quantum mechanics [17–19,25]. However, the non-Hermitian lattices that realize invisibility, discussed in Secs. III B and IV, have real-valued site energies V_n but imaginary hopping rates κ_n at some lattice sites. To implement in optics such invisible lattices, let us consider an array of evanescently coupled waveguides and assume that a suitable *longitudinal* and *periodic* modulation of both gain or loss coefficient and effective modal index, with spatial period Λ, is impressed to some waveguides in the lattice. In this case, coupled-mode equations describing the evolution of the modal amplitudes c_n of light trapped in the various waveguides read (see, for instance [26])

$$i\frac{dc_n}{dz} = \Delta_n c_{n-1} + \Delta_{n+1}c_{n+1} + [V_n + \beta_n(z) - i\gamma_n(z)]c_n, \tag{C1}$$

where Δ_n is the (real-valued) coupling rate between waveguides n and $n+1$, V_n is the propagation constant mismatch from a reference value, and $\beta_n(z)$, $\gamma_n(z)$ are the impressed longitudinal modulations of the propagation constant and loss or gain coefficient, respectively. We assume that both $\beta_n(z)$ and $\gamma_n(z)$ are periodic functions, with spatial period Λ and with zero mean. This means that, on average, a light field propagating in a *single* waveguide of the array would not be damped nor amplified. By assuming that the spatial period Λ of the modulation is much shorter than the typical coupling lengths ($\sim 1/\Delta_n$) and mismatch lengths ($\sim 1/V_n$), after introduction of the amplitudes

$$a_n(z) = c_n(z)\exp[i\varphi_n(z)], \tag{C2}$$

where

$$\varphi_n(z) = \int_0^z d\xi[\beta_n(\xi) - i\gamma_n(\xi)], \tag{C3}$$

a set of effective equations for the slowly varying amplitudes $a_n(z)$ can be derived by a multiple-scale analysis (see, for instance [27]). They read explicitly

$$i\frac{da_n}{dz} = \Delta_n\langle\exp[i\varphi_n(z) - i\varphi_{n-1}(z)]\rangle a_{n-1} + \Delta_{n+1}\langle\exp[i\varphi_n(z) - i\varphi_{n+1}(z)]\rangle a_{n+1} + V_n a_n, \tag{C4}$$

where $\langle\cdots\rangle$ denotes the average with respect to z over the spatial oscillation period Λ. Let us then assume the following:

(i) $\beta_n(z) = \rho_n\beta(z)$ and $\gamma_n(z) = \rho_n\gamma(z)$, where ρ_n can take the values 0 or 1. This means that some waveguides in the array are not modulated (those such that $\rho_n = 0$), whereas the modulated waveguides (those with $\rho_n = 1$) have the same modulation profiles of loss or gain and propagation constant, defined by the two real-valued functions $\gamma(z)$ and $\beta(z)$, respectively.

(ii) The modulation functions $\beta(z)$ and $\gamma(z)$ are chosen such that

$$\langle\exp[i\varphi(z)]\rangle = \langle\exp[-i\varphi(z)]\rangle = i\Gamma, \tag{C5}$$

where $\varphi(z)$ is defined by Eq. (C3) with $\beta_n = \beta(z)$ and $\gamma_n = \gamma(z)$, and Γ is a real-valued constant.

Under such assumptions, Eq. (C4) reduce to the following ones:

$$i\frac{da_n}{dz} = \kappa_n a_{n-1} + \kappa_{n+1}a_{n+1} + V_n a_n, \tag{C6}$$

where

$$\kappa_n = \begin{cases} \Delta_n & \text{if} \quad \rho_{n-1} = \rho_n, \\ i\Gamma\Delta_n & \text{if} \quad \rho_{n-1} \neq \rho_n, \end{cases} \tag{C7}$$

and Γ is defined by Eq. (C5). In this way, Eq. (C6) describe the dynamics in a tight-binding lattice with hopping amplitudes κ_n between adjacent sites $|n\rangle$ and $|n-1\rangle$ which can assume either real values (when the waveguides n and $n-1$ are both modulated or both not modulated) or purely imaginary values (when one of the two waveguides n or $n-1$ is modulated, but the other is not). The examples of reflectionless non-Hermitian lattices discussed in Secs. III B and IV belong to such a class of lattices. It should be noted that satisfaction of Eq. (C5) requires a proper choice of the modulation amplitudes for loss or gain and propagation constant profiles. For instance, let us assume a sinusoidal modulation

$$\gamma(z) = A_\gamma\cos(2\pi z/\Lambda), \quad \beta(z) = A_\beta\cos(2\pi z/\Lambda). \tag{C8}$$

In this case, from Eqs. (C3) and (C5) one obtains

$$i\Gamma = J_0\left(\frac{\Lambda(A_\beta - iA_\gamma)}{2\pi}\right) \tag{C9}$$

and hence the amplitudes A_β and A_γ must be chosen in such a way that the zero-order Bessel function J_0 at the complex argument $\Lambda(A_\beta + iA_g)/(2\pi)$ gives a purely imaginary value. There are several possibilities to satisfy such a condition; for instance, one could fix the product $A_\beta \Lambda$ and determine, correspondingly, the product $A_\gamma \Lambda$; for example, a choice can be

$$\frac{\Lambda A_\beta}{2\pi} \simeq 2, \quad \frac{\Lambda A_\gamma}{2\pi} \simeq 2.096, \tag{C10}$$

which yields $\Gamma \simeq 1.941$.

[1] J. B. Pendry, D. Schurig, and D. R. Smith, Science **312**, 1780 (2006).
[2] U. Leonhardt, Science **312**, 1777 (2006).
[3] D. Schurig, J. Mock, B. Justice, S. Cummer, J. Pendry, A. Starr, and D. Smith, Science **314**, 977 (2006).
[4] S. Zhang, D. A. Genov, C. Sun, and X. Zhang, Phys. Rev. Lett. **100**, 123002 (2008).
[5] I. Kay and H. E. Moses, J. Appl. Phys. **27**, 1503 (1956).
[6] P. G. Drazin and R. S. Johnson, *Solitons—An Introduction*, 2nd ed. (Cambridge University Press, Cambridge, 1989), Chap. 3.
[7] B. N. Zakharievy and V. M. Chabanov, Inverse Probl. **13**, R47 (1997).
[8] F. Cooper, A. Khare, and U. Sukhatme, *Supersymmetry in Quantum Mechanics* (World Scientific, Singapore, 2004); Phys. Rep. **251**, 267 (1995).
[9] In reflectionless potentials, synthesized by means of supersymmetric quantum mechanics, bound states introduced in the factorization procedure are responsible for the dependence of the phase of the transmitted wave on the energy [see, for instance, A. K. Grant and J. L. Rosner, J. Math. Phys. **35**, 2142 (1994)]. In this regard, it should be noted that invisibility of matter waves across nearly reflectionless and thin potentials, made of multiple thin rectangular or Pöschl-Teller barriers, has been recently proposed by S. Cordero and G. Garcia-Calderon [Phys. Rev. A **79**, 052103 (2009)]. Such scattering potentials do not sustain bound states and are not synthesized by inverse scattering or supersymmetric quantum mechanics techniques.
[10] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Mathematical Surveys and Monographs, Vol. 72 (American Mathematical Society, Providence, RI, 1970).
[11] V. Spiridonov and A. Zhedanov, Methods Appl. Anal. **2**, 369 (1995); Ann. Phys. (NY) **237**, 126 (1995).
[12] S. N. M. Ruijsenaars, J. Nonlinear Math. Phys. **8**, 106 (2001); **8**, 240 (2001).
[13] A. A. Sukhorukov, Opt. Lett. **35**, 989 (2010).
[14] N. Hatano and D. R. Nelson, Phys. Rev. Lett. **77**, 570 (1996); Phys. Rev. B **56**, 8651 (1997).
[15] F. Dogan, W. Kim, C. M. Blois, and F. Marsiglio, Phys. Rev. B **77**, 195107 (2008).
[16] M. S. Rudner and L. S. Levitov, Phys. Rev. Lett. **102**, 065703 (2009).
[17] O. Bendix, R. Fleischmann, T. Kottos, and B. Shapiro, Phys. Rev. Lett. **103**, 030402 (2009).
[18] L. Jin and Z. Song, Phys. Rev. A **80**, 052107 (2009); **81**, 032109 (2010).
[19] S. Longhi, Phys. Rev. B **80**, 165125 (2009); Phys. Rev. Lett. **103**, 123601 (2009).
[20] C. M. Bender, Rep. Prog. Phys. **70**, 947 (2007).
[21] It should be noted that the synthesis of *complex* reflectionless potentials for the continuous non-Hermitian Schrödinger equation, based on the application of the supersymmetric (Darboux) transformation, has been previously discussed by B. Bagchi, F. Cannata, and C. Quesne [Phys. Lett. A **269**, 79 (2000)]; F. Cannata, J.-P. Dedonder, and A. Ventura [Ann. Phys. (NY) **322**, 397 (2007)]; A. A. Andrianov, F. Cannata, and A. V. Sokolov [Nucl. Phys. B **773**, 107 (2007)]. The interplay bewteen $\mathcal{PT}$ symmetry and a class of reflectionless potentials has been also investigated by Z. Ahmed, C. M. Bender, and M. V. Berry [J. Phys A **38**, L627 (2005)]. However, the application of the Darboux transformation to non-Hermitian lattices and the issue of invisibility was not considerred in these previous works.
[22] For $\kappa > 0$ and $0 < q < \pi$, an asymptotic plane wave $\psi_n \sim \exp(-iqn)$ in the lattice propagates with a group velocity $v_g = -(dE/dq) = 2\kappa \sin(q) > 0$, and it is thus a progressive wave; conversely, a plane wave $\psi_n \sim \exp(iqn)$ propagates in the lattice with a group velocity $-2\kappa \sin(q)$, and it is thus a regressive wave.
[23] F. Cannata, J.-P. Dedonder, and A. Ventura, Ann. Phys. (NY) **322**, 397 (2007); A. Mostafazadeh, Phys. Rev. Lett. **102**, 220402 (2009).
[24] The reflectionless lattices proposed in Ref. [13] are a special case of a more general class of reflectionless potentials for the discrete Schrödinger equation introduced in the mathematic literature (see [11,12]).
[25] R. El-Ganainy, K. G. Makris, D. N. Christodoulides, and Z. H. Musslimani, Opt. Lett. **32**, 2632 (2007); K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani, Phys. Rev. Lett. **100**, 103904 (2008); A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, *ibid.* **103**, 093902 (2009); C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Nature Phys. **6**, 192 (2010).
[26] D. N. Christodoulides, F. Lederer, and Y. Silberberg, Nature (London) **424**, 817 (2003); S. Longhi, Laser Photon. Rev. **3**, 243 (2009).
[27] S. Longhi, Phys. Rev. B **77**, 195326 (2008).