Proposed test of macroscopic quantum contextuality

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We show that, for any system with a number of levels which can be identified with n qubits, there is an inequality for the correlations between three compatible dichotomic measurements which must be satisfied by any noncontextual theory, but is violated by any quantum state. Remarkably, the violation grows exponentially with n, and the tolerated error per correlation also increases with n, showing that state-independent quantum contextuality is experimentally observable in complex systems.

I. INTRODUCTION

Quantum mechanics (QM) is universal, that is, applicable to any physical system on which experiments can be made. However, a widely held opinion is that the kinds of phenomena that makes QM striking, like the nonexistence of definite outcomes and the need of superpositions to describe the state of a system, are relevant only for relatively simple "microscopic" systems. The more complex the system is, the greater the effect of noise and decoherence; thus these quantum phenomena soon become unobservable. In this paper we describe how one of these phenomena, quantum contextuality, is "easier" to observe in a complex system than in a simple one.

Quantum contextuality, that is, the impossibility of reproducing QM with noncontextual hidden variable (HV) models in which observables have preassigned results which are independent of any compatible measurements being carried out on the same system, was independently discovered by Kochen and Specker (KS) [1,2], and Bell [3]. Noncontextuality is motivated by the observation that, when two compatible observables are sequentially measured any number of times on the same system, their results do not change. Locality is a particular form of noncontextuality supported by the assumption that the result of a measurement does not depend on spacelike separated events. An interesting feature of quantum contextuality is that, while quantum nonlocality is specific for some states of composite systems, quantum contextuality can be observed for any state of any system with more than two compatible observables. Indeed, state-independent quantum contextuality (SIQC) is experimentally testable [4,5]. So far, it has only been observed in two-qubit systems: pairs of ions with two internal levels [6], polarization and path of single photons [7], and two-qubit nuclear magnetic resonance systems [8]. Related quantum contextuality experiments for specific states have been performed using also two-qubit systems: spin and path of single neutrons [9], and polarization of two-photon systems [10]. A natural question is whether SIQC can be observed in larger physical systems. In order to answer it, in this paper we address two problems: (a) Given an n-qubit system, and for three sequential measurements, is there a better inequality to observe SIQC, in the sense of providing a violation more resistant against experimental imperfections? (b) How does the robustness against imperfections scale with n?

II. RESISTANCE TO IMPERFECTIONS

We will focus on inequalities of the form [4,6–8]

$$\chi := \sum_{i=1}^{S} \langle \mathcal{C}_i \rangle - \sum_{i=S+1}^{N} \langle \mathcal{C}'_i \rangle \leqslant b, \qquad (1)$$

where $\langle \mathcal{C}_i \rangle$ and $\langle \mathcal{C}'_i \rangle$ are the mean values of the product of three compatible observables measured sequentially. A set of compatible observables is called a context. An experimental test of (1) requires testing N contexts: S of them are "positive" contexts, defined as those in which the product of the operators which represent the observables is the identity $\mathbb{1}$ (therefore, according to QM, the product of the outcomes must be 1); $N - S$ of them are "negative" contexts, defined as those in which the product is $-\mathbb{1}$ (therefore, the product of the outcomes must be -1). b is the upper bound of χ in any noncontextual HV theory; it can be obtained by examining all possible combinations of noncontextual outcomes. Alternatively, it can be easily seen that $b = 2s - N$, where s is the maximum number of quantum predictions which can be simultaneously satisfied by a noncontextual HV model. For instance, in the inequality tested in [6–8], there are $N = 6$ predictions and a noncontextual HV model can only satisfy $s = 5$ of them at most. Therefore, the bound is $b = 4$. The quantum prediction (assuming an ideal experiment without imperfections) maximally violates inequality (1), that is, $\chi_{QM} = N$. In practice, the experimental values are in the range $\chi_{expt} = 5.2$–5.5 [6–8], instead of $\chi_{QM} = 6$, since the experimental values for the correlations are $|\langle \mathcal{C}_i \rangle| = 1 - \epsilon_i$, with $0 < \epsilon_i \ll 2$, instead of the quantum predictions for an ideal experiment, $|\langle \mathcal{C}_i \rangle| = 1$; therefore, $\chi_{QM} \rightarrow \chi_{expt} = \chi_{QM} - \sum_{i=1}^{N} \epsilon_i$. The experiments do not reach χ_{QM} for different reasons, for example, nonperfect unitary operations and entangling gates [6], and nonperfect alignment of the interferometric setups [7].

Experimental imperfections can also be interpreted as a failure of the assumption of perfect compatibility under which the bound b is valid, and force us to correct this bound. This correction takes the form $b \rightarrow b' = b + \sum_{i=1}^{N} \phi_i$, where $\phi_i > 0$ can be obtained from additional experiments [11].

Assuming that all correlations are affected by similar errors, i.e., that $\sum_{i=1}^{N} \epsilon_i = N\epsilon$ and $\sum_{i=1}^{N} \phi_i = N\phi$, we can define the error per correlation as $\varepsilon = \epsilon + \phi$. A natural measure of robustness of a quantum violation of the inequality (1) against

imperfections is the tolerated error per correlation, which can be expressed as

$$\varepsilon = \frac{\chi_{\text{QM}} - b}{N}. \tag{2}$$

If ϕ is negligible, then ε is the maximum difference that can be tolerated (still violating the inequality) between the experimental value of a correlation and the quantum value for an ideal experiment. In general, ϕ is not negligible, but is similar for experiments with sequential measurements of the same length, then ε is a good measure to compare the resistance to imperfections of inequalities involving correlations between the same number of measurements. A different argument supporting this conclusion can be found in [12].

The inequality tested in [6–8] involves correlations between sequential measurements of three observables on two-qubit systems and tolerates an error per correlation of $\varepsilon = 1/3 \approx 0.33$. However, none of these experiments on SIQC is free of the compatibility loophole [11]. With the imperfections in the sequential measurements of these experiments we would need an inequality tolerating $\varepsilon \approx 0.48$. The fact that some experiments using specific states are free of this loophole [6] suggests that there is no fundamental reason why we could not perform a loophole-free experiment on SIQC. One way to face the problem is to improve the experimental techniques. Another approach is to find inequalities with a higher degree of robustness against imperfections. Previously proposed inequalities are not good enough for this purpose. For instance, the inequality based on the proof of the KS theorem in dimension 3 with the fewest number of contexts [13] tolerates $\varepsilon = 2/17 \approx 0.12$. This explains why it is not reasonable to expect conclusive experimental violations of any of these inequalities and raises the problem of whether these hypothetical more robust inequalities do actually exist. A different approach to deal with the loopholes created by some of the assumptions in which inequality is based (perfect compatibility between sequential measurements or one-to-one correspondence between actual measurements and projective measurements) is to combine sequential measurements on a single system with additional measurements on a highly correlated distant ancillary system [14].

III. OPTIMAL TWO-QUBIT INEQUALITY

We will first construct a more robust inequality for two qubits, and then prove that this inequality is the one with the highest ε. For inequalities of the form (1), $\varepsilon = 2(N - s)/N$. Therefore, to obtain a more robust inequality, we must increase the ratio of predictions which cannot be satisfied with a noncontextual HV model. This can be done as follows. Let us start with the proof of the KS theorem in Table I. It has nine observables and $N = 6$ predictions, but only $s = 5$ of them can be reproduced with a noncontextual HV model. Therefore, the corresponding inequality is

$$\sum_{i=1}^{3} \langle R_i \rangle - \sum_{j=1}^{3} \langle C_j \rangle \leqslant 4, \tag{3}$$

and tolerates $\varepsilon = \frac{1}{3} \approx 0.33$, since the quantum prediction is 6.

TABLE I. Proof of the KS theorem for two qubits. $X_1 Y_2 = \sigma_x^{(1)} \otimes \sigma_y^{(2)}$. Each row R_i contains a positive context and each column C_i a negative one. It is impossible to assign predefined noncontextual outcomes (-1 or $+1$) to the nine observables and satisfy the six predictions of QM.

	$C_1 :=$	$C_2 :=$	$C_3 :=$	$\prod$
$R_1 :=$	$X_1 X_2$	$Y_1 Z_2$	$Z_1 Y_2$	$= \mathbb{1}$
$R_2 :=$	$Y_1 Y_2$	$Z_1 X_2$	$X_1 Z_2$	$= \mathbb{1}$
$R_3 :=$	$Z_1 Z_2$	$X_1 Y_2$	$Y_1 X_2$	$= \mathbb{1}$
$\prod$	$= -\mathbb{1}$	$= -\mathbb{1}$	$= -\mathbb{1}$	

If we preserve one row and one column of Table I, we can construct a proof of the KS theorem by adding single-qubit observables. For example, preserving R_2 and C_1, we obtain the proof in Table II [15,16], which leads to the inequality tested in [6–8].

From Table I we can obtain eight other tables by keeping a row and a column and adding single-qubit observables. If we consider the resulting ten tables (Table I plus the nine ones like Table II), we have a proof of the KS theorem involving 15 observables, 15 contexts, and ten critical proofs of the KS theorem. The point is that, in this proof the maximum number of predictions that can be simultaneously satisfied using a noncontextual HV model is 12 out of 15. Therefore, if we consider the inequality involving all 15 contexts,

$$\sum_{i=1}^{3} \langle R_i \rangle + \sum_{p,q \in \{X,Y,Z\}} \langle L_{pq} \rangle - \sum_{j=1}^{3} \langle C_j \rangle \leqslant 9, \tag{4}$$

then, $\varepsilon = 0.4$, since the quantum prediction is 15. Indeed, by checking all possibilities, it can be seen that (4) is the inequality which tolerates the maximum amount of errors, under the assumptions that only sequences of three measurements are considered, that these measurements are of the form $\sigma_i^{(1)} \otimes \sigma_j^{(2)}$, where $i, j = 0, x, y, z$ and $\sigma_0 = \mathbb{1}$, and that all the correlations appear with the same weight.

IV. INEQUALITY FOR n-QUBIT SYSTEMS

How does the resistance to imperfection scale with the number of qubits when the experiments are limited to sequences of *three* measurements? The natural way to scale up inequality (4) to a system of $n > 2$ qubits is by considering all n-qubit observables represented by n-fold tensor products of the form $M_1 \otimes \cdots \otimes M_n$, where M_i is either the 2×2 identity matrix $I = \sigma_0$ or one of the Pauli matrices $X = \sigma_x$, $Y = \sigma_y$, and $Z = \sigma_z$. For a given n, there are $4^n - 1$ observables. For simplicity's sake, we will use the following notation:

TABLE II. Proof of the KS theorem tested in [6–8].

	$C_1 :=$	$L_{ZX} :=$	$L_{XZ} :=$	$\prod$
$L_{XX} :=$	$X_1 X_2$	X_2	X_1	$= \mathbb{1}$
$R_2 :=$	$Y_1 Y_2$	$Z_1 X_2$	$X_1 Z_2$	$= \mathbb{1}$
$L_{ZZ} :=$	$Z_1 Z_2$	Z_1	Z_2	$= \mathbb{1}$
$\prod$	$= -\mathbb{1}$	$= \mathbb{1}$	$= \mathbb{1}$	

$IXYZ = \sigma_0 \otimes \sigma_x \otimes \sigma_y \otimes \sigma_z$. We also need to consider all possible contexts such that the product of three compatible observables is $\mathbb{1}$ (positive contexts) or $-\mathbb{1}$ (negative contexts). Hereafter, by "observables" and "contexts" we will mean these kinds of observables and contexts.

Lemma 1. For a given n, the total number of contexts is

$$N(n) = \tfrac{1}{3}(4^n - 1)(4^{n-1} - 1). \quad (5)$$

Proof. $N(n)$ is equal to the number of pairs of compatible observables $P(n)$ divided by 3 (since there are three pairs in every trio). There are $4^n - 1$ observables and each of them is compatible with $2(4^{n-1} - 1)$ observables. This can be proven as follows: two observables are compatible if and only if they have an even number (including zero) of qubits in which both have different Pauli matrices. For instance, XXX is compatible with XXI (since they have zero qubits with different Pauli matrices), and with XYZ (since they have two qubits with different Pauli matrices). Given an observable O, half of the 4^n observables (including $\mathbb{1}$) have an even number (including zero) of qubits with different Pauli matrices. Two of them are O itself and $\mathbb{1}$. Therefore, there are $\frac{4^n}{2} - 2$ compatible observables with O. Therefore, $P(n) = (4^n - 1)(4^{n-1} - 1)$. ∎

Lemma 2. For a given n, the number of negative contexts is

$$N(n) - S(n) = \frac{1}{6} \sum_{c=0}^{n-2} \sum_{a,b} \binom{n}{c}\binom{n-c}{a}\binom{n-c-a}{b}$$
$$\times 3^{2n-a-b-2c}, \quad (6)$$

where $a, b \geqslant 0$, $a + b$ is even, $\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor$ (where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x) is odd, and the sum extends to all a and b such that $a + b + c \leqslant n$.

Proof. We can represent each negative context by a three-row n-column table in which each row represents an n-qubit observable, and column i contains qubit i's Pauli matrices of the three observables. Then, each column belongs to one of the following classes: It has (i) three I's, (ii) one Pauli matrix and two I's, (iii) two identical Pauli matrices and one I, (iv) three different Pauli matrices in "counterclockwise" order (i.e., $X \rightarrow Y \rightarrow Z$), or (v) three different Pauli matrices in "clockwise" order (i.e., $X \rightarrow Z \rightarrow Y$). We will denote by a, b, c, and d the number of columns of the type (v), (iv), (i), and (ii), respectively. If the three observables are compatible, $a + b$ must be even. In order to form a negative context, $d = 0$ and $\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor$ must be odd. Then, Eq. (6) is the result of counting all possible tables with these restrictions. The factor $1/6$ is due to the fact that six different tables represent the same context. Factors 3^a and 3^b are due to the fact that there are three columns of the types (v) and (iv), and factor $3^{2(n-a-b-c)}$ is due to the fact that there are nine columns of the type (iii). ∎

Lemma 3. For a given n, any noncontextual HV theory must satisfy

$$\Xi(n) := \sum_{i=1}^{S(n)} \langle \mathcal{C}_i \rangle - \sum_{i=S(n)+1}^{N(n)} \langle \mathcal{C}'_i \rangle \leqslant 2S(n) - N(n). \quad (7)$$

Proof. The bound is two times the maximum number of predictions that a (noncontextual HV) model can reproduce minus the total number of predictions. The quantum predictions corresponding to the $S(n)$ positive contexts can be

reproduced with a model in which all the outcomes are $+1$. However, this model fails to reproduce the predictions for the $N(n) - S(n)$ negative contexts. Indeed, $S(n)$ is the maximum number of predictions that a model can reproduce: For $n = 2$, the $N(2) = 15$ contexts can be grouped in ten tables like Tables I and II. Each context appears in four of them, and none of the tables admit a model. In any model which reproduces $S(2) = 12$ predictions, any prediction which is not reproduced by the model is in one of these tables together with five predictions which are reproduced by the model. Therefore, no model can reproduce more predictions. For $n > 2$, the $N(n)$ predictions can also be grouped in tables like Tables I and II, such that none of these tables admit a model. Any positive (negative) context and any table in $n > 2$ can be univocally obtained from one of the 12 positive (three negative) contexts and ten tables in $n = 2$. Similarly, any model in $n > 2$ which satisfies $S(n)$ predictions can be obtained from a model in $n = 2$ which satisfies $S(2) = 12$ predictions. In any of these models for $n > 2$, any prediction which is not reproduced by the model belongs to a table with five predictions which are reproduced by the model. Therefore, no model can reproduce more predictions. ∎

According to QM, any state of n qubits violates inequality (7) by the same amount, $\Xi_{QM}(n) = N(n)$. The remarkable feature is that the degree of violation defined as the ratio between the quantum prediction and the bound for noncontextual HV theories, $D(n) = N(n)/[2S(n) - N(n)]$, grows *exponentially* with the number of qubits n (see Fig. 1). Similarly, the tolerated error per correlation goes as

$$\varepsilon(n) = \frac{2\,[N(n) - S(n)]}{N(n)}, \quad (8)$$

which is $\frac{2}{5} = 0.4$ for $n = 2$, $\frac{4}{7} \approx 0.57$ for $n = 3$, $\frac{424}{595} \approx 0.71$ for $n = 4$, $\frac{4720}{5797} \approx 0.81$ for $n = 5$, and approaches 1 as n approaches infinity (see Fig. 1). This happens because the ratio between the maximum number of quantum predictions which can be satisfied by a noncontextual HV model and the number of quantum predictions approaches $\frac{1}{2}$ as n approaches infinity. Indeed, if we eliminate any context in (7), then the resulting inequality presents a lower resistance to noise.
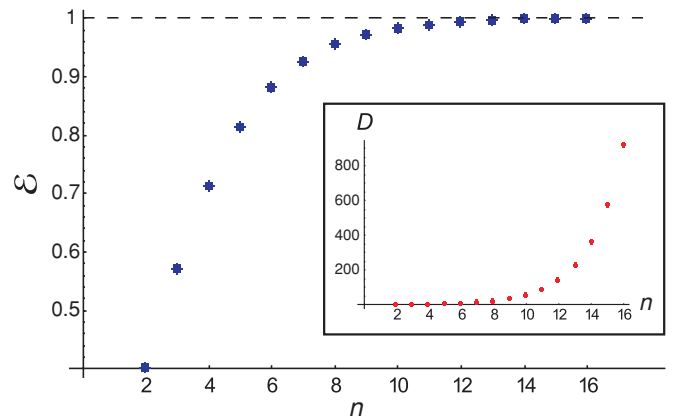

FIG. 1. (Color online) Tolerated error per correlation (still violating the inequality), ε, and degree of violation, D, of the inequality (7), as a function of the number of qubits, n.

V. EXPERIMENTAL IMPLICATIONS

A better choice for a loophole-free experiment of SIQC on a two-qubit system than the inequality tested in previous experiments [6–8] is inequality (4). This inequality requires to test 15 correlations, something which is feasible with the same techniques used in previous experiments.

For a given n, any n-qubit quantum state maximally violates (7), and the most interesting prediction is that the violation grows exponentially with n, and the tolerated error per correlation also grows with n. An experiment of inequality (7) for $n = 3$, requires to test 315 correlations, something which is within the range of what is experimentally feasible. Moreover, if we choose a system in which all three qubits are equivalent in the sense that the error in a measurement involving only qubits 1 and 2 is similar to the error in a similar measurement on any other pair of qubits, and in which any experiment ABC has a similar error, regardless of whether A, B, and C are X, Y, or Z, then the experiment can be simplified invoking this symmetry, since the 315 contexts can be divided in seven classes: (I) 27 like $\{XII, IXI, XXI\}$, (II) 81 like $\{XII, IXX, XXX\}$, (III) nine like $\{XXI, YYI, ZZI\}$, (IV) 27 like $\{XXI, XIX, IXX\}$, (V) 81 like $\{XXI, YZX, ZYX\}$, (VI) nine negative like $\{XXI, YYI, ZZI\}$, and (VII) 81 negative like $\{XXI, YYX, ZZX\}$, and a good estimation of the experimental violation of the inequality can be obtained by testing a few contexts of each class. A similar approach can be followed for $n = 4$. Actually, the most interesting experiment would be to test the exponentially-growing-with-size violation by testing the inequality (1) considering an increasing number of qubits on the same physical system. Such a test seems feasible with current technology.

VI. CONCLUSIONS

SIQC is universal in the sense that there is always an inequality valid for noncontextual HV models which is violated for any state of the system [5]. However, so far, for all previously known inequalities, the violation became smaller as the system became more complex, making the violation unobservable even for relatively simple systems, and suggesting that SIQC would be observable only on very simple systems, and that a loophole-free experiment on SIQC would not be feasible with current technology. We have shown that, on the contrary, the violation can increase along with the size of the system, making it observable for larger systems.

In contrast with Bell experiments, where an exponential violation requires correlations between n separated measurements on n distant qubits [17], and the violation decreases if the state suffers decoherence [18], the exponential violation of inequality (7) requires only correlations between three measurements, and is robust against decoherence.

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