Probing light polarization with the quantum Chernoff bound

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We recall the framework of a consistent quantum description of polarization of light. Accordingly, the degree of polarization of a two-mode state $\hat{\rho}$ of the quantum radiation field can be defined as a distance of a related state $\hat{\rho}_b$ to the convex set of all SU(2)-invariant two-mode states. We explore a distance-type polarization measure in terms of the quantum Chernoff bound and derive its explicit expression. A comparison between the Chernoff and Bures degrees of polarization leads to interesting conclusions for some particular states chosen as illustrative examples.

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I. INTRODUCTION

Polarized states of the quantum electromagnetic field are basic resources in many experiments in quantum optics and quantum-information processing, e.g., Bell inequalities [1], quantum tomography [2], quantum cryptography [3,4], quantum teleportation [5,6], superdense coding [7], entanglement swapping [8], entanglement purification for quantum communication [9], and quantum computation [10].

In classical optics, the degree of polarization is defined in terms of the Stokes parameters [11]. The classical definition was adapted to quantum optics, where the Stokes parameters have been replaced by the expectation values of the Stokes operators [12]. However, this polarization measure contains only second-order correlations of the field, which are not sufficient for a complete description of all quantum-optics problems, where higher-order correlations play an important role. An idea to eliminate this drawback is due to Luis, who quantified the polarization in terms of the variance over S^2 of the SU(2) Q function for the given field state [13–15]. Alternatively, the degree of polarization has been defined as the minimal overlap between the given state and any state obtained from it via an SU(2) transformation [16,17]. Other attempts have been made to introduce a polarization measure for electromagnetic near fields by using the Gell-Mann matrices [18–20]. Recently the degree of polarization has been defined as a distance between the field state in question and the set of unpolarized states. Several metrics, e.g., the Hilbert-Schmidt and Bures metrics, have been used for evaluating the polarization of some field states [21–23].

In this work we introduce a distance-type degree of polarization defined in terms of the quantum Chernoff bound. In a seminal paper, Chernoff investigated the problem of discriminating two probability distributions and found an upper bound on the minimal error probability $P_{\min}^{(N)}$ in the asymptotic case ($N \to \infty$) [24]. This is known as the classical Chernoff bound and has many applications in statistical decision theory. After some 55 years, this bound was generalized to the quantum case. First, Ogawa and Hayashi proposed three promising candidates for a quantum expression [25]. After some other subsequent progress [26], the quantum Chernoff bound was proven to coincide with one of their formulas. This important result was established through the conjugate efforts of two groups of researchers: Nussbaum and Szkoła, and Audenaert *et al.* [27,28]. The quantum scenario is as follows: N identical copies of a quantum system are prepared in the same unknown state, which is either $\hat{\rho}$ or $\hat{\sigma}$. The task at hand is to determine the minimal probability of error by testing the copies in order to draw a conclusion about the identity of the state. When the two states are equiprobable, the minimal error probability of discriminating them in a measurement performed on N independent copies is [26,29]

$$P_{\min}^{(N)}(\hat{\rho},\hat{\sigma}) = \tfrac{1}{2}\left(1 - \tfrac{1}{2}\|\hat{\rho}^{\otimes N} - \hat{\sigma}^{\otimes N}\|_1\right), \tag{1.1}$$

where $\|\hat{A}\|_1 := \mathrm{Tr}\sqrt{\hat{A}^\dagger \hat{A}}$ is the trace norm of a trace-class operator $\hat{A}$. In the special case when both states are pure (denoted by $|\Phi\rangle$ and $|\Psi\rangle$), the minimal error probability (1.1) reads [26]

$$P_{\min}^{(N)}(|\Phi\rangle\langle\Phi|, |\Psi\rangle\langle\Psi|) = \tfrac{1}{2}\left(1 - \sqrt{1 - |\langle\Phi|\Psi\rangle|^{2N}}\right).$$

For an optimal asymptotic testing ($N \to \infty$), an upper bound $P_{\mathrm{QCB}}^{(N)}$ of the minimal probability of error (1.1) was found to decrease exponentially with N [28,29]:

$$P_{\mathrm{QCB}}^{(N)}(\hat{\rho},\hat{\sigma}) \sim \exp[-N\xi_{\mathrm{QCB}}(\hat{\rho},\hat{\sigma})], \quad (N \gg 1),$$

where the positive quantity

$$\xi_{\mathrm{QCB}}(\hat{\rho},\hat{\sigma}) := -\ln\Big[\min_{s\in[0,1]} \mathrm{Tr}(\hat{\rho}^s\hat{\sigma}^{1-s})\Big] \tag{1.2}$$

is called *quantum Chernoff bound* [27–29].

We find it convenient to introduce the function

$$Q(\hat{\rho},\hat{\sigma}) := \min_{s\in[0,1]} \mathrm{Tr}(\hat{\rho}^s\hat{\sigma}^{1-s}), \tag{1.3}$$

which is manifestly symmetric, $Q(\hat{\rho},\hat{\sigma}) = Q(\hat{\sigma},\hat{\rho})$, and is referred to in what follows as the quantum Chernoff overlap of the states $\hat{\rho}$ and $\hat{\sigma}$ [30]. Its maximal value is reached when the states $\hat{\rho}$ and $\hat{\sigma}$ coincide. In the body of the paper we intensively employ the quantities

$$Q_s(\hat{\rho},\hat{\sigma}) := \mathrm{Tr}(\hat{\rho}^s\hat{\sigma}^{1-s}), \tag{1.4}$$

which are the quantum analogs of the classical Rényi overlaps discussed in Ref. [31] as being distinguishability measures in their own right. According to Eqs. (1.2) and (1.3), their minimum over s determines the quantum Chernoff bound, which has many applications in various branches of physics. Calsamiglia *et al.* have employed it as a measure of distinguishability between qubit states and between single-mode Gaussian states of the radiation field [32]. Hiai *et al.* have analyzed the asymptotic discrimination of two states with measurements that are invariant under some symmetry group of the system [33]. Recently, the quantum Chernoff overlap was employed to evaluate the degree of nonclassicality for one-mode Gaussian states [30]. Pirandola and Lloyd have found upper bounds for the error probability of discrimination of Gaussian states of n bosonic modes [34]. They combined Minkowski's inequality and the quantum Chernoff bound and derived computable bounds. The quantum Chernoff bound was used for asymptotic discrimination between two states of an infinite-lattice system in the fermionic case [35], as well as in the bosonic one [36]. The quantum Chernoff bound is also applied to the theory of quantum phase transitions. Abasto *et al.* have evaluated the quantum Chernoff metric for the XY model at finite temperature [37]. By use of the quantum Chernoff bound, discrimination between two ground states or two thermal states of the one-dimensional quantum Ising model was recently addressed by Invernizzi and Paris [38].

The present article deals with two-mode states of the quantum radiation field. Its purpose is to investigate a distance-type degree of polarization that involves the quantum Chernoff overlap. The paper is organized as follows. In Sec. II we review the recently formulated requirements to be fulfilled by any acceptable measure of polarization [15,21]. We here insist on the physical significance of these general requirements that change the current view on the way of evaluating the degree of polarization for a two-mode state. Section III is devoted to the Chernoff degree of polarization for which a general formula is derived and discussed. A parallel treatment of the Bures degree of polarization is then presented. In Sec. IV the obtained formulas are specialized to pure states. The Chernoff and Bures degrees of polarization are compared for two families of states, each of them having just two nonvanishing photon-number probabilities. Our conclusions are outlined in Sec. V.

II. QUANTUM DEGREE OF POLARIZATION

The polarization transformations are an essential ingredient in linear optics. They are carried out by lossless linear optical devices while transmitting a quasimonochromatic light beam between a pair of planes transverse to its travel direction. We give here two examples. The first one is that of a compensator which introduces a phase difference between two perpendicular components of the oscillating electric field. A second device to be mentioned is called rotator because it produces a rotation of the electric-field vector about the beam propagation axis.

From the mathematical point of view, the class of linear polarization transformations is a group of unitary operators $\hat{U}_{\rm pol}$ on the two-mode Hilbert space $\mathcal{H}_H \otimes \mathcal{H}_V$. They are generated by three Stokes operators:

$$\hat{S}_1 := \hat{a}_H^\dagger \hat{a}_V + \hat{a}_H \hat{a}_V^\dagger, \quad \hat{S}_2 := \frac{1}{i}(\hat{a}_H^\dagger \hat{a}_V - \hat{a}_H \hat{a}_V^\dagger), \quad \hat{S}_3 := \hat{a}_H^\dagger \hat{a}_H - \hat{a}_V^\dagger \hat{a}_V, \tag{2.1}$$

built with the amplitude operators of the horizontal (H) and vertical (V) modes. Accordingly, the operators $\hat{U}_{\rm pol}$ form an infinite-dimensional unitary representation of the group SU(2) and can be parametrized in terms of the Euler angles ϕ, θ, ψ, as follows:

$$\hat{U}_{\rm pol}(\phi,\theta,\psi) = \exp\left(-i\frac{\phi}{2}\hat{S}_3\right)\exp\left(-i\frac{\theta}{2}\hat{S}_2\right)\exp\left(-i\frac{\psi}{2}\hat{S}_3\right). \tag{2.2}$$

Any SU(2) polarization transformation (2.2) preserves the total number of photons, which is described by the fourth Stokes operator,

$$\hat{S}_0 := \hat{a}_H^\dagger \hat{a}_H + \hat{a}_V^\dagger \hat{a}_V. \tag{2.3}$$

A state $\hat{\tau}$ that remains invariant under any polarization transformation (2.2) is unpolarized [39]. It is known for a long time that a two-mode state $\hat{\tau}$ is SU(2) invariant if and only if it has the spectral decomposition [39–42]

$$\hat{\tau} = \sum_{N=0}^{\infty} \pi_N \frac{1}{N+1} \hat{P}_N, \tag{2.4}$$

where

$$\hat{P}_N := \sum_{n=0}^{N} |n,N-n\rangle\langle n,N-n| \tag{2.5}$$

is the projection operator onto the vector subspace of the N-photon states, called the Nth excitation manifold. We have denoted $|n,N-n\rangle := |n\rangle_H \otimes |N-n\rangle_V$. Further, π_N are the photon-number probabilities in the SU(2)-invariant state $\hat{\tau}$, and they satisfy the normalization condition

$$\sum_{N=0}^{\infty} \pi_N = 1. \tag{2.6}$$

In order to describe the polarization properties of an arbitrary two-mode state $\hat{\rho}$, we make use of its photon-number-ordered Fock expansion

$$\hat{\rho} = \sum_{M=0}^{\infty}\sum_{N=0}^{\infty}\sum_{m=0}^{M}\sum_{n=0}^{N} |m,M-m\rangle\langle m,M-m| \times \hat{\rho}|n,N-n\rangle\langle n,N-n|. \tag{2.7}$$

The above expansion can be split into the sum of the block-diagonal terms ($M=N$) and that of the off-block-diagonal ones ($M \neq N$). The former sum is the block-diagonal density matrix $\hat{\rho}_b$ associated with the given state $\hat{\rho}$,

$$\hat{\rho}_b := \sum_{N=0}^{\infty} p_N \hat{\rho}_N. \tag{2.8}$$

In Eq. (2.8), p_N is the probability of the Nth excitation manifold

$$p_N = \mathrm{Tr}(\hat{\rho}\hat{P}_N) = \sum_{n=0}^{N} \rho_{nn}^{(N)}, \tag{2.9}$$

where

$$\rho_{mn}^{(N)} := \langle m, N-m|\hat{\rho}|n, N-n\rangle \tag{2.10}$$

are the entries of a positive semidefinite matrix $\rho^{(N)} \in \mathcal{M}_{N+1}(\mathbb{C})$. Further, $\hat{\rho}_N$ is an N-photon state determined by the matrix $\rho^{(N)}$ with a nonvanishing trace p_N:

$$\hat{\rho}_N := \frac{1}{p_N}\hat{P}_N\hat{\rho}\hat{P}_N = \frac{1}{p_N}\sum_{m=0}^{N}\sum_{n=0}^{N}|m, N-m\rangle\rho_{mn}^{(N)}\langle n, N-n|, \quad p_N > 0. \tag{2.11}$$

Recall now the requirements we need to quantify the polarization of a two-mode state $\hat{\rho}$. There are three conditions to be satisfied by its degree of polarization $\mathbb{P}(\hat{\rho})$ [23]:

(a) $\mathbb{P}(\hat{\rho}) = 0$ if and only if $\hat{\rho}$ is unpolarized. This is only natural: for an unpolarized state the degree of polarization vanishes, and, conversely, a state with zero degree of polarization is unpolarized.

(b) The degree of polarization is invariant under polarization transformations:

$$\mathbb{P}(\hat{U}_{\mathrm{pol}}\,\hat{\rho}\,\hat{U}_{\mathrm{pol}}^{\dagger}) = \mathbb{P}(\hat{\rho}). \tag{2.12}$$

(c) The degree of polarization is not affected by coherences between different excitation manifolds. In fact, all polarization properties of a given two-mode state $\hat{\rho}$ are not influenced by its coherent terms between vector subspaces with different numbers of photons, displayed in Eq. (2.7). Excluding them, we ascribe the description of polarization to the block-diagonal density matrix $\hat{\rho}_b$, Eq. (2.8). Accordingly, we adopt a new definition for the degree of polarization of the state (2.7):

$$\mathbb{P}(\hat{\rho}) := \mathbb{P}(\hat{\rho}_b). \tag{2.13}$$

Equation (2.13) implies that all two-mode states with the same block-diagonal part $\hat{\rho}_b$ are equally polarized. In particular, any unpolarized state $\hat{\sigma}$ has an SU(2)-invariant block-diagonal part $\hat{\sigma}_b$ [39]:

$$\hat{\sigma}_b = \sum_{N=0}^{\infty} \pi_N \frac{1}{N+1}\hat{P}_N. \tag{2.14}$$

We refer here only to type I unpolarized light [41]. Note that, except for the vacuum, any unpolarized state is mixed.

The block-diagonal state $\hat{\rho}_b$ occurring in definition (2.13) has a significant operational meaning. Indeed, the observable (2.3),

$$\hat{N} := \hat{N}_H + \hat{N}_V = \sum_{N=0}^{\infty} N\hat{P}_N, \tag{2.15}$$

is a random variable that commutes with any polarization transformation $\hat{U}_{\mathrm{pol}}$. Consequently, a polarization measurement of an arbitrary state does not alter its photon-number distribution. Now, when we perform a von Neumann measurement of the total number of photons, we obtain the outcome N with the probability p_N, while the state $\hat{\rho}$ collapses into the N-photon state $\hat{\rho}_N$, Eq. (2.11). We measure the total number of photons for each member of an ensemble of identical states described by $\hat{\rho}$ and do not select any result. In this way, we eventually get another ensemble of states described by the mixture $\hat{\rho}_b = \sum_{N=0}^{\infty} p_N\hat{\rho}_N$. Note that the block-diagonal state $\hat{\rho}_b$ has the same photon-number distribution as the given state $\hat{\rho}$. This happens because $\hat{\rho}_b$ is deliberately built with the ensemble of states provided by the corresponding von Neumann measurement. To sum up, an ideal nonselective measurement of the total number of photons is a quantum operation [43] (or quantum channel) $\mathcal{B}$ whose output is $\hat{\rho}_b$:

$$\hat{\rho} \xrightarrow{\mathcal{B}} \hat{\rho}_b = \sum_{N=0}^{\infty} \hat{P}_N\hat{\rho}\hat{P}_N. \tag{2.16}$$

The quantum operation $\mathcal{B}$ preserves the photon-number distribution. Remark first that any output $\hat{\rho}_b$ of the channel $\mathcal{B}$ commutes with the output $\hat{\sigma}_b$, Eq. (2.14), of an arbitrary unpolarized state $\hat{\sigma}$:

$$[\hat{\rho}_b, \hat{\sigma}_b] = 0. \tag{2.17}$$

This is not generally true for the input states $\hat{\rho}$ and $\hat{\sigma}$. As a consequence of the commutation relation (2.17), most polarization-measure candidates $\mathbb{P}(\hat{\rho}_b)$ depend only on the photon-number probabilities p_N and the eigenvalues $\lambda_{N,n}$ of the density matrices $\frac{1}{p_N}\rho^{(N)}$ that determine the N-photon states $\hat{\rho}_N$ entering the convex decomposition (2.8). Since all these quantities are SU(2) invariant, it follows that the candidates themselves fulfill the SU(2)-invariance condition (2.12) and are therefore admissible as adequate measures of polarization [21].

III. CHERNOFF DEGREE OF POLARIZATION

A. Definition

In view of its outstanding distinguishability properties, the quantum Chernoff bound can be used to define a polarization measure similar to other distance-type ones [21,23]. We therefore introduce the Chernoff degree of polarization

$$\mathbb{P}_{\mathrm{C}}(\hat{\rho}) := 1 - \max_{\hat{\sigma}\in\mathcal{U}} Q(\hat{\rho}_b, \hat{\sigma}_b), \tag{3.1}$$

built with the Chernoff overlap (1.3). Here $\hat{\rho}_b$ is the block-diagonal state (2.8) and $\mathcal{U}$ stands for the set of all unpolarized two-mode states. Let us denote

$$\tilde{Q} := \max_{\hat{\sigma}\in\mathcal{U}} Q(\hat{\rho}_b, \hat{\sigma}_b), \tag{3.2}$$

in order to write simply: $\mathbb{P}_{\mathrm{C}}(\hat{\rho}) = 1 - \tilde{Q}$.

It is important to check that definition (3.1) fulfills the three requirements stated in Sec. II. The "if" part of property (a) is obvious, so that we are left to prove its "only if" part.

To this end, let us consider an arbitrary block-diagonal state $\hat{\rho}_b$ which is polarized. As already mentioned, we have denoted by $\lambda_{N,n}$ the eigenvalues of any N-photon density matrix $\frac{1}{p_N}\rho^{(N)}, (p_N > 0)$. Let ν_N be the rank of the matrix $\rho^{(N)}$, Eq. (2.10), i.e., the number of its positive eigenvalues $p_N\lambda_{N,n}$:

$$\nu_N := \mathrm{rank}\,\rho^{(N)}, \quad \rho^{(N)} \in \mathcal{M}_{N+1}(\mathbb{C}), \quad 1 \leqq \nu_N \leqq N+1. \tag{3.3}$$

For subsequent use, we introduce the quantity

$$\xi_N^{(s)} := \sum_{n=0}^{N} (\lambda_{N,n})^s, \quad p_N > 0, \tag{3.4}$$

which is a decreasing function of s from the limit $\xi_N^{(0)} = \nu_N$ to the value $\xi_N^{(1)} = 1$.

The commuting density operators $\hat{\rho}_b$, Eq. (2.8), and $\hat{\sigma}_b$, Eq. (2.14), have the eigenvalues $p_N \lambda_{N,n}$ and $\pi_N \frac{\delta_{nn}}{N+1}$, respectively. Therefore, a Rényi overlap of the states $\hat{\rho}_b$ and $\hat{\sigma}_b$ reads

$$Q_s(\hat{\rho}_b, \hat{\sigma}_b) = \sum_{N=0}^{\infty} \sum_{n=0}^{N} (p_N \lambda_{N,n})^s \left(\pi_N \frac{\delta_{nn}}{N+1} \right)^{1-s}, \quad 0 \leqq s \leqq 1. \tag{3.5}$$

Obviously,

$$Q_0(\hat{\rho}_b, \hat{\sigma}_b) \leqq 1, \quad Q_1(\hat{\rho}_b, \hat{\sigma}_b) \leqq 1. \tag{3.6}$$

For $0 < s < 1$, we apply Hölder's inequality [44]:

$$\sum_n a_n b_n \leqq \left[\sum_m (a_m)^p \right]^{\frac{1}{p}} \left[\sum_n (b_n)^q \right]^{\frac{1}{q}}. \tag{3.7}$$

In Eq. (3.7), $a_n \geqq 0$, $b_n \geqq 0$, and $\{p, q\}$ is a pair of *conjugate exponents*, i.e., positive real numbers such that $p + q = pq$ or, equivalently, $\frac{1}{p} + \frac{1}{q} = 1$. Equation (3.7) becomes an equality if and only if a_n and b_n are components of proportional vectors. When p is conjugate to itself $(p = q = 2)$, Hölder's inequality (3.7) reduces to Cauchy's inequality.

We specialize Eq. (3.7) by taking

$$a_{N,n} = (p_N \lambda_{N,n})^s, \quad b_{N,n} = \left(\pi_N \frac{\delta_{nn}}{N+1} \right)^{1-s},$$
$$p = \frac{1}{s}, \quad q = \frac{1}{1-s}, \quad 0 < s < 1,$$

to get the inequality:

$$Q_s(\hat{\rho}_b, \hat{\sigma}_b) < \left(\sum_{M=0}^{\infty} \sum_{m=0}^{M} p_M \lambda_{M,m} \right)^s \left(\sum_{N=0}^{\infty} \sum_{n=0}^{N} \pi_N \frac{\delta_{nn}}{N+1} \right)^{1-s} = 1, \quad 0 < s < 1. \tag{3.8}$$

In Eq. (3.8), a strict inequality holds because the states $\hat{\rho}_b$ and $\hat{\sigma}_b$ cannot coincide: the first one is polarized and the second is not. The same strict inequality is still valid for the maximum of the Rényi overlap occurring in Eq. (3.8):

$$\max_{\hat{\sigma} \in \mathcal{U}} Q_s(\hat{\rho}_b, \hat{\sigma}_b) < 1, \quad 0 < s < 1. \tag{3.9}$$

Taking into account the identity

$$\max_{\hat{\sigma} \in \mathcal{U}} \min_{s \in [0,1]} Q_s(\hat{\rho}_b, \hat{\sigma}_b) = \min_{s \in [0,1]} \max_{\hat{\sigma} \in \mathcal{U}} Q_s(\hat{\rho}_b, \hat{\sigma}_b), \tag{3.10}$$

an inspection of Eqs. (3.6) and (3.9) leads to the inequality to be proven:

$$\mathbb{P}_C(\hat{\rho}) = 1 - \tilde{Q} > 0. \tag{3.11}$$

Equation (3.11) is then true for any state $\hat{\rho}$ whose block-diagonal part $\hat{\rho}_b$ is polarized.

Property (b) is immediate. Indeed, any polarization transformation $\hat{U}_{\text{pol}}$ is the orthogonal sum of all the SU(2) irreducible representations and their carrier spaces are just the corresponding N-photon eigensubspaces. Consequently, the block-diagonal part of the state $\hat{U}_{\text{pol}}\, \hat{\rho}\, \hat{U}_{\text{pol}}^\dagger$ factors as follows:

$$(\hat{U}_{\text{pol}}\, \hat{\rho}\, \hat{U}_{\text{pol}}^\dagger)_b = \hat{U}_{\text{pol}}\, \hat{\rho}_b \hat{U}_{\text{pol}}^\dagger. \tag{3.12}$$

By use of the invariance of the Chernoff overlap under unitary transformations [29],

$$Q(\hat{U} \hat{\rho}_1 \hat{U}^\dagger, \hat{U} \hat{\rho}_2 \hat{U}^\dagger) = Q(\hat{\rho}_1, \hat{\rho}_2),$$

we get

$$Q(\hat{U}_{\text{pol}}\, \hat{\rho}_b\, \hat{U}_{\text{pol}}^\dagger, \hat{\sigma}_b) = Q(\hat{\rho}_b, \hat{U}_{\text{pol}}^\dagger\, \hat{\sigma}_b\, \hat{U}_{\text{pol}}) = Q(\hat{\rho}_b, \hat{\sigma}_b). \tag{3.13}$$

The last equality in Eq. (3.13) follows from the SU(2)-invariant formula (2.14) corresponding to any unpolarized two-mode state $\hat{\sigma}$. Hence we obtain the SU(2)-invariance property

$$\mathbb{P}_C(\hat{U}_{\text{pol}}\, \hat{\rho}\, \hat{U}_{\text{pol}}^\dagger) = \mathbb{P}_C(\hat{\rho}). \tag{3.14}$$

Property (c) is fulfilled by definition.

B. General expression

Our task here is to evaluate the parameters $\tilde{\pi}_N$ of the unpolarized state for which the maximum in Eq. (3.1) is obtained. Determining $\tilde{Q}$ is equivalent to finding the saddle point of the function $Q_s(\hat{\rho}_b, \hat{\sigma}_b)$. We start by writing the Rényi overlap $Q_s(\hat{\rho}_b, \hat{\sigma}_b)$, Eq. (3.5), in an equivalent form:

$$Q_s(\hat{\rho}_b, \hat{\sigma}_b) = \sum_{N=0}^{\infty} (p_N)^s \xi_N^{(s)} \left(\frac{\pi_N}{N+1} \right)^{1-s}, \quad 0 \leqq s \leqq 1. \tag{3.15}$$

Let us treat first the case $s > 0$. The maximum of the Rényi overlap $Q_s(\hat{\rho}_b, \hat{\sigma}_b)$ with respect to the variables π_N under the constraint (2.6) can be found by applying the method of the Lagrange multipliers. One readily gets the N-photon probabilities $\tilde{\pi}_N^{(s)}$ that maximize function (3.15):

$$\tilde{\pi}_N^{(s)} = \frac{p_N \left(\xi_N^{(s)}\right)^{1/s} (N+1)^{1-\frac{1}{s}}}{\sum_{M=0}^{\infty} p_M \left(\xi_M^{(s)}\right)^{1/s} (M+1)^{1-\frac{1}{s}}}. \tag{3.16}$$

They characterize the closest unpolarized state $\hat{\tilde{\sigma}}_b$ to the state $\hat{\rho}_b$,

$$Q_s(\hat{\rho}_b, \hat{\tilde{\sigma}}_b) := \max_{\hat{\sigma} \in \mathcal{U}} Q_s(\hat{\rho}_b, \hat{\sigma}_b). \tag{3.17}$$

Insertion of Eq. (3.16) into Eq. (3.15) gives the explicit formula

$$Q_s(\hat{\rho}_b, \hat{\tilde{\sigma}}_b) = \left[\sum_{N=0}^{\infty} p_N (N+1) \left(\frac{\xi_N^{(s)}}{N+1} \right)^{1/s} \right]^s, \quad 0 < s \leqq 1. \tag{3.18}$$

It is convenient to denote by $\tilde{N}(s)$ the value of N that maximizes the ratio $\frac{\xi_N^{(s)}}{N+1}$:

$$\frac{\xi_{\tilde{N}(s)}^{(s)}}{\tilde{N}(s)+1} := \max_{0 \leqq N < \infty} \frac{\xi_N^{(s)}}{N+1}. \tag{3.19}$$

Equations (3.18) and (3.19) imply the inequality

$$\max_{\hat{\sigma}\in\mathcal{U}} Q_s(\hat{\rho}_b,\hat{\sigma}_b) \leqq \frac{\xi^{(s)}_{\tilde{N}(s)}}{\tilde{N}(s)+1}\left(\langle N\rangle+1\right)^s, \quad 0 < s \leqq 1. \tag{3.20}$$

We are now ready to handle the limit case $s = 0$. Recalling that $\xi_N^{(0)} = \nu_N$ and setting $\tilde{N} := \tilde{N}(0)$, Eq. (3.19) reads for $s = 0$

$$\frac{\nu_{\tilde{N}}}{\tilde{N}+1} := \max_{0\leqq N<\infty} \frac{\nu_N}{N+1}. \tag{3.21}$$

The inequality (3.20) has therefore the limit

$$\lim_{s\to 0} Q_s(\hat{\rho}_b,\hat{\bar{\sigma}}_b) \leqq \frac{\nu_{\tilde{N}}}{\tilde{N}+1}. \tag{3.22}$$

If we consider the unpolarized $\tilde{N}$-photon state

$$\hat{\sigma}_{\tilde{N}} = \frac{1}{\tilde{N}+1}\hat{P}_{\tilde{N}}, \tag{3.23}$$

i.e., with $\pi_N = \delta_{N,\tilde{N}}$, then, according to Eq. (3.15) we get

$$Q_s(\hat{\rho}_b,\hat{\sigma}_{\tilde{N}}) = (p_{\tilde{N}})^s \xi^{(s)}_{\tilde{N}} \left(\frac{1}{\tilde{N}+1}\right)^{1-s}, \quad 0 \leqq s \leqq 1. \tag{3.24}$$

The limit $s = 0$ of Eq. (3.24) then reads

$$\lim_{s\to 0} Q_s(\hat{\rho}_b,\hat{\sigma}_{\tilde{N}}) = \frac{\nu_{\tilde{N}}}{\tilde{N}+1}. \tag{3.25}$$

Equations (3.22) and (3.25) show that for $s = 0$ the unpolarized state (3.23) is the closest to $\hat{\rho}_b$. Therefore, the explicit formula (3.18) can be extended to the limit case $s = 0$, so that the Chernoff degree of polarization has the general expression

$$\mathbb{P}_{\rm C}(\hat{\rho}) = 1 - \min_{s\in[0,1]} \left[\sum_{N=0}^{\infty} p_N \left(\xi_N^{(s)}\right)^{1/s} (N+1)^{1-\frac{1}{s}}\right]^s. \tag{3.26}$$

It is well known [28,29] that the Chernoff overlap is closely related to the Uhlmann fidelity. This suggests that a comparison between the Chernoff degree of polarization and the one based on the Bures distance would be interesting. The Bures degree of polarization has been defined in Refs. [21,23] as

$$\mathbb{P}_{\rm B}(\hat{\rho}) := 1 - \max_{\hat{\sigma}\in\mathcal{U}} \sqrt{\mathcal{F}(\hat{\rho}_b,\hat{\sigma}_b)}, \tag{3.27}$$

where $\mathcal{F}$ is the fidelity between two states,

$$\mathcal{F}(\hat{\rho}_1,\hat{\rho}_2) := \left[{\rm Tr}\sqrt{\hat{\rho}_1^{1/2}\,\hat{\rho}_2\,\hat{\rho}_1^{1/2}}\right]^2. \tag{3.28}$$

Owing to the commutation relation (2.17) the following identity holds:

$$[\mathcal{F}(\hat{\rho}_b,\hat{\sigma}_b)]^{1/2} = Q_{1/2}(\hat{\rho}_b,\hat{\sigma}_b). \tag{3.29}$$

We take advantage of Eq. (3.29) to specialize Eq. (3.16) for the closest unpolarized state,

$$\bar{\pi}_N^{(1/2)} = \frac{p_N(N+1)^{-1}\left[\xi_N^{(1/2)}\right]^2}{\sum_{M=0}^{\infty} p_M(M+1)^{-1}\left[\xi_M^{(1/2)}\right]^2}, \tag{3.30}$$

and Eq. (3.18) to write the maximal fidelity $\mathcal{F}(\hat{\rho}_b, \hat{\bar{\sigma}}_b)$:

$$\mathcal{F}(\hat{\rho}_b,\hat{\bar{\sigma}}_b) = \sum_{N=0}^{\infty} \frac{p_N}{N+1}\left[\xi_N^{(1/2)}\right]^2. \tag{3.31}$$

Therefore, the Bures degree of polarization (3.27) has the expression [23]

$$\mathbb{P}_{\rm B}(\hat{\rho}) = 1 - \sqrt{\sum_{N=0}^{\infty} \frac{p_N}{N+1}\left[\xi_N^{(1/2)}\right]^2}. \tag{3.32}$$

We stress that the polarization measures $\mathbb{P}_{\rm C}(\hat{\rho})$, Eq. (3.26), and $\mathbb{P}_{\rm B}(\hat{\rho})$, Eq. (3.32), depend only on the photon-number probabilities p_N and on the eigenvalues $\lambda_{N,n}$ of the N-photon density matrices $\frac{1}{p_N}\rho^{(N)}$, ($p_N > 0$). Hence both of them are nice examples for the discussion at the end of Sec. II. Note finally the inequality

$$\mathbb{P}_{\rm C}(\hat{\rho}) \geqq \mathbb{P}_{\rm B}(\hat{\rho}). \tag{3.33}$$

IV. APPLICATIONS

A. Pure states

Let us now analyze the case of a pure state, $\hat{\rho} = |\Psi\rangle\langle\Psi|$:

$$|\Psi\rangle = \sum_{N=0}^{\infty}\sum_{n=0}^{N} c_{N,n}|n,N-n\rangle, \quad \sum_{N=0}^{\infty}\sum_{n=0}^{N} |c_{N,n}|^2 = 1. \tag{4.1}$$

Its block-diagonal part is a convex combination of N-photon pure states,

$$[|\Psi\rangle\langle\Psi|]_b = \sum_{N=0}^{\infty} p_N |\Psi^{(N)}\rangle\langle\Psi^{(N)}|, \tag{4.2}$$

which is expressed in terms of the photon-number probabilities

$$p_N = \sum_{n=0}^{N} |c_{N,n}|^2 \tag{4.3}$$

and the N-photon state vectors

$$|\Psi^{(N)}\rangle := \frac{1}{\sqrt{p_N}} \sum_{n=0}^{N} c_{N,n}|n,N-n\rangle, \quad p_N > 0. \tag{4.4}$$

Each N-photon pure state $\hat{\rho}_N = |\Psi^{(N)}\rangle\langle\Psi^{(N)}|$ entering the convex decomposition (4.2) has the eigenvalues $\lambda_{N,n} = \delta_{n0}$, for $n = 0,1,\ldots,N$. Accordingly, Eqs. (3.26) and (3.32) simplify to

$$\mathbb{P}_{\rm C}(|\Psi\rangle\langle\Psi|) = 1 - \min_{s\in[0,1]} \left[\sum_{N=0}^{\infty} p_N (N+1)^{1-\frac{1}{s}}\right]^s \tag{4.5}$$

and, respectively,

$$\mathbb{P}_{\rm B}(|\Psi\rangle\langle\Psi|) = 1 - \left(\sum_{N=0}^{\infty} \frac{p_N}{N+1}\right)^{1/2}. \tag{4.6}$$

As already remarked in Ref. [23], for a pure state, $\hat{\rho} = |\Psi\rangle\langle\Psi|$, the Chernoff and Bures degrees of polarization are determined solely by its photon-number distribution, regardless of the nature of the N-photon state vectors (4.4).

We further specialize the above formulas to the case of a pure state with N photons, $\hat{\rho}_N = |\Psi^{(N)}\rangle\langle\Psi^{(N)}|$, whose photon-number probabilities are $p_M = \delta_{MN}$. Hence Eqs. (4.5) and (4.6) reduce to

$$\mathbb{P}_{\rm C}(|\Psi^{(N)}\rangle\langle\Psi^{(N)}|) = \frac{N}{N+1}, \tag{4.7}$$

since the minimum over s is reached at $\tilde{s} = 0$, and, respectively,

$$\mathbb{P}_{\rm B}(|\Psi^{(N)}\rangle\langle\Psi^{(N)}|) = 1 - \left(\frac{1}{N+1}\right)^{1/2}. \tag{4.8}$$

Both degrees of polarization are strictly increasing functions of N from the lowest value $\mathbb{P}_{\rm C} = \mathbb{P}_{\rm B} = 0$, for the vacuum, to the large-photon-number limit

$$\lim_{N\to\infty} \mathbb{P}_{\rm C} = \lim_{N\to\infty} \mathbb{P}_{\rm B} = 1.$$

B. States with a given photon-number distribution

Let us consider the set of all two-mode states (pure and mixed) with a given photon-number distribution $\{p_N\}_{N=0,1,2,3,\ldots}$. According to Eqs. (3.26) and (3.32), such a state is maximally polarized if and only if its block-diagonal part $\hat{\rho}_b$ is a convex combination (2.8) of pure N-photon states:

$$\hat{\rho}_b = \sum_{N=0}^{\infty} p_N |\Psi^{(N)}\rangle\langle\Psi^{(N)}|. \tag{4.9}$$

A significant example is that of the pure state

$$|\Psi\rangle = \sum_{N=0}^{\infty} \sqrt{p_N}\, |\Psi^{(N)}\rangle \tag{4.10}$$

that has the property (4.2). Therefore, the maximal Chernoff and Bures degrees of polarization are those for a pure state, i.e., they are given by Eqs. (4.5) and (4.6), respectively.

In what follows we analyze two families of states, each of them having only two nonvanishing N-photon probabilities. The first one is a one-parameter family of pure states, while the second one consists of Fock-diagonal mixed states.

1. Superposition of two pure N-photon states

Suppose that N_1 and N_2 are fixed numbers of photons, and $N_1 < N_2$. We investigate the family of pure states

$$|\Psi\rangle = \sqrt{p}|\Psi^{(N_1)}\rangle + \sqrt{1-p}|\Psi^{(N_2)}\rangle, \tag{4.11}$$

depending on the probability $p \in [0,1]$. The block-diagonal part (4.2) of a given state is

$$[|\Psi\rangle\langle\Psi|]_b = p|\Psi^{(N_1)}\rangle\langle\Psi^{(N_1)}| + (1-p)|\Psi^{(N_2)}\rangle\langle\Psi^{(N_2)}|, \tag{4.12}$$

so that the Rényi overlap $Q_s(\hat{\rho}_b, \hat{\sigma}_b)$, Eq. (3.15), reads

$$Q_s(p,\pi_{N_1}) = p^s \left(\frac{\pi_{N_1}}{N_1+1}\right)^{1-s} + (1-p)^s \left(\frac{1-\pi_{N_1}}{N_2+1}\right)^{1-s}. \tag{4.13}$$

In the limit cases $p = 0$ and $p = 1$, the state vector (4.11) reduces to $|\Psi^{(N_2)}\rangle$ and $|\Psi^{(N_1)}\rangle$, respectively. According to Eqs. (4.7) and (4.8), we write

$$\mathbb{P}_{\rm C}(|\Psi\rangle\langle\Psi|) = \frac{N_2}{N_2+1}, \quad \mathbb{P}_{\rm B}(|\Psi\rangle\langle\Psi|) = 1 - \frac{1}{\sqrt{N_2+1}}, \quad p = 0, \tag{4.14}$$

and

$$\mathbb{P}_{\rm C}(|\Psi\rangle\langle\Psi|) = \frac{N_1}{N_1+1}, \quad \mathbb{P}_{\rm B}(|\Psi\rangle\langle\Psi|) = 1 - \frac{1}{\sqrt{N_1+1}}, \quad p = 1. \tag{4.15}$$

In the case $0 < p < 1$, it is convenient to write the optimal value (3.16) of the parameter π_{N_1},

$$\tilde{\pi}_{N_1}^{(s)} = \left[1 + \frac{1-p}{p}\left(\frac{N_1+1}{N_2+1}\right)^{\frac{1}{s}-1}\right]^{-1}, \tag{4.16}$$

as well as the maximum over π_{N_1}, Eq. (3.18), of the Rényi overlap (4.13),

$$Q_s(p,\tilde{\pi}_{N_1}^{(s)}) = \left[p(N_1+1)^{1-\frac{1}{s}} + (1-p)(N_2+1)^{1-\frac{1}{s}}\right]^s. \tag{4.17}$$

By use of Eqs. (4.5) and (4.6), we get

$$\mathbb{P}_{\rm C}(|\Psi\rangle\langle\Psi|) = 1 - \min_{s\in[0,1]} \left[p(N_1+1)^{1-\frac{1}{s}} + (1-p)(N_2+1)^{1-\frac{1}{s}}\right]^s, \tag{4.18}$$

and, respectively,

$$\mathbb{P}_{\rm B}(|\Psi\rangle\langle\Psi|) = 1 - \left(\frac{p}{N_1+1} + \frac{1-p}{N_2+1}\right)^{1/2}. \tag{4.19}$$

The Bures degree of polarization (4.19) strictly decreases with the probability p.

We are left to find the minimum over s in Eq. (3.26). The necessary condition for minimum reduces to the transcendental equation

$$\begin{aligned} &p(N_1+1)^{1-\frac{1}{\tilde{s}}} \ln\left\{(N_1+1)\left[p(N_1+1)^{1-\frac{1}{\tilde{s}}} + (1-p)(N_2+1)^{1-\frac{1}{\tilde{s}}}\right]^{\tilde{s}}\right\} \\ &+ (1-p)(N_2+1)^{1-\frac{1}{\tilde{s}}} \ln\left\{(N_2+1)\left[p(N_1+1)^{1-\frac{1}{\tilde{s}}} + (1-p)(N_2+1)^{1-\frac{1}{\tilde{s}}}\right]^{\tilde{s}}\right\} = 0. \end{aligned} \tag{4.20}$$

Equation (4.20) has no solution for $p \geqslant \frac{1}{N_1+1}$, when there is no saddle point of the Rényi overlap (4.13). The minimum over s in Eq. (3.26) is reached in $\tilde{s} = 0$. Further, Eqs. (4.16) and (4.17) give $\tilde{\pi}_{N_1} = 1$ and $\tilde{Q} = \frac{1}{N_1+1}$, respectively. The Chernoff degree of polarization is independent of the probability p:

$$\mathbb{P}_{\rm C}(|\Psi\rangle\langle\Psi|) = \frac{N_1}{N_1+1}, \quad \frac{1}{N_1+1} \leqslant p < 1. \tag{4.21}$$

In the opposite situation, $p < \frac{1}{N_1+1}$, Eq. (4.20) has a solution $\tilde{s} \in (0,1)$. This corresponds to a saddle point of the Rényi

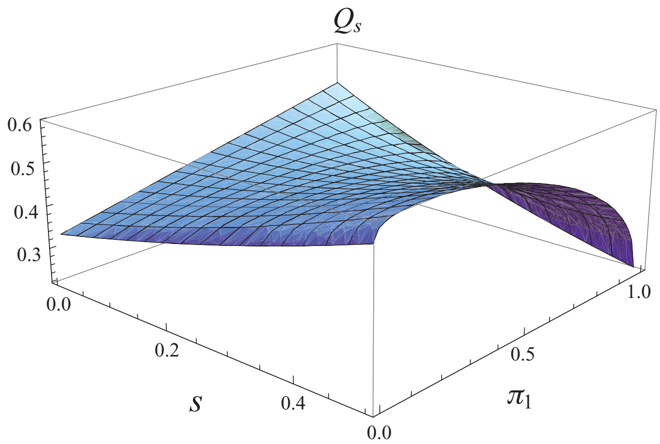

FIG. 1. (Color online) Displaying the saddle-point evaluation of the Chernoff degree of polarization $\mathbb{P}_C(|\Psi\rangle\langle\Psi|)$ for a state (4.11) with $N_1 = 1$, $N_2 = 2$, and $p = 0.1$. The Rényi overlap, Eq. (4.13), is plotted vs s and π_1. The saddle point has the coordinates $\tilde{s} = 0.124$ and $\tilde{\pi}_1 = 0.634$. The Chernoff overlap, Eq. (4.17), is $\tilde{Q} = 0.431$, so the degree of polarization is $\mathbb{P}_C = 0.569$.

overlap (4.13). The Chernoff degree of polarization (3.26) depends on the probability p, taking values in the interval

$$\mathbb{P}_C(|\Psi\rangle\langle\Psi|) \in \left(\frac{N_1}{N_1+1}, \frac{N_2}{N_2+1}\right), \quad 0 < p < \frac{1}{N_1+1}. \tag{4.22}$$

The above analysis is illustrated in Fig. 1 for a superposition with lower photon numbers at a fixed value of the probability p. The numerical calculation of the Chernoff degree of polarization by the saddle-point method is straightforward and can be performed with great accuracy. Figure 2 displays the comparison between the maximal (pure-state) Chernoff and Bures degrees of polarization as functions of the probability p.

2. *Mixture of two mixed N-photon states*

We consider again a pair of fixed numbers of photons, N_1 and N_2, such that $N_1 < N_2$, and examine a mixture

$$\hat{\tau} = p\,\hat{\rho}_{N_1} + (1-p)\,\hat{\rho}_{N_2}, \tag{4.23}$$

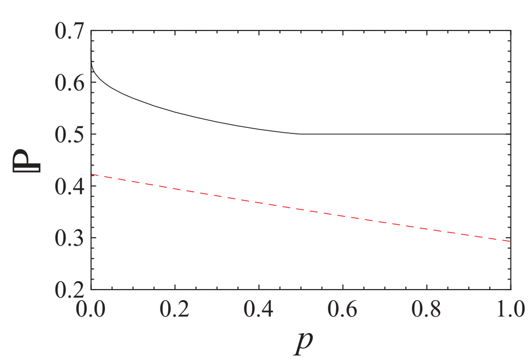

FIG. 2. (Color online) Degree of polarization of the pure states (4.11) characterized by $N_1 = 1$, $N_2 = 2$ as a function of the probability p: the Chernoff measure (black full line) and the Bures measure (red dashed line).

where the states $\hat{\rho}_{N_1}$ and $\hat{\rho}_{N_2}$ are Fock-diagonal. Obviously, $\hat{\tau}_b = \hat{\tau}$. In the particular case when $N_1 = 1$ and $N_2 = 2$, we choose density matrices $\frac{1}{p}\,\rho^{(1)}$ and $\frac{1}{1-p}\,\rho^{(2)}$ with nonvanishing diagonal entries:

$$\frac{1}{p}\,\rho^{(1)} = \begin{pmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{pmatrix}, \qquad \frac{1}{1-p}\,\rho^{(2)} = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 1-\beta-\gamma \end{pmatrix}. \tag{4.24}$$

The Rényi overlap (3.15) specializes to

$$Q_s(p,\pi_1) = \left(\frac{\pi_1}{2}\right)^{1-s} p^s[\alpha^s + (1-\alpha)^s] + \left(\frac{1-\pi_1}{3}\right)^{1-s} \times (1-p)^s[\beta^s + \gamma^s + (1-\beta-\gamma)^s]. \tag{4.25}$$

The Chernoff degree of polarization, Eq. (3.26), reads

$$\mathbb{P}_C(\hat{\tau}) = 1 - \min_{s\in[0,1]} \{2^{1-1/s} p[\alpha^s + (1-\alpha)^s]^{1/s} + 3^{1-1/s}(1-p) \times [\beta^s + \gamma^s + (1-\beta-\gamma)^s]^{1/s}\}^s. \tag{4.26}$$

We further write the Bures measure of polarization, Eq. (3.32):

$$\mathbb{P}_B(\hat{\tau}) = 1 - \left\{\frac{p}{2}[\alpha^{1/2} + (1-\alpha)^{1/2}]^2 + \frac{1-p}{3} \times [\beta^{1/2} + \gamma^{1/2} + (1-\beta-\gamma)^{1/2}]^2\right\}^{1/2}. \tag{4.27}$$

Figure 3 presents the saddle-point evaluation of the Chernoff degree of polarization $\mathbb{P}_C(\hat{\tau})$ of a state (4.23) with lower photon numbers. For the same family of states, a comparison between

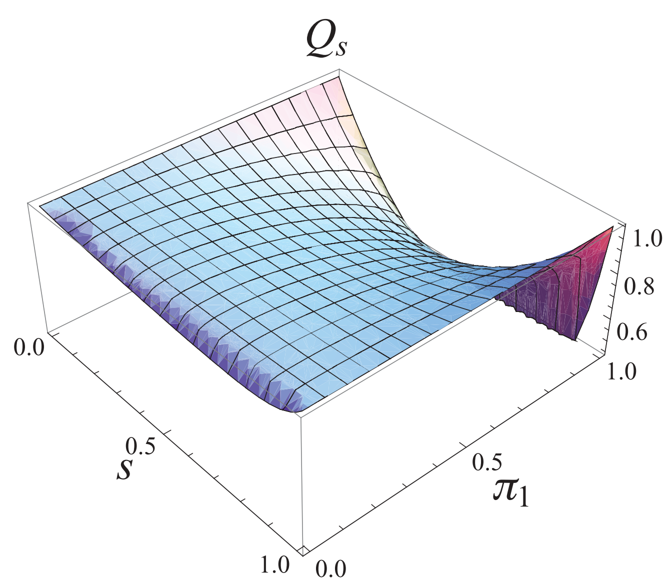

FIG. 3. (Color online) Saddle-point evaluation of the Chernoff degree of polarization $\mathbb{P}_C(\hat{\tau})$ for a state (4.23) with $N_1 = 1$, $N_2 = 2$, $p = 0.1$, $\alpha = 0.1, \beta = 0.01$, and $\gamma = 0.04$. The Rényi overlap Q_s, Eq. (4.25), is plotted vs s and π_1. The saddle point is reached at $\tilde{s} = 0.434$ and $\tilde{\pi}_1 = 0.209$. The optimal value $\tilde{Q}$ is 0.544, so that the degree of polarization is $\mathbb{P}_C(\hat{\tau}) = 0.251$. For the same state, the Bures degree of polarization, Eq. (4.27), is $\mathbb{P}_B(\hat{\tau}) = 0.247$.

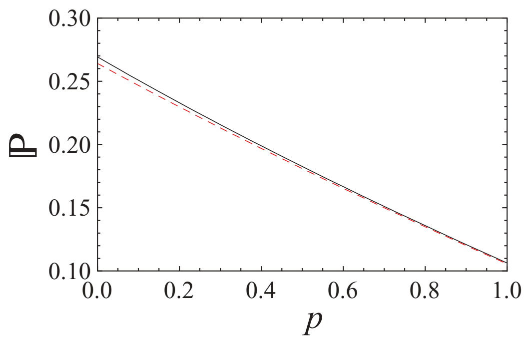

FIG. 4. (Color online) Degree of polarization of the mixed states (4.23) characterized by the same parameters $N_1, N_2, \alpha, \beta, \gamma$ as in Fig. 3 vs the mixing coefficient p: the Chernoff measure (black full line) and the Bures measure (red dashed line).

the Chernoff and Bures degrees of polarization as functions of the mixing parameter p is made in Fig. 4. Unlike the couple of maximal degrees of polarization drawn in Fig. 2, their graphs are here very close.

V. SUMMARY AND CONCLUSIONS

In this paper we have exploited the quantum Chernoff bound in order to introduce a distance-type polarization measure for the quantum radiation field. This measure fulfills the requirements for a genuine degree of polarization, put forward quite recently [15,23]. We have derived a general expression of the Chernoff degree of polarization, Eq. (3.26), that allows its computation. Moreover, a comparison between the Chernoff and Bures degrees of polarization proved to be very useful. For instance, Fig. 2 displays both degrees of polarization for a one-parameter family of pure states that are superpositions of a fixed pair of pure N-photon states. The Bures polarization measure distinguishes between all the states of this family because it is strictly decreasing with the probability of one of the N-photon states. On the contrary, the predicted existence of a plateau of the Chernoff degree of polarization starting from a threshold of the same probability is displayed. Although considerably larger than the Bures polarization measure, the Chernoff measure cannot discriminate between the corresponding states. On the other hand, Fig. 4 points out that for a one-parameter mixture of two given mixed N-photon states, the Bures and Chernoff degrees of polarization happen to be very close.

We stress that the Rényi overlaps $Q_s(\hat{\rho}_b, \hat{\hat{\sigma}}_b)$, with $0 < s < 1$ [Eq. (3.18)], can themselves be employed as reliable measures of polarization. The symmetric one ($s = \frac{1}{2}$) yields the Bures degree of polarization via Eq. (3.29) and has a privileged position owing to its significant meaning in quantum mechanics. To conclude, the Chernoff polarization measure, Eq. (3.26), deserves special attention because it is the maximal Rényi distance-type polarization measure.

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