Fisher information in a quantum-critical environment

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We consider a process of parameter estimation in a spin-j system surrounded by a quantum-critical spin chain. Quantum Fisher information lies at the heart of the estimation task. We employ Ising spin chain in a transverse field as the environment which exhibits a quantum phase transition. Fisher information decays with time almost monotonously when the environment reaches the critical point. By choosing a fixed time or taking the time average, one can see the quantum Fisher information presents a sudden drop at the critical point. Different initial states of the environment are considered. The phenomenon that the quantum Fisher information, namely, the precision of estimation, changes dramatically can be used to detect the quantum criticality of the environment. We also introduce a general method to obtain the maximal Fisher information for a given state.

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I. INTRODUCTION

Fisher information lies at the heart of a parameter estimation theory that was originally introduced by Fisher [1]. It provides in particular a bound to distinguish the members of a family of probability distributions. When quantum systems are involved, especially for problems in which the quantity of interest is not directly accessible, the optimal measurement may be found using tools from quantum estimation theory. The quantum version of the Cramér-Rao inequality has been established [2–5] and the lower bound is imposed by quantum Fisher information (QFI) [5]. Fisher information becomes a useful tool for evaluating the accuracy limits of quantum measurements; moreover, it has important applications in quantum technology such as quantum frequency standards [6,7], measurement of gravity accelerations [8], and clock synchronization [9].

It is significant to consider the relation between QFI and other important concepts such as quantum entanglement and squeezing. Recently, in Ref. [10] QFI is employed to understand multipartite entanglement. The researchers introduce an operational interpretation for pure-state global multipartite entanglement based on a locally depolarizing channel. After that Pezzé and Smerzi [11] introduced a sufficient condition for N-particle entanglement by making use of QFI, which is more general than the spin squeezing condition. Concerning open quantum systems and nonunitary processes, QFI has been applied to finite dimensional systems [12] to optimally estimate the noise parameter of depolarizing [13] and amplitude-damping channels [14]. In the estimation of the loss parameter of a bosonic channel, by calculating the QFI, people found that Gaussian squeezed probes can improve the estimation [15]. In a noisy quantum system, researchers identified an optimal quantum measurement which maximizes the QFI [16]. Recently, many people have turned to the problem of parameter estimation in systems with quantum phase transition (QPT). In a transverse Ising chain, optimal estimation of coupling constants quantified by QFI is possible at values of the field equal to the critical point [17]. QFI can be also used to distinguish and characterize behaviors of the ground state of the Lipkin-Meskhov-Glick model, which displays a second-order quantum phase transition [18].

In this article, we consider parameter estimation in a spin-j system surrounded by a quantum-critical environment which is described by a quantum Ising chain with transverse field. In most of the previous studies about QFI in noisy systems, people have focused on how the properties of input states, such as entanglement or squeezing, influence the QFI. We think it is also significant to consider that the noisy channel itself possesses special properties, for example, QPTs. The aim of this article is to find how the quantum criticalities of the environment change the QFI and, accordingly, influence the precision of estimation in the center system.

It is known that QPT takes place at zero temperature, at which the thermal fluctuations vanish. Therefore, quantum fluctuation plays the major role in QPT. At the critical point, a qualitative change occurs in the ground state and long-range correlation also develops. Actually, the QPT of the surrounding indeed affects the properties of the center system; for example, it enhances the decoherence of the system [19–21], accelerates the disentanglement [22], and also helps to induce entanglement [23,24] of the system. In this article, we observe that QFI decays with time almost monotonically when the environment approaches the critical point, which implies that the precision of the estimation is depressed evidently by the critical fluctuation. For a fixed time or by taking the time average, QFI presents a sudden drop at the critical point. We stress that the critical fluctuation of the environment enhances the decoherence process and consequently reduces the QFI greatly, that is, it reduces the precision of estimation. Therefore, the critical properties of the environment are harmful to parameter estimation in the coupled system, which is in contrary to the case in Ref. [17], where estimation of the coupling constant is improved by quantum criticality.

From another point of view, the reduction of the estimation precision also reflects that the information captured by us is being lost. We recall another concept—purity, which is closely related to QFI. During some parameter estimation processes, QFI can characterize the purity of the measured state [25,26].

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Therefore, purity is decreased by decoherence as well as QFI and indicates information loss. It is known that, in non-Markovian processes, there exist revivals of coherence [27,28], which can be understood as the reversed flow of information from the environment to the open system [29,30]. In a sense, the results shown by us imply that the critical fluctuation can remarkably prevent the reversed flow of information. Another interesting point of this article is the introduction of a method to detect QPT by coupling the quantum-critical system to the external spin-j system. The region where the estimation accuracy changes dramatically indicates the quantum-critical region of the environment.

The rest of this article is organized as follows: In Sec. II, we introduce the parameter estimation and the definition of QFI. Also a general method to obtain the maximal QFI is be presented. In Sec. III, we describe the driven Hamiltonian and give the time evolution operator. In Sec. IV, the reduced density matrix of the system is obtained and all the elements are calculated in detail. For the small systems of $j = 1/2$ and 1, the analytical results of the maximal QFI are shown. Two initial states of the environment are considered in this section. We also calculate the QFI numerically. Finally, the significant results are given in Sec.V.

II. THE MAXIMAL QFI UNDER UNITARY TRANSFORMATION

A typical problem of quantum estimation is to ask what is the best observable. For example, in order to estimate the true value of parameter θ provided that the system is in one state of the family $\{\rho_\theta\}$, an observable $\hat{\theta}$ is called to be the unbiased estimator, that is, the expectation of the estimator should satisfy $\mathrm{Tr}(\rho_\theta\hat{\theta}) = \theta$, and in general the estimator $\hat{\theta}$ is not unique. Among the unbiased estimators, we should select a good one. Fortunately, an effective tool is presented by the quantum Cramér-Rao (QCR) inequality,

$$\Delta\hat{\theta} \geqslant (\Delta\theta)_{\mathrm{QCR}} = \frac{1}{\sqrt{\nu I(\rho_\theta)}}, \tag{1}$$

which holds for any unbiased estimator of θ and provides a bound to limit the precision of our estimation. In Eq. (1), ν is the number of trails and $I(\rho_\theta)$ is the QFI. In some cases, the state ρ_θ is obtained as the output during a uniform process, $\rho_\theta = \rho_{\mathrm{out}} = U_\theta\rho_{\mathrm{in}}U_\theta^\dagger$, where $U_\theta = \exp(i\theta\hat{K})$ and $\hat{K}$ is a generator. Generally speaking, an interferometer is quantum mechanically described as a collective, linear rotation of the input state by an angle; then the generator $\hat{K} = \hat{J}_{\vec{n}}$ is the angular momentum operator along direction $\vec{n}$. To obtain the QFI, the standard procedure starts by solving for the symmetric logarithmic derivative L_θ, defined as any Hermitian operator that satisfies the equation

$$\partial_\theta\rho_\theta = \tfrac{1}{2}(L_\theta\rho_\theta + \rho_\theta L_\theta). \tag{2}$$

The QFI does not depend on the particular choice of L_θ and is given by

$$I_\theta = \mathrm{Tr}\big(\rho_\theta L_\theta^2\big). \tag{3}$$

For pure states, the QFI is just proportional to the variance of $\hat{J}_{\vec{n}}$ [31], that is, $I(\rho_{\mathrm{in}},\hat{J}_{\vec{n}}) = 4(\Delta\hat{J}_{\vec{n}})^2$. Therefore, besides increasing experimental times ν, we can improve the estimation precision $\Delta\hat{\theta}$ by choosing the proper states ρ_{in} for a given $\hat{J}_{\vec{n}}$. It is found that entangled input states usually induce larger variance of $\hat{J}_{\vec{n}}$ than separable states; thus entangled states are useful to improve parameter sensitivity [32–39]. On the other hand, for a given state ρ_{in}, we can optimize the QFI by choosing a proper rotated direction $\vec{n}$.

Now let us introduce a general method to obtain maximal QFI for a given state ρ [40].

A. Pure state case

The angular momentum in $\vec{n}$ direction is

$$J_{\vec{n}} = \sum_{k=1}^{3} J_k n_k, \tag{4}$$

where the normalized direction $\vec{n} = (n_1,n_2,n_3)$ and the angular momentum operator $J_{1(2,3)}$ corresponds to $J_{x(y,z)}$. The QFI is equal to the variance

$$\begin{aligned}(\Delta J_{\vec{n}})^2 &= \frac{1}{2}\sum_{kl}[\langle J_kJ_l\rangle + \langle J_lJ_k\rangle]n_kn_l - \sum_{kl}\langle J_k\rangle\langle J_l\rangle n_kn_l\\ &= \vec{n}C\vec{n}^T,\end{aligned} \tag{5}$$

where the symmetry covariance matrix

$$C_{kl} = \mathrm{Cov}(J_k,J_l) = \tfrac{1}{2}[\langle J_kJ_l\rangle + \langle J_lJ_k\rangle] - \langle J_k\rangle\langle J_l\rangle. \tag{6}$$

Therefore the variance can be written as

$$(\Delta J_{\vec{n}})^2 = \vec{n}O(O^TCO)O^T\vec{n}^T = \vec{n}'C^d\vec{n}'^T, \tag{7}$$

where O is an orthogonal 3×3 matrix, C^d is the diagonal form of C,

$$C^d = \mathrm{diag}(E_1,E_2,E_3), \tag{8}$$

where we set $E_1 \geqslant E_2 \geqslant E_3$, and the rotated direction $\vec{n}' = \vec{n}O = (n_1',n_2',n_3')$. Now, the maximal variance

$$\max(\Delta J_{\vec{n}})^2 = \max\left(E_1n_1'^2 + E_2n_2'^2 + E_3n_3'^2\right) \leqslant E_1, \tag{9}$$

and in the above equation, the rotated direction is also normalized and satisfies the condition $n_1'^2 + n_2'^2 + n_3'^2 = 1$. To attain $E_{\max} = E_1$, we choose $n_1' = 1$ and $n_2' = n_3' = 0$; therefore, $\vec{n} = \vec{n}'O^T$.

B. Mixed state case

In the case of mixed state ρ, for a rotated angle θ, the operator L_θ in Eq. (2) can be described in the eigenspace of ρ,

$$L_{i,j} = \langle i|L_\theta|j\rangle = \frac{2i(p_j - p_i)\langle i|J_{\vec{n}}|j\rangle}{p_j + p_i}, \tag{10}$$

in which the eigenstate satisfies $\rho|i\rangle = p_i|i\rangle$; during a uniform process $\rho_\theta = U_\theta\rho U_\theta^\dagger$, the eigenvalue p_i will not change. Then by using Eq. (3) we obtain the Fisher information

$$\begin{aligned}I_\theta &= \sum_{i,j}\frac{2(p_i - p_j)^2}{p_i + p_j}|\langle i|J_{\vec{n}}|j\rangle|^2\\ &= \sum_{kl}\sum_{i,j}\frac{2(p_i - p_j)^2}{p_i + p_j}\langle i|J_k|j\rangle\langle j|J_l|i\rangle n_kn_l\\ &= \sum_{kl}\tilde{C}_{kl}n_kn_l = \vec{n}\tilde{C}\vec{n}^T,\end{aligned} \tag{11}$$

where the matrix element for the symmetry matrix $\tilde{C}$ is

$$\tilde{C}_{kl} = \sum_{i,j} \frac{2(p_i - p_j)^2 \langle i|J_k|j\rangle\langle j|J_l|i\rangle}{p_i + p_j}. \quad (12)$$

Therefore, following the previous steps, we can use a unitary matrix $\tilde{O}$ to diagonalize $\tilde{C}$ and find the direction $\vec{n}$ that corresponds to the maximal Fisher information,

$$\tilde{C}^d = \tilde{O}^T \tilde{C} \tilde{O} = \mathrm{diag}(\tilde{E}_1, \tilde{E}_2, \tilde{E}_3), \quad (13)$$

where $\tilde{E}_1 \geqslant \tilde{E}_2 \geqslant \tilde{E}_3$. Therefore, $\vec{n} = \vec{n}'\tilde{O}$, with $\vec{n}' = (1,0,0)$, and the maximal Fisher information $I_{\max} = \tilde{E}_1$.

III. MODEL HAMILTONIAN AND EVOLUTION OPERATOR

We choose the engineered environment system to be an Ising spin chain in a transverse field which displays a QPT. One spin is transversely coupled to the chain. The corresponding Hamiltonian reads

$$H = -\left[\sum_{l=-M}^{M} \sigma_l^x \sigma_{l+1}^x + (\lambda + gJ_z) \sum_{l=-M}^{M} \sigma_l^z\right], \quad (14)$$

where λ characterizes the strength of the transverse field, J_z denotes the z component of the spin operator with the quantum number j. g denotes the coupling strength between the Ising chain and the spin J_z, $\sigma_l^\alpha (\alpha = x,y,z)$ are the Pauli operators defined on the lth site, and the total number of spins in the Ising chain is $L = 2M + 1$. The Ising model is an important model which exhibits QPT and can be exactly calculated.

In order to diagonalize the Hamiltonian, first we notice that $[J_z, \sigma_l^\alpha] = 0$; thus it is convenient to define an operator-valued parameter,

$$\lambda_m = \lambda + gm, \quad (15)$$

with m denoting the eigenvalues of J_z. When we diagonalize the Ising spin chain, the parameter λ_m can be treated as a c number with different values corresponding to the eigenvalues of J_z in the system space.

By combining a Jordan-Wigner transformation and a Fourier transformation in the momentum space [41], the Hamiltonian can be written as [22,42]

$$H_m = \sum_{k>0} \left(\Omega_{k,m} e^{i\frac{\theta_{k,m}}{2}\sigma_{kx}} \sigma_{kz} e^{-i\frac{\theta_{k,m}}{2}\sigma_{kx}}\right) + (\lambda_m - 1)\sigma_{0z}, \quad (16)$$

where we have used the following pseudospin operators $\sigma_{k\alpha}(\alpha = x,y,z)$,

$$\begin{aligned} \sigma_{kx} &= d_k^\dagger d_{-k}^\dagger + d_{-k} d_k, (k = 1,2,\ldots,M), \\ \sigma_{ky} &= -i d_k^\dagger d_{-k}^\dagger + i d_{-k} d_k, \\ \sigma_{kz} &= d_k^\dagger d_k + d_{-k}^\dagger d_{-k} - 1, \\ \sigma_{0z} &= 2 d_0^\dagger d_0 - 1, \end{aligned} \quad (17)$$

and $d_k^\dagger, d_k$ $\{k = 0,1,2,\ldots\}$ denote the fermionic creation and annihilation operators in the momentum space, respectively. Here,

$$\Omega_{k,m} = -2\sqrt{[\lambda_m - \cos(2\pi k/L)]^2 + \sin^2(2\pi k/L)}, \quad (18)$$

$$\theta_{k,m} = \arcsin\left[\frac{2\sin(2\pi k/L)}{\Omega_{k,m}}\right]. \quad (19)$$

From Eq. (16) and the units where $\hbar = 1$, the time evolution operator is obtained as

$$U_m(t) = e^{i(1-\lambda_m)\sigma_{0z}t} \prod_{k>0} e^{i\frac{\theta_{k,m}}{2}\sigma_{kx}} e^{-it\Omega_{k,m}\sigma_{kz}} e^{-i\frac{\theta_{k,m}}{2}\sigma_{kx}}. \quad (20)$$

IV. DECOHERENCE FACTOR AND MAXIMAL QFI

We choose the initial state of the whole system in the following form,

$$|\Psi(0)\rangle = |\psi\rangle_s \otimes |\psi\rangle_E, \quad (21)$$

where the states $|\psi\rangle_s$ and $|\psi\rangle_E$ correspond to the system and environment, respectively. In this article we consider that the system initially starts from a coherent spin state which is the eigenstate of J_x with the highest value j and can be expanded in the eigenspace of J_z [43],

$$|\psi\rangle_s = |j\rangle_x = \frac{1}{2^j} \sum_{m=-j}^{j} C_m |m\rangle, \quad (22)$$

in which $J_x|j\rangle_x = j|j\rangle_x$, $|m\rangle$ are the eigenstates of J_z with the eigenvalues m, and the coefficient $C_m = \sqrt{(2j)!/[m!(2j-m)!]}$. The initial state of the whole system is

$$|\Psi(0)\rangle = \frac{1}{2^j} \sum_{m=-j}^{j} C_m |m\rangle \otimes |\psi\rangle_E. \quad (23)$$

Driven by the time evolution operator U_m [in Eq. (20)], the state at time t becomes

$$|\Psi(t)\rangle = \frac{1}{2^j} \sum_{m=-j}^{j} C_m |m\rangle \otimes U_m |\psi\rangle_E, \quad (24)$$

then we can obtain the reduced density matrix at time t,

$$\rho_s(t) = \mathrm{Tr}_E[|\Psi(t)\rangle\langle\Psi(t)|] = \frac{1}{4^j} \sum_{m,n=-j}^{j} C_m C_n^* |m\rangle\langle n| F_{m,n}(t), \quad (25)$$

where the time dependent factor is

$$F_{m,n}(t) = {}_E\langle\psi|U_n^\dagger U_m|\psi\rangle_E, \quad (26)$$

which is the so-called decoherence factor.

A. The first initial state of environment

The decoherence factor (26) contains the information of the system and also depends on the initial state of the environment. Thus we will consider a different initial state $|\psi\rangle_E$. First, we assume it to be the vacuum state in the momentum space, namely,

$$|\psi\rangle_E = |0\rangle_{k=0} \otimes_{k>0} |0\rangle_k |0\rangle_{-k}, \quad (27)$$

and the k_{th} vacuum state $|0\rangle_k$ satisfies $d_k|0\rangle_k = 0$. Then the decoherence factor is obtained as

$$\begin{aligned} F_{m,n} = e^{itg(m-n)} \prod_{k>0} \{ & (\Omega_{k,m}t)\cos(\Omega_{k,n}t) \\ & + \sin(\Omega_{k,m}t)\sin(\Omega_{k,n}t)\cos(\theta_{k,n}-\theta_{k,m}) \\ & + i\cos(\Omega_{k,m}t)\sin(\Omega_{k,n}t)\cos\theta_{k,n} \\ & - i\sin(\Omega_{k,m}t)\cos(\Omega_{k,n}t)\cos\theta_{k,m}\}, \end{aligned} \tag{28}$$

in which $\Omega_{k,m}(\Omega_{k,n})$ and $\theta_{k,m}(\theta_{k,n})$ are determined in Eqs. (18) and (19), and the norm of the factor is [22]

$$\begin{aligned} |F_{m,n}| = \prod_{k>0} \{ & 1 - [\cos(\Omega_{k,m}t)\sin(\Omega_{k,n}t)\sin\theta_{k,n} \\ & - \sin(\Omega_{k,m}t)\cos(\Omega_{k,n}t)\sin\theta_{k,m}]^2 \\ & - \sin^2(\Omega_{k,n}t)\sin^2(\Omega_{k,m}t)\sin^2(\theta_{k,n}-\theta_{k,m})\}^{\frac{1}{2}}, \end{aligned} \tag{29}$$

This is one of our main results. Clearly, the norm $|F_{m,n}|$ consists of the terms $F^k_{m,n}$ corresponding to the k_{th} space which is not more than 1. Therefore it can be expected that in the large L limit, $|F_{m,n}|$ will go to zero under some reasonable conditions.

By carrying out an analysis similar to that in Ref. [19], we introduce a cutoff number K_c and define the partial product for the decoherence factor in Eq. (29) as

$$|F_{m,n}|_{K_c} = \prod_{k>0}^{K_c} F^k_{m,n} \geqslant |F_{m,n}|, \tag{30}$$

from which we obtain the corresponding partial sum

$$S(t) = \ln|F_{m,n}|_{K_c} \equiv -\sum_{k>0}^{K_c} \left|\ln F^k_{m,n}\right|. \tag{31}$$

For the case of small k and large L, we have $\Omega_{k,m} \approx -2|1-\lambda_m|$, $\sin\theta_{k,m} \approx -2\pi k/(L|1-\lambda_m|)$, $\cos\theta_{k,m} \approx (1-\lambda_m)/|1-\lambda_m|$, and consequently $\sin^2(\theta_{k,m}-\theta_{k,n}) \approx 4k^2\pi^2 g^2(m-n)^2/[L^2(1-\lambda_m)^2(1-\lambda_n)^2]$.

As a result, if L is large enough and g is a small perturbation, the approximation of S can be obtained as

$$\begin{aligned} S(t) \approx & -E(K_c)(1-\lambda_m)^{-2}(1-\lambda_n)^{-2} \\ & \times \{g^2(m-n)^2\sin^2(2|1-\lambda_m|t)\sin^2(2|1-\lambda_n|t) \\ & + [\sin(2|1-\lambda_m|t)\cos(2|1-\lambda_n|t)|1-\lambda_n| \\ & - \sin(2|1-\lambda_n|t)\cos(2|1-\lambda_m|t)|1-\lambda_m|]^2\}, \end{aligned} \tag{32}$$

where $E(K_c) = 2\pi^2K_c(K_c+1)(2K_c+1)/(6L^2)$. In the derivation of this equation, we have used $\ln(1-x) \approx -x$ for small x and $\sum_{k=1}^n k^2 = n(n+1)(2n+1)/6$. Let us consider a limit case where $\lambda \to \lambda_c = 1$, with a small enough g and finite number $(m-n)^2$, then $S_c(t) \approx -16E(K_c)(m-n)^2g^2t^4$; that is, the behavior of the factor can be approximated as

$$|F_{m,n}|_{K_c} \approx e^{-\gamma_c t^4},$$

with $\gamma_c = 16E(K_c)g^2(m-n)^2$. This means that $|F_{m,n}|$ decays nearly as a monotonous function of time when λ tends to the critical point. While beyond the critical region, $|F_{m,n}|$ may present periodicity. Next we consider another limit $\lambda \to 0$, with $g \ll 1$, we have $|1-\lambda_{m(n)}| \approx 1$, then $S_0(t) \approx -E(K_c)(m-n)^2g^2\sin^4(2t)$. Here it should be noted that, since λ is very small, the sum $S_0(t)$ obtained by a small k approximation cannot give a good description of $|F_{m,n}|$; however, it gives an upper limit, that is, $|F_{m,n}| \leqslant e^{-S_0(t)}$. At the strong strength limit $\lambda \to \infty$, we have $S_\infty(t) \to 0$, that is, $|F_{m,n}| \to 1$. From the analysis for the three limits $\lambda \to 0$, 1, and ∞, by choosing a finite time $t > 1$, one can find the bounds $e^{-S_c(t)} < e^{-S_0(t)}$ and $e^{-S_c(t)} < e^{-S_\infty(t)}$. Thus it can be expected that the critical strength $\lambda_c = 1$ may induce the minimal value of $|F_{m,n}|$. Although the previous analysis is rough, it is heuristic and brings us some important behaviors of $|F_{m,n}|$, such as that $|F_{m,n}|$ decays with time almost monotonously and at a finite long time it may present a drop near the critical point. In the following, we will calculate the maximal QFI $I_{\max}$ and find its direct relation to the decoherence factors. Therefore, one can see that $I_{\max}$ presents special properties at the critical point.

Now let us calculate the maximal QFI. When $j = 1/2$, the maximal QFI $I_{\max}$ is absolutely determined by the norm of the decoherence factor:

$$I_{\max} = |F_{1/2,-1/2}|^2. \tag{33}$$

From the previous analysis, when $\lambda \to 1$ we have $I_{\max}$ bounded by $e^{-2\gamma t^4}$ with $\gamma = 16E(K_c)g^2$. In Eq. (33), $I_{\max}$ is just the Loschmidt echo [19]. The weak coupling strength g acts as the perturbation; thus the $I_{\max}$, that is, the Loschmidt echo, will present a sudden drop at the critical point [19]. However, it is more complicated for the larger size system. When $j = 1$, the initial state $|\psi\rangle_s = \frac{1}{2}(|1\rangle + \sqrt{2}|0\rangle + |-1\rangle)$, and the reduced density is

$$\rho_s = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{2}F_{1,0} & F_{1,-1} \\ \sqrt{2}F^*_{1,0} & 2 & \sqrt{2}F_{0,-1} \\ F^*_{1,-1} & \sqrt{2}F^*_{0,-1} & 1 \end{pmatrix}. \tag{34}$$

Based on this expression, obtaining the analytical result of the maximal QFI is nearly impossible. Fortunately, in this initial state $|\psi\rangle_E = |0\rangle_{k=0} \otimes_{k>0} |0\rangle_k|0\rangle_{-k}$, by choosing a small enough parameter $g < 10L^{-1}$ (L is the size of Ising chain), we can approximately treat the factors as $F_{1,-1} \to |F_{1,-1}|$ and $F_{1,0}, F_{0,-1} \to |F_{1,0}|$. Consequently, a good approximation of the maximal QFI is obtained as

$$I_{\max} = \max(E_1, 2E_2), \tag{35}$$

where

$$\begin{aligned} E_1 &= \frac{(1+|F_{1,-1}|)^2 + 2|F_{1,0}|^2(1-3|F_{1,-1}|)}{3-|F_{1,-1}|-2|F_{1,0}|^2}, \\ E_2 &= \frac{3|F_{1,-1}|^2 - |F_{1,-1}|^3 - 4|F_{1,-1}||F_{1,0}|^2 + 2|F_{1,0}|^2}{3-|F_{1,-1}|-2|F_{1,0}|^2}. \end{aligned} \tag{36}$$

This analytical result is more complicated than the case of $j = 1/2$. The $I_{\max}$ depends on the two factors $|F_{1,-1}|$ and $|F_{1,0}|$, and the expression is nonlinear. Nevertheless, from our analysis and the results in Ref. [19], we know that the critical behavior of the environment will enhance the decoherence process, and a sudden drop of the decoherence factor can signal the criticality. Then it is reasonable to expect that the QFI $I_{\max}$ in Eq. (35) will exhibit an extremum at the critical point. From the Eqs. (35) and (36), one can obtain $I_{\max} = 2$ when $t = 0$ or $g = 0$.With time evolving, when the factors $|F_{1,-1}|$ and $|F_{1,0}|$ tend to zero, the state becomes the maximal

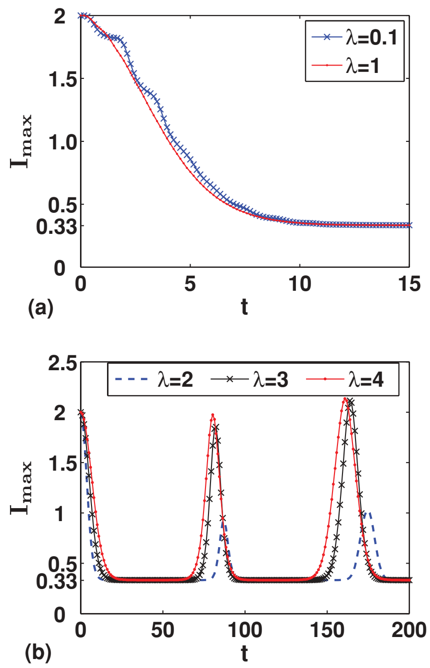

FIG. 1. (Color online) (a) $I_{\max}$ versus t for $\lambda = 0.1$ and 1. The angular momentum number of the system is $j = 1$, and the size of the environment is $L = 2M + 1$ and $M = 750$. The coupling strength is $g = 0.01$. (b) The cases of larger $\lambda = 2$, 3, and 4 are shown.

mixed state and $I_{\max}$ approaches the value $1/3$. When $\lambda \to 1$, the decoherence factor $|F_{1,0}|$ is bounded by $e^{-\gamma t^4}$ and $|F_{1,-1}|$ by $e^{-4\gamma t^4}$, which constrains the behavior of $I_{\max}$.

By numerically calculating the QFI in Fig. 1, we find the behavior of QFI $I_{\max}$ versus time is affected by the strength λ, especially at the critical point $\lambda_c = 1$. The case of $j = 1$ is presented as an example, and the time t is rescaled to be dimensionless according to the coupling constants. In Fig. 1(a), we find that only in the vicinity of $\lambda \to \lambda_c$ does $I_{\max}$ decay monotonously to a steady value of about $1/3$. And extending the time range, there is no oscillation. However, when parameter λ becomes larger than λ_c ($\lambda = 2$, 3, and 4) in Fig. 1(b), $I_{\max}$ oscillates with time. It presents like a quasiperiodic function of t, and at some times $I_{\max}$ is larger than 2. According to quantum estimation theory, the decay of QFI means the optimal precision of estimation is reduced. From our study, one can find that the precision of the estimation on the postdecohered state is lower than that on the initial state. And the critical fluctuation makes the decay process nearly monotonous. However, beyond the critical region, the oscillation of QFI with time implies the precision of estimation may rise again during some time period. In a sense, this phenomenon can be regarded as evidence of the revival of coherence, which may be understood as the reversed flow of information from the environment back to the open system [27,30]. Therefore, the results shown in Fig. 1(a) exhibit that the critical fluctuation of the environment remarkably suppresses the reversed flow of information. We stress that the monotonous decay of the precision of estimation in the critical region $\lambda \to \lambda_c$ is helpful to characterize the quantum criticality.

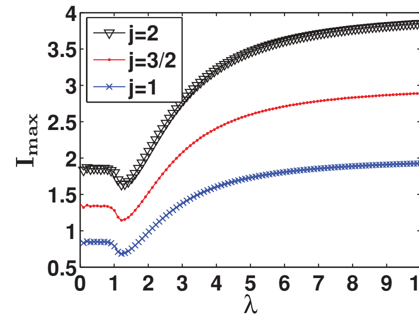

FIG. 2. (Color online) For the first initial state of the environment, it shows $I_{\max}$ versus λ at $t = 10$ for the systems $j = 1, 3/2, 2$. The size of the environment is $M = 750$. The coupling strength is $g = 0.005$.

At a fixed time $t = 10$ in Fig. 2, by changing the parameter λ, one can see the $I_{\max}$ presents a sudden drop at the critical point. We show the results for different system sizes of $j = 1$, $3/2$, and 2. Clearly, when λ tends to the critical point, $I_{\max}$ exhibits a drop, and beyond the critical point it increases to the maximal value. If we consider the time-averaging QFI in the region $[0, T]$, that is, $\bar{I}_{\max} = \frac{1}{T}\int_0^T I_{\max}(t)\,dt$, the phenomenon is similar to that in Fig. 2 and is not shown in this article. The drop of $I_{\max}$ reflects that the parameter estimation of the open system becomes more inaccurate, which can be understood as that the critical fluctuation greatly destroys the quantum coherence and, consequently, the estimation based on the quantum coherence becomes inaccurate.

Let us recall the fundamental change of the ground state of the transverse Ising chain. At the limit $\lambda = 0$, the ground state is doubly degenerate under the global spin flip by $\prod_l^L \sigma_l^z$. However, at the limit $\lambda \to \infty$, the ground state approaches a product of spins pointing to the positive z direction. The symmetry breaks at $|\lambda| = 1$. At this critical point, the structure of the ground state changes dramatically, which will greatly affect the system coupled to it. In our study, the critical properties are reflected by the QFI. When λ is large, the effect of a small coupling g will be erased, and then the decoherence factor $F_{m,n}(t) \to 1$. In other words the initial state of the system will not be driven by the Hamiltonian (14); thus for the pure state $|j\rangle_x$ [in Eq. (22)], we can easily obtain $I_{\max} = 2j$ by using of Eqs. (5) and (9), which is consistent with our numerical results.

B. The second initial state of environment

In this section, the initial state of the environment is assumed as the ground state of the transverse Ising chain,

$$|\psi\rangle_E = |1\rangle_{k=0} \otimes_{k>0} e^{i\frac{\theta_k}{2}\sigma_{kx}} |1\rangle_k |1\rangle_{-k}, \tag{37}$$

where the parameter θ_k satisfies $\sin\theta_k = 2\sin(2\pi k/L)/\Omega_k$ and the eigenenergy $\Omega_k = -2\sqrt{[\lambda - \cos(2\pi k/L)]^2 + \sin^2(2\pi k/L)}$. It should be noted that in the ground state Eq. (37), $|1\rangle_{k=0}$ is for the case $\lambda < 1$, while for $\lambda \geqslant 1$ it should be $|0\rangle_{k=0}$. The system still starts from the state (22); then we can calculate the decoherence factor. We find the function of $F_{m,n}$ is similar to the conjugate form of Eq. (28) by replacing the parameters $\theta_{k,m}(\theta_{k,n})$ with $\alpha_{k,m}(\alpha_{k,n})$, which is defined as $\alpha_{k,m} = \theta_{k,m} - \theta_k$. It should be pointed out that for the second initial state $|\psi\rangle_E$ (37), we

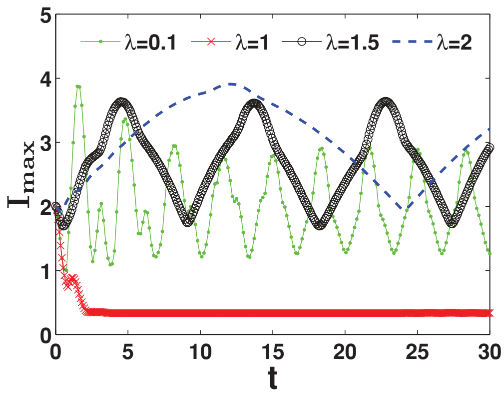

FIG. 3. (Color online) For the second initial state of the environment, it shows $I_{\max}$ versus t for the system $j = 1$. The size of the environment is $M = 750$. The coupling strength is $g = 0.05$. The different cases of $\lambda = 0.1$, 1, 1.5, and 2 are considered.

cannot obtain the analytical result of $I_{\max}$ like Eq. (35), since all the decoherence factors behave quite differently in this initial state. Only in a short-time region such as $t < 0.3$, can $I_{\max}$ be characterized by Eq. (35) approximately. Instead, we numerically study the QFI. In Fig. 3, in the system of $j = 1$, we plot $I_{\max}$ as a function of t for different λ. Clearly, one can find that when the environment tends to the critical point, QFI is remarkably suppressed to the minimal value and without rising again. While beyond the vinicity of the critical point, $I_{\max}$ oscillates with time as a periodic function and larger λ makes a longer period. It is clear that the quantum criticality of the environment significantly reduces the precision of estimation. However, beyond the critical region, the precision

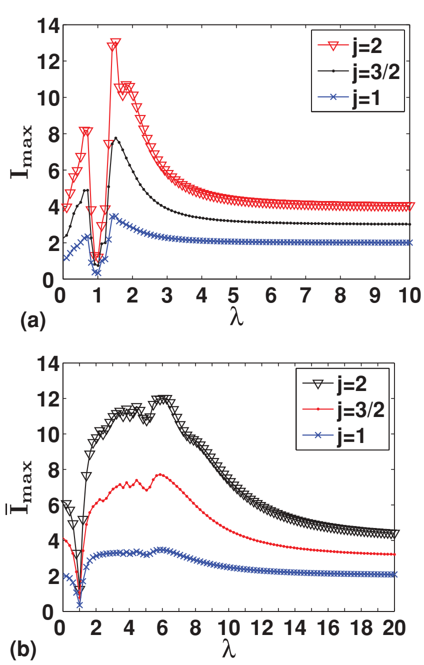

FIG. 4. (Color online) For the second initial state of the environment, panel (a) shows $I_{\max}$ versus λ at $t = 4$ for the systems $j = 1$, 3/2, and 2. The size of the environment is $M = 750$. The coupling strength is $g = 0.05$. Panel (b) shows the time-averaging $\bar{I}_{\max}$ versus λ; also the different cases of $j = 1$, 3/2, and 2 are considered.

of estimation can increase again and even be several times higher than the initial situation in some time periods.

Figure 4(a) shows $I_{\max}$ changes with different λ at a fixed time $t = 4$, and the cases of $j = 2$, 3/2, and 1 are considered. Similar to Fig. 2(a), $I_{\max}$ presents a sudden drop to the minimum at the critical point. However, differently in this case, $I_{\max}$ behaves as a very intense oscillation so that it decreases sharply to the minimum and jumps up to the maximum when λ crosses the critical region. Moreover, by enlarging the system to $j = 2$, the maximum of $I_{\max}$ is more than 10 times the minimum; that is, the precision of estimation suddenly changes by a factor of 10. Therefore, it can be expected that the larger dimension of the open system will magnify the change of the precision of estimation, which will make the phenomenon more evident and aid in the detection of the quantum criticality. In Fig. 4(b), we plot the time-averaging QFI $\bar{I}_{\max} = \frac{1}{T}\int_0^T I_{\max}(t)\,dt$, with $T = 500$. We find that the point where $\bar{I}_{\max}$ suddenly drops to the minimal value indicates the critical region more accurately than that in Fig. 4(a). All the results shown by us imply that the $I_{\max}$, that is, the precision of parameter estimation, changes dramatically when $\lambda \to 1$ and can be used to indicate the critical region of the environment.

V. CONCLUSION

By using quantum Fisher information, we have investigated the problem of parameter estimation in a spin-j system surrounded by an environment with QPT which is modeled by the quantum Ising chain with transverse field. We introduced a general method to obtain maximal QFI by choosing a proper rotation direction. For the case $j = 1/2$, we obtained the analytical result of the maximal QFI $I_{\max}$, which is the norm of the decoherence factor. For the system of $j = 1$, only when the environment initially starts from the vacuum state in the momentum space, can we approximately obtain the analytical result of $I_{\max}$, which depends on three kinds of decoherence factors. By numerical calculation, we also considered another initial state of the environment: the ground state of the transverse Ising chain. For both the two initial states, only in the critical region of $\lambda \to 1$, does $I_{\max}$ decay with time almost monotonically to the minimal value, namely, the precision of estimation decreases almost monotonically. While beyond the critical region, $I_{\max}$ begins to oscillate with time, which means the precision of estimation can rise again at some times. We conclude that the critical fluctuation of the environment seriously destroys the coherence and thus reduces the precision of estimation.

By choosing a fixed time or taking the time average, we observe a sudden drop of the maximal QFI at the critical point, which is due to the enhanced decoherence caused by the quantum criticality. Our results clearly show that the critical fluctuation of the environment is harmful to the parameter estimation. The precision of estimation presents an evident decrease due to the quantum criticality. Especially when the environment starts from its ground state, enlarging the open system to $j = 2$, the $I_{\max}$, that is, the precision of estimation, suddenly changes by a factor of 10, which makes it possible to be detected in experiment. Therefore, we introduce a method to detect QPT by coupling the quantum-critical system to the external spin-j system. The region where the estimation precision changes dramatically can indicate the quantum-critical region.

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