Entanglement-distribution maximization over one-sided Gaussian noisy channels

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We present an upper bound of the entanglement evolution for two-mode Gaussian pure states under a one-sided Gaussian map. Based on this, the optimization of entanglement evolution is studied. Even if complete information about the one-sided Gaussian noisy channel does not exist, one can still maximize the entanglement distribution by testing the channel with only two specific states.

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I. INTRODUCTION

The study of the properties of quantum entanglement has drawn much interest for a long time [1–4]. Although quantum information processing (QIP) was initially studied with discrete quantum states, it was then extended to continuous-variable quantum states [5]. So far, many concepts and results with two-level quantum systems have been extended to the continuous-variable case with parallel results, such as quantum teleportation [6], the inseparability criterion [7], the degree of entanglement [8,9], entanglement purification [10–12], entanglement sudden death [13], the characterization of Gaussian maps [14], and so on. However, this does not mean *all* results with two-level quantum systems can have parallel results for Gaussian states.

Entanglement distribution is the first step toward many novel tasks in quantum communication and QIP [1]. In practice, there is no perfect channel for entanglement distribution. Naturally, how to maximize the entanglement after distribution is an important question in practical QIP. If we distribute the quantum entanglement by sending one part of the entangled state to a remote place through a noisy channel, we can use the model of a one-sided noisy channel, or one-sided map.

Given the factorization law presented by Konrad *et al.* [15], such a maximization problem for entanglement distribution over a one-sided map does not exist for the 2×2 system because any one-sided map will produce the same entanglement on the output states provided that the entanglement of the input pure states are the same. The result was experimentally tested [16] and has also recently been extended [17]. However, such a factorization does not hold for the continuous-variable state, as shown below. In this work, we consider the following problem. Initially we have a bipartite Gaussian pure state. Given a one-sided Gaussian map (or a one-sided Gaussian noisy channel), we must find how to maximize the entanglement of the output state by taking a Gaussian unitary transformation on the input mode before it is sent to the noisy channel. We find that by testing the channel with only two different states, if a certain result is verified, then we can find the correct Gaussian unitary transformation which optimizes the entanglement evolution for any input Gaussian pure state. That is to say, we can maximize the output entanglement even though we do not have the full information of the one-sided map. In what follows, we first show by specific example that the factorization law for the 2×2 system presented by Konrad *et al.* [15] does not hold for Gaussian states. We then present an upper bound of the entanglement evolution for initial Gaussian pure states. Based on this, we study how to optimize the entanglement evolution over a one-sided Gaussian map by taking a local Gaussian unitary transformation to the mode before it is sent to the noisy channel.

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II. OUTPUT ENTANGLEMENT OF ONE-SIDED GAUSSIAN MAP AND SINGLE-MODE SQUEEZING

Most generally, a two-mode Gaussian pure state is

$$|g(U,V,q)\rangle = U \otimes V|\chi(q)\rangle \tag{1}$$

and $|\chi(q)\rangle = \sqrt{1-q^2}e^{qa_1^\dagger a_2^\dagger}|00\rangle$ $(-1 \leqslant q \leqslant 1)$ is a two-mode squeezed state (TMSS). We define map \$ as a Gaussian map which acts on one mode of the state only. A Gaussian map changes a Gaussian state to a Gaussian state only. In whatever reasonable entanglement measure, the entanglement of a Gaussian pure state in the form of Eq. (1) is uniquely determined by q. Therefore, we define the *characteristic* value of entanglement of the Gaussian pure state $\rho(q) = |g(U,V,q)\rangle\langle g(U,V,q)|$ as

$$E[\rho(q)] = |q|^2. \tag{2}$$

On the other hand, any bipartite Gaussian density operator ρ has a characteristic function of the form

$$C(\alpha_1,\alpha_2) = \mathrm{tr}[\rho\hat{D}_1(\alpha_1)\hat{D}_2(\alpha_2)] = e^{-\frac{1}{2}\bar{\alpha}\Lambda\bar{\alpha}^T}, \tag{3}$$

where $\hat{D}_k(\alpha_k) = e^{\alpha_k a_k^\dagger - \alpha_k^* a_k}$, Λ is the covariance matrix of ρ, and $\bar{\alpha} = (x_1,y_1,x_2,y_2)$ with $\alpha_k = \frac{1}{\sqrt{2}}(x_k + iy_k)$. Therefore, a Gaussian state is fully characterized by its covariance matrix (CM) [5,9]. Suppose the CM of state $U \otimes V|\chi(q)\rangle$ is

$$\Lambda = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}; \tag{4}$$

$|q|^2$ is uniquely determined by $|A|$ (the determinant of the matrix A). So, to compare the entanglement of two Gaussian pure states, we only need to compare the $|A|$ values of their covariance matrices.

We start with the projection operator $\hat{T}_k(q_\alpha)$, which acts on mode k only:

$$\hat{T}_k(q_\alpha)=\sum_{n=0}^{\infty} q_\alpha^n |n\rangle\langle n| = q_\alpha^{a_k^\dagger a_k}. \tag{5}$$

This operator has an important mathematical property,

$$\hat{T}_k(q_\alpha)(a_k^\dagger,a_k)\hat{T}_k^{-1}(q_\alpha)=(q_\alpha a_k^\dagger,a_k/q_\alpha), \tag{6}$$

which is used later in this paper. For simplicity, we sometimes omit the subscripts of states and/or operators provided that the omission does not affect the clarity.

Define the one-mode squeezed operator $\mathcal{S}(r)=e^{r(a^{\dagger 2}-a^2)}$, where r is a real number and bipartite state $|\psi_r(q_0)\rangle = I\otimes \mathcal{S}(r)|\chi(q_0)\rangle$. We have the following theorem.

Theorem 1. Consider the one-sided map $I\otimes\hat{T}(q_1)$ acting on the initial state $|\psi_r(q_0)\rangle$. The entanglement for the outcome state $I\otimes\hat{T}(q_1)|\psi_r(q_0)\rangle$ is a descending function of $|r|$. Mathematically, if $|r_1|>|r_2|$ then

$$E[I\otimes\hat{T}(q_1)|\psi_{r_1}(q_0)\rangle] < E[I\otimes\hat{T}(q_1)|\psi_{r_2}(q_0)\rangle]. \tag{7}$$

Proof. Using the Baker-Campbell-Hausdorff (BCH) formula, up to a normalization factor we have

$$|\psi_r(q_0)\rangle = e^{-\frac{1}{2}a_1^{\dagger 2}q_0^2\tanh(2r)+\frac{1}{2}a_2^{\dagger 2}\tanh(2r)+\frac{q_0 a_1^\dagger a_2^\dagger}{\cosh(2r)}}|00\rangle. \tag{8}$$

A detailed derivation of this identity is given in the Appendix. Based on Eq. (5), the one-sided map $I\otimes\hat{T}(q_1)$ changes state $|\psi_r(q_0)\rangle$ into

$$|\psi'\rangle = e^{f_1 a_1^{\dagger 2}+f_2 a_2^{\dagger 2}+f_3 a_1^\dagger a_2^\dagger}|00\rangle, \tag{9}$$

where $f_1=-\frac{1}{2}q_0^2\tanh(2r)$, $f_2=\frac{1}{2}q_1^2\tanh(2r)$, and $f_3=\frac{q_0q_1}{\cosh(2r)}$. Here we have omitted the normalization factor. Since we only need the covariance matrix of state $|\psi'\rangle$, the normalization can be disregarded because it does not change the covariance matrix. Suppose the characteristic function of state $\rho'=|\psi'\rangle\langle\psi'|$ has the form

$$C(\alpha_1,\alpha_2)=\mathrm{tr}[\rho'\hat{D}_1(\alpha_1)\hat{D}_2(\alpha_2)]=e^{-\frac{1}{2}\bar{\alpha}\Lambda'\bar{\alpha}^T}, \tag{10}$$

where Λ' is the CM of $|\psi'\rangle$ and $\hat{D}_k(\alpha_k)=e^{\alpha_k a_k^\dagger-\alpha_k^* a_k}$, $\bar{\alpha}=(x_1,y_1,x_2,y_2)$ with $\alpha_k=\frac{1}{\sqrt{2}}(x_k+iy_k)$, as defined earlier. Writing Λ' here in the form of Eq. (4), we find $A=\mathrm{diag}[b_1,b_2]$, $C=\mathrm{diag}[c_1,c_2]$, and $B=\mathrm{diag}[d_1,d_2]$ with $b_1=-\frac{1}{2}+\frac{1+2f_2}{1+2f_1+2f_2+4f_1f_2-f_3^2}$, $b_2=-\frac{1}{2}+\frac{1-2f_2}{1-2f_1-2f_2+4f_1f_2-f_3^2}$, $d_1=-\frac{1}{2}+\frac{1+2f_1}{1+2f_1+2f_2+4f_1f_2-f_3^2}$, $d_2=-\frac{1}{2}+\frac{1-2f_1}{1-2f_1-2f_2+4f_1f_2-f_3^2}$, $c_1=\frac{-f_3}{1+2f_1+2f_2+4f_1f_2-f_3^2}$, and $c_2=\frac{f_3}{1-2f_1-2f_2+4f_1f_2-f_3^2}$. The entanglement in whatever measure of state $|\psi'\rangle$ is a rising functional of $|A|$ and

$$|A|=\frac{1}{4}+\frac{2q_0^2q_1^2}{1-4q_0^2q_1^2+q_1^4+q_0^4(1+q_1^4)+(1-q_0^4)(1-q_1^4)\cosh(4r)}. \tag{11}$$

This is obviously a descending functional of $|r|$. ∎

This theorem actually shows that there is not a factorization law similar to that in 2×2 states for the continuous-variable states in any good entanglement measure.

III. UPPER BOUND OF ENTANGLEMENT EVOLUTION

Since $U\otimes I$ and $I\otimes\$$ commute, the unitary operator U plays no role in the entanglement evolution under the one-sided map $I\otimes\$$, and hence we only need to consider the initial state $|g(I,V,q)\rangle = I\otimes V|\chi(q)\rangle = |\varphi(q)\rangle$. We also define $\rho^G(q_\alpha)=I\otimes\$(|\varphi(q_\alpha)\rangle\langle\varphi(q_\alpha)|)$.

Using Eq. (6), one easily finds $|\varphi(q=q_aq_b)\rangle=\hat{T}(q_a)\otimes I|\varphi(q_b)\rangle$. Since the operator $\hat{T}(q_a)\otimes I$ and the map $I\otimes\$$ commute, we have

$$\rho^G(q=q_aq_b)=\hat{T}(q_a)\otimes I\rho^G(q_b)\hat{T}^\dagger(q_a)\otimes I. \tag{12}$$

Using entanglement of formation [9,18], we can calculate the entanglement of a two-mode Gaussian state through its optimal decomposition form [9]. Suppose $\rho^G(q_b)$ has the following optimal decomposition [9]:

$$\rho^G(q_b)=U_1\otimes U_2\rho^s(q_0)U_1^\dagger\otimes U_2^\dagger. \tag{13}$$

Here U_1,U_2 are two local Gaussian unitaries and ρ^s is in the form

$$\rho^s(q_0)=\int d^2\beta_1 d^2\beta_2 P(\beta_1,\beta_2) \times\hat{D}(\beta_1,\beta_2)|\chi(q_0)\rangle\langle\chi(q_0)|\hat{D}^\dagger(\beta_1,\beta_2), \tag{14}$$

where $P(\beta_1,\beta_2)$ is positive definite, and $\hat{D}(\beta_1,\beta_2)=\hat{D}_1(\beta_1)\otimes\hat{D}_2(\beta_2)$ is a displacement operator defined as $\hat{D}_k(\beta_k)=e^{\beta_k a_k^\dagger-\beta_k^* a_k}$. According to the definition of optimal decomposition [9,18], there are no other U_1,U_2 or positive definite functional $P(\beta_1,\beta_2)$ which can decompose $\rho^G(q_b)$ in the form of Eq. (13) with a smaller $|q_0|$. The entanglement of $\rho^G(q_b)$ is equal to that of a TMSS $|\chi(q_0)\rangle$; that is, $|q_0|^2$. For the Gaussian state $\rho^G(q_b)$ with its optimal decomposition of Eq. (13), we define the characteristic value of entanglement of $\rho^G(q_b)$ as $E[\rho^G(q_b)]=|q_0|^2$.

Lemma 1. For any local Gaussian unitary U and operator $\hat{T}(q_a)$, we can find θ, θ', and r satisfying

$$\begin{aligned}&\hat{T}(q_a)U\otimes I\cdot\hat{D}(\beta_1,\beta_2)|\chi(q_0)\rangle\\ &\quad=\mathcal{R}(\theta')\otimes\mathcal{R}(\theta)\cdot\hat{D}(\beta_1',\beta_2')\cdot\hat{T}(q_a)\mathcal{S}(r)\otimes I|\chi(q_0)\rangle,\end{aligned} \tag{15}$$

where $\mathcal{S}(r)$ is a squeezing operator defined earlier, $\mathcal{R}(\theta)$ is a rotation operator defined by $\mathcal{R}(\theta)(a^\dagger,a)\mathcal{R}^\dagger(\theta)=(e^{-i\theta}a^\dagger,e^{i\theta}a)$, and β_1',β_2' and β_1,β_2 are related by a certain linear transformation.

Proof. Any local Gaussian unitary operator U can be decomposed into the product form of $\mathcal{R}(\theta')\mathcal{S}(r)\mathcal{R}(\theta)$. Also, $\mathcal{S}(r)\mathcal{R}(\theta)\otimes I\cdot\hat{D}(\beta_1,\beta_2)=\hat{D}(\beta_1'',\beta_2'')\cdot\mathcal{S}(r)\mathcal{R}(\theta)\otimes I$. Defining $\hat{d}=\hat{T}(q_a)\otimes I\cdot\hat{D}(\beta_1'',\beta_2'')\cdot\hat{T}^{-1}(q_a)\otimes I$, we have

$$\begin{aligned}&\hat{T}(q_a)U\otimes I\cdot\hat{D}(\beta_1,\beta_2)|\chi(q_0)\rangle\\ &\quad=\hat{T}(q_a)\mathcal{R}(\theta')\mathcal{S}(r)\mathcal{R}(\theta)\otimes I\cdot\hat{D}(\beta_1,\beta_2)|\chi(q_0)\rangle\\ &\quad=\mathcal{R}(\theta')\otimes I\cdot\hat{d}\cdot\hat{T}(q_a)\mathcal{S}(r)\mathcal{R}(\theta)\otimes I|\chi(q_0)\rangle\\ &\quad=\mathcal{R}(\theta')\otimes\mathcal{R}(\theta)\cdot\hat{D}(\beta_1',\beta_2')\cdot\hat{T}(q_a)\mathcal{S}(r)\otimes I|\chi(q_0)\rangle.\end{aligned}$$

In the second equality above, we used the fact that $\hat{T}(q_a)$ and $\mathcal{R}(\theta')$ commute. Also, $\hat{d}$ there is *not* unitary. However, using the BCH formula and the vacuum state property $a_k|00\rangle = 0$, we can always construct a unitary operator $\hat{D}(\beta_1',\beta_2')$ so that the final equality above holds and β_1', β_2' are certain linear functions of β_1, β_2. This completes the proof of Eq. (15). ■

Theorem 2. Using the entanglement formation as the entanglement measure, if the entanglement of $\rho^G(q_b)$ is equal to that of TMSS $|\chi(q_0)\rangle$, the entanglement of $\rho^G(q = q_a q_b)$ must not be larger than that of TMSS $|\chi(q_a q_0)\rangle$. Mathematically, we say that if $|q| \leqslant |q_b| \leqslant 1$ we have

$$\frac{E[I \otimes \$(|\varphi(q)\rangle\langle\varphi(q)|)]}{E[I \otimes \$(|\varphi(q_b)\rangle\langle\varphi(q_b)|)]} \leqslant \frac{E[|\varphi(q)\rangle\langle\varphi(q)|]}{E[|\varphi(q_b)\rangle\langle\varphi(q_b)|]}. \qquad (16)$$

Here $|\varphi(q)\rangle = I \otimes V|\chi(q)\rangle$ as defined earlier, and V can be any Gaussian unitary operator.

Proof. Using Eqs. (12) and (13) with Eq. (15), we have

$$\begin{aligned}
&E[\rho^G(q = q_a q_b)]\\
&= E[I \otimes U_2 \cdot \hat{T}(q_a)U_1 \otimes I\rho^s U_1^\dagger \hat{T}^\dagger(q_a) \otimes I \cdot I \otimes U_2^\dagger]\\
&= E\Bigg[\mathcal{R}(\theta_1') \otimes U_2\mathcal{R}(\theta_1)\Bigg(\int d^2\beta_1 d^2\beta_2 P(\beta_1,\beta_2)\\
&\quad \times \hat{D}(\beta_1',\beta_2') \cdot \hat{T}(q_a)\mathcal{S}(r_1) \otimes I|\chi(q_0)\rangle\langle\chi(q_0)|\mathcal{S}^\dagger(r_1)\\
&\quad \hat{T}^\dagger(q_a) \otimes I \cdot \hat{D}^\dagger(\beta_1',\beta_2')\Bigg)\mathcal{R}^\dagger(\theta_1') \otimes \mathcal{R}^\dagger(\theta_1)U_2^\dagger\Bigg]\\
&\leqslant E\Bigg(\int d^2\beta_1 d^2\beta_2 P(\beta_1,\beta_2)\hat{D}(\beta_1',\beta_2') \cdot \hat{T}(q_a) \otimes I\\
&\quad \times |\chi(q_0)\rangle\langle\chi(q_0)|\hat{T}^\dagger(q_a) \otimes I \cdot \hat{D}^\dagger(\beta_1',\beta_2')\Bigg)\\
&\leqslant |q_a q_0|^2 = E[|\chi(q_a)\rangle\langle\chi(q_a)|]E[\rho^G(q_b)]. \qquad (17)
\end{aligned}$$

In the third step above we used Theorem 1 for the inequality sign. ■

Remark. Obviously, the inequality of formula (16) also holds if we replace $|\varphi(q)\rangle$ by $|g(U,V,q)\rangle$ and replace $|\varphi(q_b)\rangle$ by $|g(U',V,q_b)\rangle$, and U, U' can be arbitrary unitary operators. Theorem 2 also gives rise to the following theorem.

Theorem 3. Given the one-sided Gaussian map $I \otimes \$$, if the equality sign holds in formula (16) for two specific values q,q_b and $0 < |q| < |q_b| \leqslant 1$, then the equality sign there holds even if q,q_b are replaced by any q',q'', respectively, as long as $|q'|,|q''| \in [|q|,1]$.

Proof. For simplicity, we first consider the case where q is replaced by any q'.

(1) Suppose $|q'| \in [|q|,|q_b|]$. The left-hand side of formula (16) is equivalent to $w' \cdot z'$, and $w' = \frac{E[I\otimes\$(|\varphi(q)\rangle\langle\varphi(q)|)]}{E[I\otimes\$(|\varphi(q')\rangle\langle\varphi(q')|)]}$ and $z' = \frac{E[I\otimes\$(|\varphi(q')\rangle\langle\varphi(q')|)]}{E[I\otimes\$(|\varphi(q_b)\rangle\langle\varphi(q_b)|)]}$. The right-hand side of formula (16) is equivalent to $w \cdot z$ and $w = \frac{E[|\varphi(q)\rangle\langle\varphi(q)|]}{E[|\varphi(q')\rangle\langle\varphi(q')|]}$ and $z = \frac{E[|\varphi(q')\rangle\langle\varphi(q')|]}{E[|\varphi(q_b)\rangle\langle\varphi(q_b)|]}$. Theorem 2 itself says that $w' \leqslant w$ and $z' \leqslant z$. If the equality sign holds in formula (16), we have $w' \cdot z' = w \cdot z$; hence we must have $w = w'$ and $z = z'$, which is just Theorem 3 in the case q is replaced by q'.

(2) Suppose $|q'| > |q_b|$. As is already known, $\rho^G(q) = \hat{T}(q_a) \otimes I\rho^G(q_b)\hat{T}^\dagger(q_a) \otimes I$. Consider Eq. (15). Unitary U_1 in the optimal decomposition of Eq. (13) must be a rotation operator only (i.e., it contains no squeezing), because, otherwise, according to Theorem 1, $E[\rho^G(q)]$ is strictly less than $|q_0 q_a|^2$, which means the equality in formula (16) does not hold. We denote $q' = q_b/q_c$ and $|q_c| < 1$. We have

$$\begin{aligned}
&\rho^G(q' = q_b/q_c)\\
&= \hat{T}^{-1}(q_c) \otimes I\rho^G(q_b)(\hat{T}^{-1}(q_c) \otimes I)^\dagger\\
&= \hat{T}^{-1}(q_c) \otimes I \cdot \mathcal{R}_1 \otimes U_2\rho^s\mathcal{R}_1^\dagger \otimes U_2^\dagger \cdot \hat{T}^{-1}(q_c) \otimes I\\
&= \mathcal{R}_1 \otimes U_2 \int d^2\beta_1 d^2\beta_2 P(\beta_1,\beta_2)\hat{D}(\beta_1',\beta_2')\\
&\quad \times |\chi(q_0/q_c)\rangle\langle\chi(q_0/q_c)|\hat{D}^\dagger(\beta_1',\beta_2') \cdot \mathcal{R}_1^\dagger \otimes U_2^\dagger. \qquad (18)
\end{aligned}$$

Here we used $\hat{T}^{-1}(q_c) \otimes I|\chi(q_b = q'q_c)\rangle = |\chi(q')\rangle$. We used the optimal decomposition for $\rho^G(q_b)$ in the second equality, and Lemma 1 in the last equality above. Equation (18) is one possible decomposition of the state $\rho^G(q')$ but not necessarily the optimized decomposition. Therefore, $E[\rho^G(q' = q_b/q_c)] \leqslant |q_0|^2/|q_c|^2 = |q'|^2/|q_b|^2 \cdot E[\rho^G(q_b)]$. On the other hand, according to Theorem 2, we further obtain that $E[\rho^G(q_b = q'q_c)] \leqslant |q_b|^2/|q'|^2 \cdot E[\rho^G(q')]$.

Remark. Since here $|q'| \geqslant q_b$, the sign $\leqslant$ should be replaced by the sign $\geqslant$ in formula (16), when q is replaced by q'. These two inequalities and result of (1) lead to

$$\frac{E[\rho^G(q')]}{E[\rho^G(q_b)]} = \frac{E[|\chi(q')\rangle\langle\chi(q')|]}{E[|\chi(q_b)\rangle\langle\chi(q_b)|]} \qquad (19)$$

for any q' provided that $|q| \leqslant |q'| \leqslant 1$. Replacing the symbol q' above by the symbol q'', we have another equation. Comparing these two equations we conclude Theorem 3. ■

Lemma 2. Given any Gaussian unitaries U and V, we have

$$E[I \otimes \$(U \otimes V|\phi^+\rangle\langle\phi^+|U^\dagger \otimes V^\dagger)] = E[I \otimes \$(|\phi^+\rangle)]. \qquad (20)$$

Here $|\phi^+\rangle$ is the maximally entangled state defined as the simultaneous eigenstate of position difference $\hat{x}_1 - \hat{x}_2$ and momentum sum $\hat{p}_1 + \hat{p}_2$, with both eigenvalues being zero. Also, when $q = 1$, the state $|\chi(q)\rangle = |\phi^+\rangle$.

Proof. We prove the following fact first. For any local Gaussian unitary operators U and V, we can always find another Gaussian unitary operator $\mathcal{V}$ so that

$$U \otimes V|\phi^+\rangle = \mathcal{V} \otimes I|\phi^+\rangle. \qquad (21)$$

Proof. Any local Gaussian unitary operator can be decomposed into the product form of $\mathcal{R}(\theta')\mathcal{S}(r)\mathcal{R}(\theta)$. For any TMSS $|\chi(q)\rangle$, we have $\mathcal{R}(\theta_1) \otimes \mathcal{R}(\theta_2)|\chi(q)\rangle = I \otimes \mathcal{R}(\theta_1 + \theta_2)|\chi(q)\rangle$. For the maximally TMSS $|\phi^+\rangle$ we have $\mathcal{S}(r) \otimes \mathcal{S}(r)|\phi^+\rangle = |\phi^+\rangle$, for which both sides are the simultaneous eigenstates of position difference and momentum sum, with both eigenvalues being zero. This is confirmed by directly using Eq. (8): for the maximally TMSS $|\phi^+\rangle = \lim_{q\to 1}|\chi(q)\rangle$, we have $\mathcal{S}(r)\otimes I|\phi^+\rangle = I\otimes\mathcal{S}^\dagger(r)|\phi^+\rangle$. Suppose $V=\mathcal{R}(\theta'_B)\mathcal{S}(r_B)\mathcal{R}(\theta_B)$; then

$$U\otimes V|\phi^+\rangle = \mathcal{V}\otimes I|\phi^+\rangle, \tag{22}$$

where $\mathcal{V}=U\mathcal{R}(\theta_B)\mathcal{S}^\dagger(r_B)\mathcal{R}(\theta'_B)$. This completes the proof of Eq. (21). Then we can easily get Eq. (20) by using this fact. ∎

If the equality sign in formula (16) holds, we can apply Theorem 3 by replacing q_b by 1 and we obtain $E[\rho^G(q')] = |q'|^2\cdot E[I\otimes\$(|\phi^+\rangle)]$. On the other hand, by using Theorem 2 and Lemma 2, we have $E[\rho^G(q')]\leqslant |q'|^2\cdot E[I\otimes\$(|\phi^+\rangle)]$. This means

$$E[\rho^G(q')] = \max_{\{V'\}}\{E[I\otimes\$(|g(I,V',q')\rangle)]\}, \tag{23}$$

where $\rho^G(q') = I\otimes\$(|g(I,V,q')\rangle\langle g(I,V,q')|)$ as defined earlier, and $\{V'\}$ is the set containing all single-mode Gaussian unitaries. The equality holds for *any* q' provided that the equality of formula (16) holds for two specific values q,q_b and $|q'|\geqslant|q|$.

Theorem 3 leads to the following conclusion for the maximization of the entanglement evolution under a one-sided Gaussian map.

Conclusion. Suppose that we have a TMSS $|\chi(q')\rangle$. We want to maximize the entanglement distribution over a one-sided Gaussian map $I\otimes\$$ by taking the local Gaussian unitary operation $I\otimes V'$ before entanglement distribution. Although we do not have complete information of the map $I\otimes\$$, it is still possible for us to find out a specific Gaussian unitary operation V so that the entanglement distribution is maximized over all V', for an initial state $|\chi(q')\rangle$ with *any* $|q'|\geqslant|q|$, as long as we can find two specific values $|q_b|>|q|$ such that the equality sign in formula (16) holds. Obviously, the conclusion is also correct for any initial state which is a Gaussian pure state.

The conclusion actually says that, in verifying that V can maximize the entanglement distribution for all initial states $\{|\chi(q')\rangle \,|\, |q'|\geqslant|q|\}$, we only need to verify the equality sign of formula (16) for two specific values.

IV. EXPERIMENTAL PROPOSAL

To experimentally test our major conclusion, we can consider the following beam splitter (BS) channel: Initially, beams 1 and 2 are in a TMSS $\rho_{12}=|\chi(q')\rangle\langle\chi(q')|$, which is the initial bipartite Gaussian pure state. Writing the CM of ρ_{12} in the form of Eq. (4), we find $A=B=\mathrm{diag}[x,x], C=\mathrm{diag}[y,-y]$ with $x=\frac{1}{1-q'^2}+\frac{1}{2}, y=\sqrt{x^2-1/4}$. Beam 3 is in a squeezed thermal state $\rho_3=\tilde{S}(u_3)\rho_{\rm th}\tilde{S}^\dagger(u_3)$. Here $\tilde{S}(u)$ is a squeezing operator defined by

$$\tilde{S}(u)(\hat{x},\hat{p})\tilde{S}^\dagger(u) = (u\hat{x},\hat{p}/u)$$

and $\rho_{\rm th}$ is a thermal state whose CM is $\mathrm{diag}[b_3,b_3]$. Beam 3 together with the BS makes the one-sided Gaussian channel $I\otimes\$_B$ such that

$$\begin{aligned}\rho'_{12} &= I\otimes\$_B(|\chi(q')\rangle\langle\chi(q')|)\\ &= \mathrm{Tr}_3(I\otimes U_B\cdot\rho_{12}\otimes\rho_3\cdot I\otimes U_B^\dagger),\end{aligned}$$

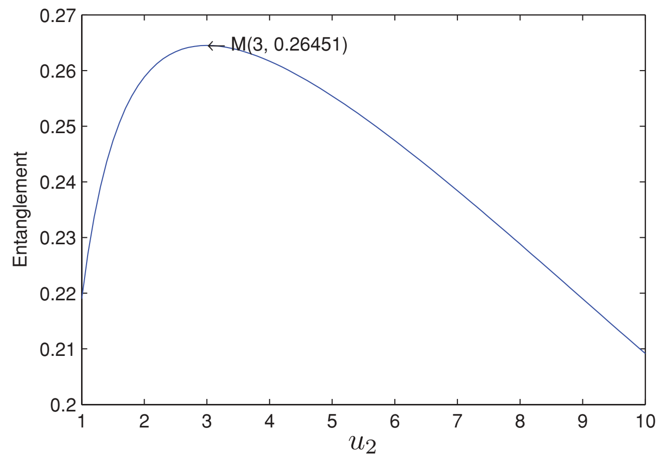

FIG. 1. (Color online) The entanglement with different squeezing factors u_2. Point M corresponds to the maximum entanglement with $u_2=u_3=3$. Here we set $u_3=3$ and $q'=2/3$, $\theta=\pi/6$, and $b_3=1$.

where ρ'_{12} is the final output state of modes 1 and 2, and Tr_3 is the partial trace operator over beam 3. A beam splitter will transform $\hat{x}_2,\hat{x}_3$ by

$$U_B(\hat{x}_2,\hat{x}_3)U_B^{-1}\longrightarrow(\hat{x}_2,\hat{x}_3)\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}.$$

Taking a squeezed transformation $\tilde{S}(u_2)$ on mode 2 before sending beams 2 and 3 into the BS, the output state $\tilde{\rho}'_{12}$ of mode 1 and 2 should be

$$\begin{aligned}\tilde{\rho}'_{12} &= I\otimes\$_B(I\otimes\tilde{S}(u_2)|\chi(q')\rangle\langle\chi(q')|I\otimes\tilde{S}^\dagger(u_2))\\ &= \mathrm{Tr}_3[I\otimes U_B\cdot I\otimes\tilde{S}(u_2)\otimes I\\ &\quad\cdot\rho_{12}\otimes\rho_3\cdot I\otimes\tilde{S}^\dagger(u_2)\otimes I\cdot I\otimes U_B^\dagger].\end{aligned}$$

The CM of $\tilde{\rho}'_{12}$ is denoted by $\tilde{\Lambda}'_{12}$. Writing $\tilde{\Lambda}'_{12}$ in the form of Eq. (4), we find $A=\mathrm{diag}[x,x]$, $C=\mathrm{diag}[y\cos\theta\sqrt{u_2},-y\cos\theta/\sqrt{u_2}]$, and $B=\mathrm{diag}[u_2x\cos^2\theta+u_3b_3\sin^2\theta, x\cos^2\theta/u_2+b_3\sin^2\theta/u_3]$. Using the results presented in [9], we find that the equality sign in formula (16) can hold with $V=\tilde{S}(u_2=u_3)$ for any two different q and q_b.

In an experiment, we can take, for example, $q=0.02$ and $q_b=0.5$, and testing with many different V we should find that the equality sign in formula (16) can hold with $V=\tilde{S}(u_2=u_3)$. Our major conclusion is verified if we can find that the same $V=\tilde{S}(u_3)$ always maximizes the output entanglement for any input state $|\chi(q')\rangle$ provided that $|q'|\geqslant 0.02$. Numerical calculation is shown in Fig. 1, from which we can find that the maximum entanglement is really attained when $V=\tilde{S}(u_3)$.

V. CONCLUDING REMARK

In summary, we present an upper bound of the entanglement evolution of a two-mode Gaussian pure state under a one-sided Gaussian map. We show that one can maximize the entanglement distribution over an unknown one-sided Gaussian noisy channel by testing the channel with only two specific states. An experimental scheme is proposed.

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APPENDIX

Here we give the details of the proof of Eq. (8). The following lemma is used.

Lemma 3 [19]. If $\mathcal{A}$ and $\mathcal{B}$ are two noncommuting operators that satisfy the conditions

$$[\mathcal{A},[\mathcal{A},\mathcal{B}]] = [\mathcal{B},[\mathcal{A},\mathcal{B}]] = 0,$$

then $e^{\mathcal{A}+\mathcal{B}} = e^{\mathcal{A}}e^{\mathcal{B}}e^{-\frac{1}{2}[\mathcal{A},\mathcal{B}]}$.

The squeezing operator $S(r) = e^{r(a^{\dagger 2}-a^2)}$ can be normally ordered as [20]

$$S(r) = \frac{1}{\sqrt{\cosh(2r)}}e^{\frac{1}{2}a^{\dagger 2}\tanh(2r)}e^{-a^\dagger a\{\ln[\cosh(2r)]\}}e^{-\frac{1}{2}a^2\tanh(2r)}.$$

We neglect the constant of normalization in all the following calculations:

$$\begin{aligned}
&I \otimes S(r)|\chi(q_0)\rangle \\
&\quad = e^{r(a_2^{\dagger 2}-a_2^2)}e^{q_0 a_1^\dagger a_2^\dagger}|00\rangle \\
&\quad = e^{q_0 a_1^\dagger[a_2^\dagger\cosh(2r)-a_2\sinh(2r)]}e^{r(a_2^{\dagger 2}-a_2^2)}|00\rangle \\
&\quad = e^{q_0 a_1^\dagger[a_2^\dagger\cosh(2r)-a_2\sinh(2r)]}e^{\frac{1}{2}a_2^{\dagger 2}\tanh(2r)}|00\rangle \\
&\quad = e^{\frac{1}{2}a_2^{\dagger 2}\tanh(2r)}e^{q_0 a_1^\dagger\{a_2^\dagger\cosh(2r)-[a_2+a_2^\dagger\tanh(2r)]\sinh(2r)\}}|00\rangle \\
&\quad = e^{\frac{1}{2}a_2^{\dagger 2}\tanh(2r)}e^{q_0 a_1^\dagger[\frac{a_2^\dagger}{\cosh(2r)}-a_2\sinh(2r)]}|00\rangle \\
&\quad = e^{\frac{1}{2}a_2^{\dagger 2}\tanh(2r)}e^{\frac{q_0 a_1^\dagger a_2^\dagger}{\cosh(2r)}}e^{-\frac{1}{2}a_1^{\dagger 2}q_0^2\tanh(2r)}|00\rangle \\
&\quad = e^{-\frac{1}{2}a_1^{\dagger 2}q_0^2\tanh(2r)+\frac{1}{2}a_2^{\dagger 2}\tanh(2r)+\frac{q_0 a_1^\dagger a_2^\dagger}{\cosh(2r)}}|00\rangle.
\end{aligned}$$

In the second-to-last equality we have used Lemma 3. This completes the proof of Eq. (8).

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