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Adiabatic cluster-state quantum computing

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Models of quantum computation (QC) are important because they change the physical requirements for achieving universal QC. For example, one-way QC requires the preparation of an entangled "cluster" state, followed by adaptive measurement on this state, a set of requirements which is different from the standard quantum-circuit model. Here we introduce a model based on one-way QC but *without measurements* (except for the final readout), instead using adiabatic deformation of a Hamiltonian whose initial ground state is the cluster state. Our results could help increase the feasibility of adiabatic schemes by using tools from one-way QC.

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Computers that can exploit the laws of quantum theory can, in principle, outperform today's classical computers. For example, quantum computers can factor efficiently [1], something classical computers are thought to be incapable of doing. Motivated by this fact, a vast amount of ongoing research focuses on figuring out exactly how to build a quantum computer. In addition to different physical media for implementing QC, numerous different *models* for how to achieve QC have been proposed. While to date each of these models provides the same computational power, they differ substantially in the requirements they place on the physical hardware. The most widely used model of QC is the quantum-circuit model, but other models include one-way (or measurement-based) QC [2], holonomic QC [3], universal adiabatic QC [4], and topological QC [5]. Here we propose a model of computing which combines ideas from all of these models. In particular, we demonstrate how one can perform one-way QC adiabatically.

One-way QC [2] is a method for QC in which one creates a specific, fixed entangled state of a quantum many-body system and then computes via a series of local measurements on the subsystems. The choice of measurements corresponds to unitary gates enacted in the QC, and these measurements are adaptive: that is, the exact measurement being executed depends on the previous measurement results. One set of states which can be used for one-way QC is the class of so-called cluster states [6], which are defined with respect to a graph (though not all graphs allow for universal one-way QC). It is the ground state of a commuting Hamiltonian with at most $(d+1)$-qubit interactions, where d is the maximum degree of the graph. Importantly, one can replace this Hamiltonian with another involving only two-qubit interactions while retaining the cluster state as an approximate ground state [7]. Thus one could imagine engineering a physical system with this Hamiltonian, cooling the system to its ground state (which can be done efficiently [8]), and then doing measurements that enact the cluster-state QC. Here we show that, instead of performing these measurements, one can simply adiabatically turn on appropriate local fields while turning off portions of the cluster state to perform the QC. Thus we can dispense with measurements in the one-way model (except, of course, for the final readout) and instead use adiabatic evolutions to enact one-way QC. This model provides many of the advantages of adiabatic control; in particular, it retains robustness to deformations of the specific adiabatic path traversed during the open-loop holonomic evolution [9].

Adiabatic dragging. The main tools we use in this paper are adiabatic changes in a Hamiltonian. Suppose that initially we have a system with Hamiltonian H_i and the system is in an energy eigenstate. Then we evolve the system under a time-varying Hamiltonian over a time period $0 \leqslant t \leqslant T$ as $H(t) = f(s) H_i + g(s) H_f$, where $f(0) = g(1) = 1$, $f(1) = g(0) = 0$, and $s = \frac{t}{T}$ is a scaled time. If we vary this evolution smoothly and there are no level crossings, then it is always possible to choose a T large enough such that at the end of this evolution we will be in the eigenstate of H_f which is continuously connected to the initially prepared eigenstate. In particular, if we choose $1/T$ to be of the order of the minimum energy gap between the instantaneous eigenstate of $H(t)$ and the nearest eigenstates, then, with a high probability, at the end of the aforementioned evolution the system will be in the connected eigenstate of the final Hamiltonian [10]. We will call such a setup and evolution an *adiabatic dragging*. Recently, adiabatic dragging between Hamiltonians with energy eigenstates that are degenerate and are quantum-error-correcting code-word states has emerged as a powerful primitive for building a quantum computer [11,12]. Here we extend these ideas to one-way QC.

One-dimensional degenerate cluster-state model. Begin by considering a line of n qubits and a degenerate variation on the one-dimensional cluster state. In particular, define the $n-1$ commuting operators

$$S_i = [Z]_i[X]_{i+1}[Z]_{i+2}, \quad 1 \leqslant i \leqslant n-2,$$
$$S_{n-1} = [Z]_{n-1}[X]_n,$$

where X and Z are the corresponding Pauli operators and we use the notation $[P]_i$ to denote the operator P acting on the ith physical qubit. These are $n-1$ of the n operators usually used to define a cluster state [6]. These operators define a *stabilizer code* consisting of the subspace of states which all have $+1$ eigenvalue for (and hence are *stabilized* by) each of the S_i.

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Standard results in the theory of stabilizer codes [13] imply that this code space is two dimensional (encodes a qubit). We can define the logical operators for this encoded qubit as

$$\bar{X} = [X]_1[Z]_2 \quad \text{and} \quad \bar{Z} = [Z]_1. \tag{1}$$

Now consider the Hamiltonian

$$H_0 = -\Delta \sum_{i=1}^{n-1} S_i. \tag{2}$$

The ground-state subspace of this Hamiltonian is exactly the stabilizer code space already defined. Quantum information in this space can be accessed by measuring or manipulating the encoded Pauli operators, which are themselves localized on the first two qubits.

Now suppose that we adiabatically turn on a local field along the $-[X]_1$ direction while turning off the S_1 term in H_0, which anticommutes with $[X]_1$. In particular, consider adiabatic dragging from H_0 to $H_0 + \Delta(S_1 - [X]_1)$. Note that while $\bar{X}$ commutes with $[X]_1$, $\bar{Z}$ does not commute with $[X]_1$. However because we are in the $+1$ eigenspace of each S_i, instead of defining the logical $\bar{Z}$ as we have done in Eq. (1), we could also define the encoded Z as $\bar{Z}' = \bar{Z}S_1 = [X]_2[Z]_3$. If we do this, then the encoded qubit commutes with the terms we are turning on and off (S_1 and $[X]_1$). Thus the quantum information in this encoded subspace is not touched. However, since S_1 anticommutes with $[X]_1$, the information in S_1 is changed. To see how this evolution proceeds, we can consider a code in which we promote S_1 into an encoded Pauli Z operator and $[X]_1$ is its conjugate encoded X operator. The adiabatic evolution is then simply between these two encoded Pauli operators (i.e., from an encoded $-\Delta\bar{Z}_a$ to an encoded $-\Delta\bar{X}_a$, where a denotes this newly defined encoded qubit). Such an evolution has no level crossing and an energy gap for reasonable adiabatic interpolations which is proportional to Δ. Thus at the end of this evolution we will be in the $+1$ eigenstate of $[X]_1$ along with all the remaining S_i. In other words, we are in the stabilizer code with stabilizer generators $[X]_1, S_2, S_3, \ldots, S_{n-1}$. The information in the degenerate subspace, which originally was represented via the encoded operators $\bar{X} = [X]_1[Z]_2$ and $\bar{Z} = [Z]_1$, is now represented by $\bar{X}' = [X]_1[Z]_2$ and $\bar{Z}' = [X]_2[Z]_3$. However, since we are in the $+1$ eigenstate of $[X]_1$, this is equivalent to the encoded operator $\bar{X}'' = [Z]_2$ and $\bar{Z}'' = [X]_2[Z]_2$. In other words, the information which was originally encoded in the first two qubits, after the aforementioned adiabatic dragging, will be in the second and third qubits. Using the same logical Pauli encoding (logical X is $[X]_i[Z]_{i+1}$ and logical Z is $[Z]_i$), we see that a Hadamard gate has been applied to this information. Thus, by turning on an $[X]_1$ term on the first qubit while turning off the term in the Hamiltonian with which it anticommuted, we have effectively moved this information one step down the line and applied a Hadamard gate to the quantum information.

Proceeding inductively, if we first adiabatically turn on $[X]_1$, then $[X]_2$, etc., while turning off the corresponding anticommuting term in the original Hamiltonian, we will end up with the qubit which was originally localized to one end of the line moved to the other end of the line, along with a sequence of Hamadard gates applied to this qubit. Throughout this piecewise evolution the energy gap will remain constant because each successive adiabatic dragging acts independently. If we proceed to turn on each $[X]_i$ all the way up to the $(n-1)$st qubit, the information originally encoded into the first two qubits will end up exactly on the last qubit. In other words, after this evolution, $\bar{X}$ is mapped to $[X]_n$ and $\bar{Z}$ is mapped to $[Z]_n$ if the chain is odd length and $\bar{X}$ is mapped to $[Z]_n$ and $\bar{Z}$ is mapped to $[X]_n$ otherwise—with these differences arising from whether an even or an odd number of Hadamards has been applied to the encoded qubit.

Single-qubit gates. We now show how to modify the preceding setup such that, in addition to propagating a single qubit of information down the one-dimensional system, we also apply gates other than the Hadamard gate to the qubit. This scheme is motivated directly by the one-way QC model where, instead of measuring the qubit along the X direction to propagate the information, we measure along a rotated direction, $M(\theta) = \cos(\theta)X + \sin(\theta)Y$. Importantly, however, our scheme proceeds *without adaptive operations*. Consider mimicking the preceding scheme, but instead of turning on successive $-\Delta[X]_i$ values while turning off the appropriate anticommuting terms in H_0 (the $-\Delta[Z]_i[X]_{i+1}[Z]_{i+2}$ terms), we instead turn on successive $-\Delta M_i$ terms, where $M_i = [M(-\theta_i)]_i$ is a set of rotated local fields, $1 \leqslant i \leqslant n-1$. We claim that this will take the qubit localized to one end of the line and propagate it to the other end of the line while applying a gate dependent on the choice of θ_i.

To analyze this scheme it is easier to work in a frame of reference in which the ith qubit has been rotated by $U(\theta_i) = \exp(-i\theta_i[Z]_i/2)$. It is convenient to take $\theta_1 = 0$, which we now assume. Consider again n qubits on a line and now define the *rotated* stabilizer code operators:

$$\begin{aligned} T_i &= [Z]_i[X^{U_{i+1}}]_{i+1}[Z]_{i+2}, \quad 1 \leqslant i \leqslant n-2, \\ T_{n-1} &= [Z]_{n-1}[X]_n\,, \end{aligned} \tag{3}$$

where we use the superscript to denote conjugation, $P^U = UPU^\dagger$, and $U_i = U(\theta_i)$. Note that this conjugation does not change the fact that these operators commute and square to identity, and therefore we can again define a code space as the joint $+1$ eigenspace of these operators. Let H_0 be the initial Hamiltonian for our system as in Eq. (2), but now with the rotated stabilizer operators T_i substituted for S_i. Again, initially we can define the information in the degenerate subspace as being localized to the first two qubits, with $\bar{X} = [X]_1[Z]_2$ and $\bar{Z} = [Z]_1$. Now imagine adiabatically dragging H_0 to $H_0 + \Delta(T_1 - [X]_1)$, then dragging to $H_0 + \Delta(T_1 + T_2 - [X]_1 - [X]_2)$, etc. We claim that at the end of this scheme we will end up with the quantum information in $\bar{X}$ and $\bar{Z}$ propagated to the last qubit, with a gate dependent on θ_i applied to this information.

To see this, we proceed in three steps. First, we show that using the rotated stabilizer operators, it is possible to write the logical qubit in a form where each X_i (except $i = n$) commutes with this information. Define the following operators for $\alpha, \beta \in \{X, Y, Z\}$: $\bar{\alpha}_i = \sum_\beta (P_i^{\alpha,\beta}[\beta]_i)^{C_{i,i+1}}$, where $C_{i,i+1}$ is the controlled phase gate between the i and the $(i+1)$st qubits except when $i = n$, in which case we define $C_{n,n+1} = I$. We claim that these new Pauli operators are, under the

rotated stabilizer code generated by the T_i values, equivalent to the logical operators $\bar{X} = [X]_1[Z]_2$, $\bar{Y} = [Y]_1[Z]_2$, and $\bar{Z} = [Z]_1$, with the condition that the $P^{\alpha,\beta}$ values are the sum of products of $[X]_j$ operators for $j < i$. This can be proven inductively. The base case corresponds to $P^{\alpha,\beta} = \delta_{\alpha,\beta} I$, where $\bar{X}_1 = \bar{X}$ and $\bar{Z}_1 = \bar{Z}$. Now assume that the hypothesis is true for the ith operators. Examine, for example, $\bar{X}_i$ and expand the controlled phase:

$$\bar{X}_i = P_i^{X,X}[X]_i[Z]_{i+1} + P_i^{X,Y}[Y]_i[Z]_{i+1} + P_i^{X,Z}[Z]_i. \quad (4)$$

Recall that the T_i operators act as identity on the code space and thus can be inserted into this sum in any manner to yield any equivalent operator (over the code). Left multiplying $\bar{X}_i$ by T_i for the last two terms yields

$$\begin{aligned}\bar{X}_i = {} & P_i^{X,X}[X]_i[Z]_{i+1} + P_i^{X,Z}[X^{U_{i+1}}]_{i+1}[Z]_{i+2} \\ & - i\,P_i^{X,Y}[X]_i[X^{U_{i+1}}]_{i+1}[Z]_{i+1}[Z]_{i+2}. \end{aligned} \quad (5)$$

Expanding $X^{U_{i+1}}$, we find that

$$\begin{aligned} P_{i+1}^{X,X} &= \cos(\theta_{i+1})P_i^{X,Z} + \sin(\theta_{i+1})[X]_i P_i^{X,Y}, \\ P_{i+1}^{X,Y} &= \sin(\theta_{i+1})P_i^{X,Z} - \cos(\theta_{i+1})[X]_i P_i^{X,Y}, \\ P_{i+1}^{X,Z} &= [X]_i P_i^{X,X}. \end{aligned} \quad (6)$$

Similar relations hold for $\bar{Y}_{i+1}$ and $\bar{Z}_{i+1}$, with the important property that the new $P_{i+1}^{\alpha,\beta}$ values are functions of the previous $P_i^{\alpha,\beta}$ and $[X]_i$ values. This proves our statement.

But these expressions also prove much more. In particular, if we restrict the preceding equivalence to the $+1$ subspace of $[X]_i$, then we see (when we calculate all nine new $P_{i+1}^{\alpha,\beta}$ values) that the relationship between the $\bar{\alpha}_i$ and the $\bar{\alpha}_{i+1}$ is $\bar{\alpha}_{i+1} = \bar{\alpha}_i^{U_{i+1}H}$. In other words, with this restriction, the effect on the encoded quantum information in this new form is as if the gate $U_{i+1}H$ has been applied to the quantum information. Note, further that in the procedure we have described for adiabatically dragging the initial Hamiltonian, we are always turning off a $-\Delta T_i$ while turning on a $-\Delta[X]_i$. Thus not only does $[\alpha]_{i+1}$ commute with these terms (because the $P_{i+1}^{\alpha,\beta}$ is made up entirely of a product of $[X]_j$'s with $j < i+1$), and hence it is untouched by the evolution, but by an argument identical to the untwisted Hamiltonian case, we end each such dragging in the $+1$ eigenvalue of $-\Delta[X]_i$. Thus we end up exactly in the subspace where the gate $U_{i+1}H$ has been applied and the quantum information shifted one site down the chain for each such adiabatic dragging. The final effect of the turning-on of all $n-1$ $[X]_i$ in order is that the sequence of gates $H\prod_{i=n-2}^{2}(U_{i+1}H)$ is applied to the quantum information.

To recap, we have shown that by starting with a Hamiltonian which is a negative sum of twisted stabilizer operators T_i and then turning off the T_i's while turning on the $[X]_i$'s sequentially, we have enacted a gate which depends on the angles θ_i. This is equivalent to using the standard cluster-state Hamiltonian from Eq. (2) with the unrotated S_i stabilizer operators as the initial Hamiltonian and using rotated magentic fields $[M(-\theta_i)]_i$ for the piecewise final Hamiltonians. Note that we did not work in a rotating frame for the final qubit, and therefore the information ends up exactly in the last qubit of this evolution. Throughout this piecewise evolution the energy gap is constant (independent of the length of the chain). The gates enacted are universal for single-qubit gates.

State preparation. In the previous section we enacted gates on the degenerate ground state of a Hamiltonian. We now show how it is possible to prepare quantum information in a particular state, with the Hamiltonian nondegenerate, and then propagate the information down the line while turning the Hamiltonian into one with a degenerate ground state where this encoded information lives. Consider, for example, our original Hamiltonian in Eq. (2), but now with the full cluster-state Hamiltonian $H_0' = H_0 - \Delta S_0$, where $S_0 = [X]_1[Z]_2$. The ground state of H_0' is now not degenerate and corresponds, in our previous picture of H_0, to being in the $+1$ eigenstate of $\bar{X}$. Consider first adiabatically dragging H_0' to $H_0' + \Delta(S_1 - [X]_1)$. At the end of this evolution we will be in the $+1$ eigenspace of $[X]_1$ as before. Since we started in the $+1$ eigenspace of $\bar{X}$ we will be in the $+1$ eigenspace of $\bar{X}' = [Z]_2$. Next adiabatically drag the Hamiltonian to $H_0' + \Delta(S_0 + S_1 + S_2 - [X]_1 - [X]_2)$. Note that we have to turn off two stabilizer generators while turning on a single field. This implies that we must *increase the degeneracy* of the ground state. We see that this second dragging, despite increasing the degeneracy, ends with the system in the $+1$ eigenstate of the $\bar{X}'' = [X]_3[Z]_4$.

To see this, note that while both S_1 and S_2 do not commute with $[X]_2$, S_1S_2 does. Thus the eigenvalue of S_1S_2 is preserved while turning on $[X]_2$. If we then rewrite $S_1 + S_2$ as $S_1(I + S_1S_2)$, if we are in the -1 eigenspace of S_1S_2, this term vanishes, but if we are in the $+1$ eigenspace, in this space the operator effectively acts as $2S_1$ (or, equivalently, $2S_2$). We can then consider the code where we promote S_1 to an encoded Z operator and $[X]_2$ to an encoded X operator, and then at the end of the evolution we are in the $+1$ eigenstate of $[X]_2$, and we are also in the $+1$ eigenstate of S_1S_2 (due to this operator commuting with $[X]_2$). Translating this into the coding language, we are in the $+1$ eigenstate of a stabilizer code with generators $[X]_1,[X]_2,[X]_3[Z]_4,S_3,\ldots,S_n$, which is equivalent to saying that we are in the $+1$ eigenstate of the one-dimensional cluster state with $n-2$ qubits but prepared in the $+1$ eigenstate of the encoded $\bar{X}$ at one end of this chain. If we wish to apply gates to this information, we can proceed as before by applying rotated local fields or rotating the stabilizer Hamiltonian. It is important to realize that the preceding evolution has gone from a nondegenerate to a degenerate ground state, so that the energy gap vanishes. However, over the subspaces defined by the conserved quantity S_1S_2, the energy gap is *constant*. Therefore, the adiabatic theorem implies that a sufficiently large (but constant) amount of time is sufficient to guarantee that this evolution produces the desired outcome to within a fixed accuracy. In fact, the same situation occurs in the creation of anyon in topological QC [14].

Two-qubit gates. Let us now show how to apply two-qubit gates. The idea, just as in one-way QC, is to use a Hamiltonian which has a coupling between two chains that support single qubits. To see how this works let us analyze a cluster-state Hamiltonian with a degenerate ground subspace and a single coupling between two encoded qubits. Consider the six-qubit

initial Hamiltonian

$$H_2 = -\Delta([Z]_{1,a}[X]_{2,a}[Z]_{3,a}[Z]_{2,b} + [Z]_{2,a}[X]_{3,a}) + (a \leftrightarrow b),$$

where the encoded qubits will be associated with a and b, and $(a \leftrightarrow b)$ denotes the same term with the a and b labels reversed. This Hamiltonian is degenerate, but now there are two qubits of degeneracy, corresponding to logical operators $\bar{X}_\gamma = [X]_{1,\gamma}[Z]_{2,\gamma}$ and $\bar{Z}_\gamma = [Z]_{1,\gamma}$ with $\gamma \in \{a,b\}$. Now suppose that we turn on $-\Delta([X]_{1,a} + [X]_{1,b} + [X]_{2,a} + [X]_{2,b})$ while turning off H_2 (we could proceed by turning each of these on separately and achieve similar results). Using the four stabilizer terms in the preceding Hamiltonian we can rewrite the encoded operators as $\bar{X}'_\gamma = [X]_{1,\gamma}[X]_{3,\gamma}$ and $\bar{Z}'_\gamma = [X]_{2,\gamma}[Z]_{3,\gamma}[X]_{3,\neg\gamma}$, where $\neg a = b$ and $\neg b = a$. Using an argument similar to that for the single-qubit gates, we end up in the $+1$ eigenstate of the $X_{i,\gamma}$ operators, $i \in \{1,2\}$. Over this eigenspace, the logical operators become $\bar{X}_f = [X]_{3,\gamma}$ and $\bar{Z}_f = [Z]_{3,\gamma}[X]_{3,\bar{\gamma}}$. This is equivalent to performing a Hadamard on each encoded qubit, a controlled phase gate between them, then a Hadamard on each.

Adiabatic cluster-state QC. We now see how to build a quantum computer using piecewise adiabatic evolutions from a Hamiltonian whose ground state is a cluster state to a Hamiltonian consisting of local fields (we note that this initial state can also be piecewise adiabatically prepared [8]). Consider a quantum circuit made up of gates from a universal gate set such as $\{HU(\frac{\pi}{4}), H, (H \otimes H)C_{i,j}\}$ (other sets are also possible) along with the preparation in the $+1$ eigenstate of the Pauli X operator. Then one can map the graph of this circuit onto a cluster-state graph using the preceding elements in such a way that one can also prescribe local fields which, when turned on piecewise, enact the quantum circuit (or, equivalently, one can use a twisted cluster-state Hamiltonian and local fields all along X).

Conclusion. We have shown how to perform one-way QC on a cluster state using only piecewise adiabatic evolutions. This scheme shares many of the traits of the recently introduced primitive of adiabatic gate teleportation [12]: it has a robustness to the adiabatic path, for example. In light of connections between teleportation and one-way QC [15], it is perhaps not surprising that one can obtain an adiabatic scheme for one-way QC. However, we note that these prior connections exploited measurement-based equivalent circuits and did not work at a fundamental level with the stabilizer operator equivalences that were essential in our adiabatic construction. Further, as in [12] we can use perturbation theory gadgets [7] to implement this entire scheme using only two-qubit interactions instead of the four-qubit interactions we have presented. One way to do this calculation would proceed exactly as outlined in [7] and [12]. Our model shows the novelty of starting with a global entangled ground state and then piecewise turning on local fields to do QC. We have also shown how it is possible to use cluster states and their parent Hamiltonians to perform QC without resorting to adaptive measurements. Adiabatic cluster-state QC thus opens up a new way to adapt the numerous results of one-way QC to viable adiabatic architectures.

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