Delayed birth of distillable entanglement in the evolution of bound entangled states

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The dynamical creation of entanglement between three-level atoms coupled to the common vacuum is investigated. For the class of bound entangled initial states, we show that the dynamics of closely separated atoms generates stationary distillable entanglement of asymptotic states. We also find that the effect of delayed sudden birth of distillable entanglement occurs in the case of atoms separated by a distance comparable with the radiation wavelength.

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I. INTRODUCTION

Dynamical creation of entanglement by the indirect interaction between otherwise decoupled systems has been studied by many researchers mainly in the case of two-level atoms that interact with the common vacuum. When the two atoms are separated by a small distance compared to the radiation wavelength λ, there is a substantial probability that a photon emitted by one atom will be absorbed by the other, and the resulting process of photon exchange produces correlations between the atoms. Such correlations may cause that initially separable states become entangled (see, e.g., Refs. [1–4]). The case of three-level atoms is very interesting for many reasons. First of all, in a system of coupled multilevel atoms, which have closely lying energy states and that interact with the vacuum, quantum interference between different radiative transitions can occur, which results in coherences in a system that are known as *vacuum-induced coherences*. In addition to the usual effects, such as collective damping and dipole-dipole interaction, which involves nonorthogonal transition dipole moments [5,6], here, radiative coupling can produce a new interference effect in the spontaneous emission. This effect manifests by the cross coupling between radiative transitions with *orthogonal* dipole moments [7] and is strongly dependent on the relative orientation of the atoms [8,9]. All such collective properties of the system influence the quantum dynamics, which can significantly differ from a corresponding single atom dynamics. On the other hand, the theory of entanglement between the pairs of such atoms is much more complex than in the case of qubits. As is well known, there is no simple necessary and sufficient condition of entanglement, since the Peres-Horodecki separability criterion [10,11] only shows that the states that are not positive after partial transposition [(NPPT) states] are entangled. However, there can exist entangled states that are positive after this operation, and such states are not distillable [12]. Hence, all entangled states can be divided into two classes; one contains free entangled states that can be distilled by using local operations and classical communication (LOCC), and the other consists of bound entangled states for which no LOCC strategy can be used to extract pure state entanglement. Since many effects in quantum information exploit pure maximally entangled states, only distillable states can directly be used for quantum communication. To the contrary, nondistillable states involve some kind of irreversibility: We need pure entanglement to create them, but no pure entanglement can be obtained back from them [13].

The dynamics of entanglement between three-level atoms with vacuum-induced coherences was previously investigated mainly in the context of creation or degradation of NPPT states [14,15]. In particular, it was shown that, for small distances between the atoms, the system decays to a stationary state that can be entangled, even if the initial state was separable [14]. On the other hand, if the distance is comparable to the radiation wavelength, the dynamics brings all initial states into the asymptotic state in which both atoms are in their ground states, but there still can be some transient entanglement between the atoms [15].

In the present paper, we investigate the dynamics of bound entangled initial states. There are some results that concern decoherence and disentanglement of bound entangled states [16]; but, here, we focus on the process of dynamical creation of distillable entanglement due to the collective damping and cross coupling between the three-level atoms. For the specific bound entangled initial state and the small interatomic distance, we show that the asymptotic state is both entangled and distillable. Thus, we obtain a stationary-free entanglement. The same result is also valid for other initial states, which include separable states. So, this dynamics of three-level atoms distinguishes distillable states, since all nontrivial asymptotic entangled states are also distillable. For larger distances, the dynamics of the bound entangled initial state is very peculiar: The system very quickly disentangles, and only after some finite time, does a distillable entanglement suddenly appear. (The similar phenomenon of delayed sudden birth of entanglement was observed in the case of two-level atoms [17].) So also, in this situation, the physical process of spontaneous emission can create some transient distillable entanglement out of the initially prepared bound entanglement.

II. MIXED-STATE ENTANGLEMENT AND DISTILLATION

A. Distillability of entanglement

Distillability of mixed entangled state ρ is the property that enables to convert n copies of ρ into a smaller number of

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k copies of a maximally entangled pure state by means of LOCC [18]. It is known that all pure entangled states can be reversibly distilled [19], and any mixed two-qubit entangled state is also distillable [20]. In the general case, the following necessary and sufficient condition for entanglement distillation was shown in Ref. [12]: The state ρ is distillable if and only if there exists n such that ρ is n-copy distillable (i.e., $\rho^{\otimes n}$ can be filtered to a two-qubit entangled state). However, this condition is hard to apply, since conclusions based on a few copies may be misleading [21]. A more practical but not necessary condition is based on the reduction criterion of separability [22]. The criterion can be stated as follows: If a bipartite state ρ of a compound system AB is separable, then

$$\rho_A \otimes \mathbb{1} - \rho \geqslant 0 \quad \text{and} \quad \mathbb{1} \otimes \rho_B - \rho \geqslant 0, \tag{1}$$

where

$$\rho_A = \mathrm{tr}_B\, \rho, \quad \rho_B = \mathrm{tr}_A\, \rho. \tag{2}$$

As shown in Ref. [23], any state that violates Eq. (1) is distillable, so if

$$\rho_A \otimes \mathbb{1} - \rho \not\geqslant 0 \quad \text{or} \quad \mathbb{1} \otimes \rho_B - \rho \not\geqslant 0, \tag{3}$$

the state ρ can be distilled. The condition Eq. (3) is easy to check, and we will use it in our discussion of dynamical aspects of distillability.

B. Peres-Horodecki criterion and bound entanglement

To detect entangled states of two qubits, we apply the Peres-Horodecki criterion of separability [10,11]. From this criterion follows that any state ρ for which its partial transposition ρ^{PT} is nonpositive (NPPT state) is entangled. One also defines *negativity* of the state ρ as

$$N(\rho) = \frac{\|\rho^{\mathrm{PT}}\|_{\mathrm{tr}} - 1}{2}. \tag{4}$$

$N(\rho)$ is equal to the absolute value of the sum of the negative eigenvalues of ρ^{PT} and is an entanglement monotone [24], however, it cannot detect entangled states that are positive under partial transposition [(PPT) states]. Such states exist [25] and, as shown in Ref. [12], are not distillable. They are called *bound entangled PPT states*. Up to now, it is not known if there exist bound entangled NPPT states [26].

To detect some of the bound entangled PPT states, we can use the realignment criterion of separability [27,28]. The criterion states that, for any separable state ρ of a compound system, the matrix $R(\rho)$ with elements,

$$\langle m| \otimes \langle \mu| R(\rho) |n\rangle \otimes |\nu\rangle = \langle m| \otimes \langle n|\, \rho\, |\mu\rangle \otimes |\nu\rangle \tag{5}$$

has a trace norm not greater than 1. So, if the *realignment negativity* defined by

$$N_R(\rho) = \max\left(0, \frac{\|R(\rho)\|_{\mathrm{tr}} - 1}{2}\right) \tag{6}$$

is greater then zero, the state ρ is entangled. In the case of two qubits, the measure Eq. (6) cannot detect all NPPT states [29]; but, for larger dimensions, the criterion is capable of detecting some bound entangled PPT states [27].

III. TIME EVOLUTION OF THREE-LEVEL ATOMS

To study the dynamics of entanglement between three-level atoms, we consider the model introduced by Agarwal and Patnaik [7]. We start with the short description of the model. Consider two identical three-level atoms (A and B) in the V configuration. The atoms have two near-degenerate excited states $|1_\alpha\rangle$, $|2_\alpha\rangle$ ($\alpha = A, B$) and ground states $|3_\alpha\rangle$. Assume that the atoms interact with the common vacuum and that transition dipole moments of atom A are parallel to the transition dipole moments of atom B. Due to this interaction, the process of spontaneous emission from two excited levels to the ground state takes place in each individual atom, but a direct transition between excited levels is not possible. Moreover, the coupling between two atoms can be produced by the exchange of the photons, but, in such an atomic system, the radiative process in which atom A in excited state $|1_A\rangle$ loses its excitation, which, in turn, excites atom B to state $|2_B\rangle$ is also possible. This effect manifests by the cross coupling between radiation transitions with orthogonal dipole moments. The evolution of this atomic system can be described by the following master equation [7]:

$$\frac{d\rho}{dt} = i\,[H, \rho] + (L^A + L^B + L^{AB})\rho, \tag{7}$$

where

$$H = \sum_{k=1}^{2} \Omega_{k3}\left(\sigma_{k3}^A \sigma_{3k}^B + \sigma_{k3}^B \sigma_{3k}^A\right) + \sum_{\alpha=A,B} \Omega_{vs}\left(\sigma_{23}^\alpha \sigma_{31}^{\neg\alpha} + \sigma_{32}^\alpha \sigma_{13}^{\neg\alpha}\right), \tag{8}$$

and, for $\alpha = A, B$,

$$L^\alpha \rho = \sum_{k=1}^{2} \gamma_{k3}\left(2\sigma_{3k}^\alpha \rho \sigma_{k3}^\alpha - \sigma_{a3}^\alpha \sigma_{3k}^\alpha \rho - \rho \sigma_{k3}^\alpha \sigma_{3k}^\alpha\right). \tag{9}$$

Moreover,

$$L^{AB}\rho = \sum_{k=1}^{2} \sum_{\alpha=A,B} \Gamma_{k3}\left(2\sigma_{3k}^\alpha \rho \sigma_{k3}^{\neg\alpha} - \sigma_{k3}^{\neg\alpha}\sigma_{3k}^\alpha \rho - \rho \sigma_{k3}^{\neg\alpha}\sigma_{3k}^\alpha\right) + \sum_{\alpha=A,B} \Gamma_{vc}\left(2\sigma_{31}^\alpha \rho \sigma_{23}^{\neg\alpha} - \sigma_{23}^{\neg\alpha}\sigma_{31}^\alpha \rho - \rho \sigma_{23}^{\neg\alpha}\sigma_{31}^\alpha + 2\sigma_{32}^\alpha \rho \sigma_{13}^{\neg\alpha} - \sigma_{13}^{\neg\alpha}\sigma_{32}^\alpha \rho - \rho \sigma_{13}^{\neg\alpha}\sigma_{32}^\alpha\right). \tag{10}$$

In Eqs. (8)–(10), $\neg\alpha$ is A for $\alpha = B$ and B for $\alpha = A$, σ_{jk}^α is the transition operator from $|k_\alpha\rangle$ to $|j_\alpha\rangle$, and the coefficient γ_{j3} represents the single atom spontaneous-decay rate from state $|j\rangle$ ($j = 1,2$) to state $|3\rangle$. Since states $|1_\alpha\rangle$ and $|2_\alpha\rangle$ are closely lying, the transition frequencies ω_{13} and ω_{23} satisfy

$$\omega_{13} \approx \omega_{23} = \omega_0. \tag{11}$$

Similarly, the spontaneous-decay rates:

$$\gamma_{13} \approx \gamma_{23} = \gamma. \tag{12}$$

The coefficients Γ_{j3} and Ω_{j3} are related to the coupling between two atoms and are the collective damping and the dipole-dipole interaction potential, respectively. The coherence terms Γ_{vc} and Ω_{vc} are cross-coupling coefficients, which couple a pair of orthogonal dipoles. Detailed analysis shows the cross

coupling between two atoms strongly depends on the relative orientation of the atoms, and there are such configurations of the atomic system that $\Gamma_{vc} = \Omega_{vc} = 0$ and the other configurations for which $\Gamma_{vc} \neq 0$, $\Omega_{vc} \neq 0$. Moreover, all the coupling coefficients are small for large distance R between the atoms and tend to zero for $R \to \infty$. On the other hand, when $R \to 0$, Ω_{13}, Ω_{23}, and Ω_{vc} diverge, whereas

$$\Gamma_{13}, \Gamma_{23} \to \gamma \quad \text{and} \quad \Gamma_{vc} \to 0. \tag{13}$$

The time evolution of the initial state of the system is given by the semigroup $\{T_t\}_{t \geqslant 0}$ of completely positive mappings, which act on density matrices, generated by the Hamiltonian Eq. (8) and the dissipative part $L^A + L^B + L^{AB}$. The properties of the semigroup crucially depend on the distance R between the atoms and the geometry of the system. Irrespective of the geometry, when R is large compared to the radiation wavelength, the semigroup $\{T_t\}_{t \geqslant 0}$ is uniquely relaxed with the asymptotic state $|3_A\rangle \otimes |3_B\rangle$. Thus, for any initial state, its entanglement asymptotically approaches 0. However, there can still be some transient entanglement between the atoms. On the other hand, in the strong correlation regime (when $R \to 0$), the semigroup is not uniquely relaxed, and the asymptotic stationary states are nontrivial and depend on initial conditions. The explicit form of the asymptotic state ρ_{as}, for any initial state ρ with matrix elements ρ_{kl} (with respect to the canonical basis), was found in Ref. [14]. It is given by

$$\rho_{\text{as}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & z & -x & -z & w \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{z} & 0 & 0 & y & -\overline{z} & -y & v \\ 0 & 0 & -x & 0 & 0 & -z & x & z & -w \\ 0 & 0 & -\overline{z} & 0 & 0 & -y & \overline{z} & y & -v \\ 0 & 0 & \overline{w} & 0 & 0 & \overline{v} & -\overline{w} & -\overline{v} & t \end{pmatrix}, \tag{14}$$

where

$$\begin{aligned} x &= \tfrac{1}{8}(\rho_{22} + 2\rho_{33} + \rho_{44} + 2\rho_{77} - 2\mathrm{Re}\,\rho_{24} - 4\mathrm{Re}\,\rho_{37}), \\ z &= \tfrac{1}{4}(\rho_{36} - \rho_{38} - \rho_{76} + \rho_{78}), \\ w &= \tfrac{1}{4}(\rho_{26} + \rho_{28} + 2\rho_{39} - \rho_{46} - \rho_{48} - 2\rho_{79}), \\ y &= \tfrac{1}{8}(\rho_{22} + \rho_{44} + 2\rho_{66} + 2\rho_{88} - 2\mathrm{Re}\,\rho_{24} - 4\mathrm{Re}\,\rho_{68}), \\ v &= \tfrac{1}{4}(-\rho_{23} - \rho_{27} + \rho_{43} + \rho_{47} + 2\rho_{69} - 2\rho_{89}), \end{aligned} \tag{15}$$

and

$$t = 1 - 2x - 2y. \tag{16}$$

By depending on the initial state, the asymptotic state Eq. (14) can be separable or entangled. In Sec. IV, we will study distillability of ρ_{as} for some initial states.

IV. GENERATION OF STATIONARY DISTILLABLE ENTANGLEMENT

As shown in Ref. [14], the negativity of the asymptotic states Eq. (14) can be obtained analytically in the case of diagonal (i.e., separable) initial states. For such states, only the parameters x, y, and t are nonzero, and the asymptotic negativity reads

$$N(\rho_{\text{as}}) = \tfrac{1}{2}[\sqrt{4(x^2 + y^2) + t^2} - t]. \tag{17}$$

Note that every nontrivial asymptotic state from that class is entangled. Now, we show that this entanglement is free (i.e., all such asymptotic states are distillable). To do this, we show that ρ_{as}, which corresponds to diagonal initial states, violates reduction criterion Eq. (1). Indeed, since for such states,

$$\mathrm{tr}_B\, \rho_{\text{as}} \otimes \mathbb{1} - \rho_{\text{as}} = \begin{pmatrix} x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & 0 \\ 0 & 0 & x & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}, \tag{18}$$

where

$$a = 1 - 2x - y, \quad b = 1 - x - 2y, \quad c = x + y, \tag{19}$$

and the matrix on the right-hand side of Eq. (18) has two negative leading principal minors (other minors are positive), so

$$\mathrm{tr}_B\, \rho_{\text{as}} \otimes \mathbb{1} - \rho_{\text{as}} \not\geqslant 0. \tag{20}$$

Similarly,

$$\mathbb{1} \otimes \mathrm{tr}_A\, \rho_{\text{as}} - \rho_{\text{as}} \not\geqslant 0. \tag{21}$$

The interesting examples of nontrivial asymptotic states are given by the separable initial states, where one atom is in the excited state and the other is in the ground state, or two atoms are in different excited states. In all such cases, the created entanglement is free and can be distilled.

Now, we consider the possibility for creating free stationary entanglement from the bound initial entanglement. As the initial states, we take the family [30],

$$\rho_\alpha = \frac{2}{7}|\Psi_0\rangle\langle\Psi_0| + \frac{\alpha}{7}P_+ + \frac{5-\alpha}{7}P_-, \quad 3 < \alpha \leqslant 4, \tag{22}$$

where

$$|\Psi_0\rangle = \frac{1}{\sqrt{3}}\sum_{j=1}^{3} |j_A\rangle \otimes |j_B\rangle, \tag{23}$$

$$P_+ = \tfrac{1}{3}\big(P_{|1_A\rangle\otimes|2_B\rangle} + P_{|2_A\rangle\otimes|3_B\rangle} + P_{|3_A\rangle\otimes|1_B\rangle}\big), \tag{24}$$

and

$$P_- = \tfrac{1}{3}\big(P_{|2_A\rangle\otimes|1_B\rangle} + P_{|3_A\rangle\otimes|2_B\rangle} + P_{|1_A\rangle\otimes|3_B\rangle}\big). \tag{25}$$

The states Eq. (22) are constructed as follows: We prepare the maximally entangled state $|\Psi_0\rangle$ and add some specific noise, which results in the mixing of $|\Psi_0\rangle$ with separable states P_+ and P_-. For a special choice of mixing parameter,

such prepared states have positive partial transposition but are entangled, as can be shown by computing the realignment negativity. For ρ_α, it is given by

$$N_R(\rho_\alpha) = \frac{1}{21}(\sqrt{3\alpha^2 - 15\alpha + 19} - 1), \tag{26}$$

and is obviously positive for $3 < \alpha \leqslant 4$. So the states Eq. (22) are bound entangled, and the entanglement initially present in $|\Psi_0\rangle$ cannot be extracted from them for any number of copies of the states. It is worth noticing that, recently, the bound entanglement was created experimentally in the system of three qubits [31].

Although the states Eq. (22) are not diagonal, one can check that the corresponding asymptotic states have the same form as in the diagonal case. In fact, for all initial states ρ_α, there is only one asymptotic state $\rho_{\rm as}$ given by

$$x = y = \frac{5}{56} \quad \text{and} \quad t = \frac{9}{14}. \tag{27}$$

By the previous discussion, this state is entangled; and, moreover, its entanglement is distillable. Thus, we have shown that the physical process of spontaneous emission in the radiatively coupled three-level atoms can transform initial bound entanglement into free distillable entanglement of the asymptotic state.

V. DELAYED CREATION OF DISTILLABLE ENTANGLEMENT

In this section, we study, in detail, the evolution of entanglement of the bound entangled initial states Eq. (22), for $3 < \alpha \leqslant 4$, beyond the strong correlation regime. In that case, the asymptotic state is trivial, but some transient entanglement between the atoms can be produced. For simplicity, we consider such atomic configuration for which the cross-coupling coefficients are equal to zero. One can check that the initial states Eq. (22) will evolve into the states of the form

$$\rho_\alpha(t) = \begin{pmatrix} \rho_{11} & 0 & 0 & 0 & \rho_{15} & 0 & 0 & 0 & \rho_{19} \\ 0 & \rho_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{33} & 0 & 0 & 0 & \rho_{37} & 0 & 0 \\ 0 & 0 & 0 & \rho_{44} & 0 & 0 & 0 & 0 & 0 \\ \rho_{51} & 0 & 0 & 0 & \rho_{55} & 0 & 0 & 0 & \rho_{59} \\ 0 & 0 & 0 & 0 & 0 & \rho_{66} & 0 & \rho_{68} & 0 \\ 0 & 0 & \rho_{73} & 0 & 0 & 0 & \rho_{77} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_{86} & 0 & \rho_{88} & 0 \\ \rho_{91} & 0 & 0 & 0 & \rho_{95} & 0 & 0 & 0 & \rho_{99} \end{pmatrix}, \tag{28}$$

where all nonzero matrix elements are time dependent.

Numerical analysis indicates that, during the time evolution, the realignment negativity Eq. (6) of the initial state very rapidly progresses to zero, so the system almost immediately disentangles. To consider possible creation of free entanglement, let us first check if the PPT condition can be violated during such evolution. After taking the partial transposition, the state Eq. (28) becomes

$$\rho_\alpha(t)^{\rm PT} = \begin{pmatrix} \rho_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_{37} \\ 0 & \rho_{22} & 0 & \rho_{15} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{33} & 0 & 0 & 0 & \rho_{19} & 0 & 0 \\ 0 & \rho_{51} & 0 & \rho_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_{55} & 0 & 0 & 0 & \rho_{68} \\ 0 & 0 & 0 & 0 & 0 & \rho_{66} & 0 & \rho_{59} & 0 \\ 0 & 0 & \rho_{91} & 0 & 0 & 0 & \rho_{77} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_{95} & 0 & \rho_{88} & 0 \\ \rho_{37} & 0 & 0 & 0 & \rho_{86} & 0 & 0 & 0 & \rho_{99} \end{pmatrix}. \tag{29}$$

One can check that determinant d of the matrix Eq. (29) equals

$$d = (\rho_{22}\rho_{44} - |\rho_{15}|^2)(\rho_{33}\rho_{77} - |\rho_{19}|^2)(\rho_{66}\rho_{88} - |\rho_{59}|^2) \times (\rho_{11}\rho_{55}\rho_{99} - \rho_{55}|\rho_{37}|^2 - \rho_{11}|\rho_{86}|^2). \tag{30}$$

We can show numerically that Eq. (30) changes sign, since the last factor is positive for all $t < t_N$ and becomes negative if $t > t_N$, for some $t_N > 0$, and the remaining factors are positive. Moreover, all other leading principal minors of the matrix Eq. (29) are always positive. So, the evolution of the bound entangled state Eq. (22) has the interesting property: For all $t < t_N$, the states $\rho_\alpha(t)$ are PPT; and, then, suddenly they become NPPT states (see Fig. 1). Now, we discuss distillability of the states $\rho_\alpha(t)$. Since we cannot exclude the possibility that there are NPPT states, which are nondistillable, we try to apply the reduction criterion of entanglement. As we know, from the discussion in Sec. II, any state that violates this criterion is necessarily distillable. By direct computations, we show that the matrix,

$$\mathrm{tr}_B\, \rho_\alpha(t) \otimes \mathbb{1} - \rho_\alpha(t) \tag{31}$$

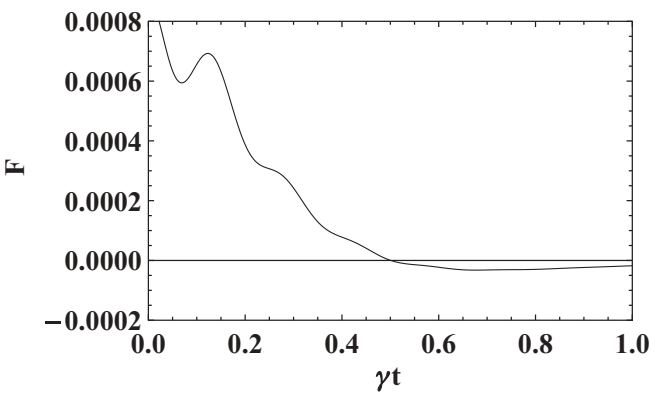

FIG. 1. The time evolution of the factor $F = \rho_{11}\rho_{55}\rho_{99} - \rho_{55}|\rho_{37}|^2 - \rho_{11}|\rho_{86}|^2$ in Eq. (30) for the initial state Eq. (22) with $\alpha = 3.6$ and the interatomic distance $R = 0.2\,\lambda$.

equals

$$\begin{pmatrix} r_{11} & 0 & 0 & 0 & -\rho_{15} & 0 & 0 & 0 & -\rho_{19} \\ 0 & r_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{33} & 0 & 0 & 0 & -\rho_{37} & 0 & 0 \\ 0 & 0 & 0 & r_{44} & 0 & 0 & 0 & 0 & 0 \\ -\rho_{15} & 0 & 0 & 0 & r_{55} & 0 & 0 & 0 & -\rho_{59} \\ 0 & 0 & 0 & 0 & 0 & r_{66} & 0 & -\rho_{68} & 0 \\ 0 & 0 & -\rho_{73} & 0 & 0 & 0 & r_{77} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\rho_{86} & 0 & r_{88} & 0 \\ -\rho_{91} & 0 & 0 & 0 & -\rho_{95} & 0 & 0 & 0 & r_{99} \end{pmatrix}, \tag{32}$$

where

$$r_{kk} = \begin{cases} \rho_{11} + \rho_{22} + \rho_{33} - \rho_{kk}, & k = 1,2,3 \\ \rho_{44} + \rho_{55} + \rho_{66} - \rho_{kk}, & k = 4,5,6. \\ \rho_{77} + \rho_{88} + \rho_{99} - \rho_{kk}, & k = 7,8,9 \end{cases} \tag{33}$$

We compute leading principal minors of the matrix Eq. (32), which can change sign during the evolution, and can find the following expressions:

$$\begin{aligned} m_5 &= r_{22}r_{33}r_{44}(r_{11}r_{55} - |\rho_{15}|^2), \\ m_6 &= r_{22}r_{33}r_{44}r_{66}(r_{11}r_{55} - |\rho_{15}|^2), \\ m_7 &= r_{22}r_{44}r_{66}(r_{11}r_{55} - |\rho_{15}|^2)\,(r_{33}r_{77} - |\rho_{37}|^2), \\ m_8 &= r_{22}r_{44}(r_{11}r_{55} - |\rho_{15}|^2)\,(r_{33}r_{77} - |\rho_{37}|^2)\,(r_{66}r_{88} - |\rho_{68}|^2), \end{aligned} \tag{34}$$

where m_k for $k = 5,6,7,8$ are determinants of principal $k \times k$ submatrices of the matrix Eq. (32). It turns out that the factor $r_{11}r_{55} - |\rho_{15}|^2$ is always positive, but, as follows from the numerical analysis, $r_{33}r_{77} - |\rho_{37}|^2$ as well as $r_{66}r_{88} - |\rho_{68}|^2$ change sign during the evolution (see Fig. 2). Let t_D be the time at which the factor $r_{66}r_{88} - |\rho_{68}|^2$ changes sign. We see that $t_D > 0$, so only after that time does the matrix Eq. (32) become nonpositive. It means that, for $t > t_D$, states $\rho_\alpha(t)$ are necessarily distillable. One can check that $t_D > t_N$, and we see that the initial bound entangled state Eq. (22) evolves in a remarkable way: For all $t \leqslant t_N$, it is PPT, for $t_N < t \leqslant t_D$, it is NPPT but *a priori* can be nondistillable and only after t_D, does it become distillable. To show this in the explicit way,

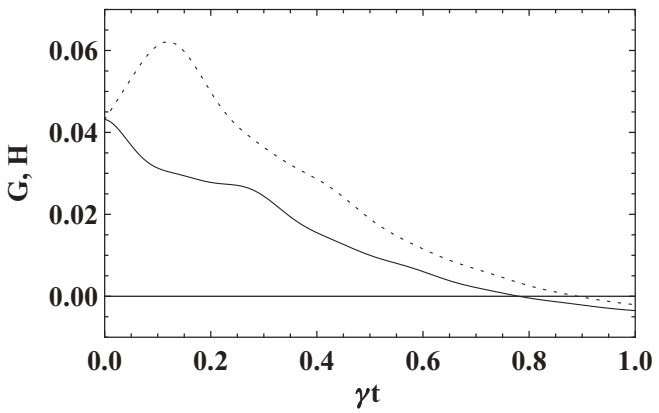

FIG. 2. The time evolution of the factors $G = r_{33}r_{77} - |\rho_{37}|^2$ (dotted line) and $H = r_{66}r_{88} - |\rho_{68}|^2$ (solid line) in Eq. (34) for the initial state Eq. (22) with $\alpha = 3.6$ and $R = 0.2\lambda$.

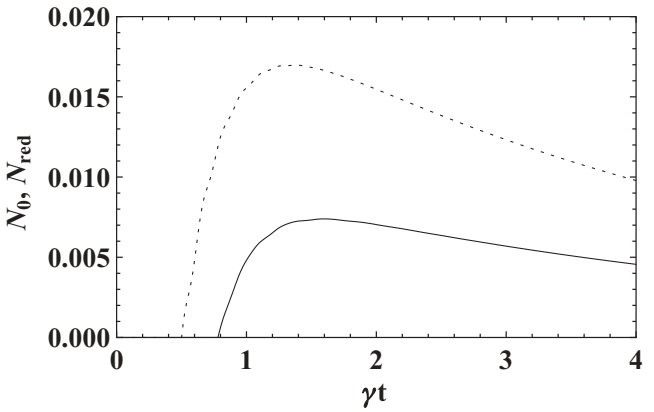

FIG. 3. The time-evolution negativity N_0 (dotted line) and reduction negativity $N_{\rm red}$ (solid line) for the initial state Eq. (22) with $\alpha = 3.6$ and $R = 0.2\lambda$.

let us introduce the measure of the violation of the reduction criterion, defined as

$$N_{\rm red}(\rho) = \max\left(0, -\lambda_{\rm min}^{\rm red}\right), \tag{35}$$

where $\lambda_{\rm min}^{\rm red}$ is the minimal eigenvalue of the matrix:

$$\rho_{\rm red} = {\rm tr}_B\, \rho \otimes \mathbb{1} - \rho. \tag{36}$$

The quantity Eq. (35) can be called the reduction negativity of state ρ. For the bound entangled initial state Eq. (22), the evolution of negativity and reduction negativity is given as follows (Fig. 3). So, in the system, we observe the phenomenon of delayed sudden birth of distillable entanglement. The physical reason for the appearance of this phenomenon can be explained as follows. During the time evolution of the system, the process of the photon exchange produces coherence between states $|1_A\rangle \otimes |3_B\rangle$ and $|3_A\rangle \otimes |1_B\rangle$, so the value of $|\rho_{37}|$ starts to grow. Similarly, the same process causes the production of coherence between states $|2_A\rangle \otimes |3_B\rangle$ and $|3_A\rangle \otimes |2_B\rangle$, so $|\rho_{68}|$ also grows. Notice that the nonzero value of $|\rho_{37}|$ or $|\rho_{68}|$ is necessary for the possibility of creation of distillable entanglement. However, this condition is not sufficient, since the populations of states of the two-atomic system also evolve in time. We see from the formula Eq. (34) that there is a threshold for the reduction negativity Eq. (35) at which the system becomes distillable.

The numerical value of t_D depends on the choice of the parameter α and the interatomic distance R. For the initial state with $\alpha = 3.6$ and the distance $R = 0.2\lambda$, we obtain $t_D\gamma \approx 0.78$, whereas $t_N\gamma \approx 0.49$.

VI. CONCLUSIONS

We have studied the dynamics of entanglement in the system of three-level atoms in the V configuration coupled to the common vacuum. In the case of small (compared to the radiation wavelength) separation between the atoms, the system has nontrivial asymptotic states that can be entangled even if the initial states are separable. For the large class of separable initial states, the asymptotic states are not only entangled but also distillable. The same is true for some class of bound entangled initial states. Thus, we have shown that the

dynamics of the system can transform bound entanglement into the free distillable entanglement of stationary states. For the atoms separated by larger distances, only some transient entanglement can exist, but still, the dynamical generation of entanglement is possible. Also, we have shown that this happens for the class of bound entangled initial states. Moreover, we have demonstrated that such states evolve in a very peculiar way: They almost immediately disentangle after the atoms begin to interact with the vacuum, then, for some finite period of time, there is no entanglement, and suddenly, at some time, the entanglement starts to build up. However, this entanglement *a priori* can be nondistillable. We have analyzed this problem by using the reduction criterion of separability and found that the free entanglement surely appears in the system after some additional period of time.

[1] L. Jakóbczyk, J. Phys. A **35**, 6383 (2002); **36**, 1537 (2003), Corrigendum.
[2] Z. Ficek and R. Tanaś, J. Mod. Opt. **50**, 2765 (2003).
[3] R. Tanaś and Z. Ficek, J. Opt. B: Quantum and Semiclass. Opt. **6**, S90 (2004).
[4] L. Jakóbczyk and J. Jamróz, Phys. Lett. A **318**, 318 (2003).
[5] G. S. Agarwal, *Quantum Statistical Theories of Spontaneous Emission and Their Relation to Other Approaches* (Springer, Berlin, 1974).
[6] Z. Ficek and S. Swain, *Quantum Interference and Coherence: Theory and Experiments* (Springer, Berlin, 2005).
[7] G. S. Agarwal and A. K. Patnaik, Phys. Rev. A **63**, 043805 (2001).
[8] J. Evers, M. Kiffner, M. Macovei, and C. H. Keitel, Phys. Rev. A **73**, 023804 (2006).
[9] S. I. Schmid and J. Evers, Phys. Rev. A **77**, 013822 (2008).
[10] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[11] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[12] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).
[13] D. Yang, M. Horodecki, R. Horodecki, and B. Synak-Radtke, Phys. Rev. Lett. **95**, 190501 (2005).
[14] Ł. Derkacz and L. Jakóbczyk, J. Phys. A **41**, 205304 (2008).
[15] Ł. Derkacz and L. Jakóbczyk, Phys. Lett. A **372**, 7117 (2008).
[16] Z. Sun, X. G. Wang, Y. B. Gao, and C. P. Sun, Eur. Phys. J. D **46**, 521 (2008).
[17] Z. Ficek and R. Tanaś, Phys. Rev. A **77**, 054301 (2008).
[18] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Phys. Rev. Lett. **76**, 722 (1996).
[19] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
[20] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **78**, 574 (1997).
[21] J. Watrous, Phys. Rev. Lett. **93**, 010502 (2004).
[22] N. J. Cerf, C. Adami, and R. M. Gingrich, Phys. Rev. A **60**, 898 (1999).
[23] M. Horodecki and P. Horodecki, Phys. Rev. A **59**, 4206 (1999).
[24] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[25] P. Horodecki, Phys. Lett. A **232**, 333 (1997).
[26] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[27] K. Chen and L.-A. Wu, Quantum Inf. Comput. **3**, 193 (2003).
[28] O. Rudolph, J. Phys. A **36**, 5825 (2003).
[29] O. Rudolph, Phys. Rev. A **67**, 032312 (2003).
[30] P. Horodecki, M. Horodecki, and R. Horodecki, Phys. Rev. Lett. **82**, 1056 (1999).
[31] H. Kampermann, D. Bruss, X. Peng, and D. Suter, Phys. Rev. A **81**, 040304(R) (2010).