Isospectral potentials from modified factorization

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Factorization of quantum-mechanical potentials has a long history extending back to the earliest days of the subject. In the present article, the nonuniqueness of the factorization is exploited to derive new isospectral nonsingular potentials. Many one-parameter families of potentials can be generated from known potentials using a factorization that involves superpotentials defined in terms of excited states of a potential. For these cases an operator representation is available. If ladder operators are known for the original potential, then a straightforward procedure exists for defining such operators for its isospectral partners. The generality of the method is illustrated with a number of examples which may have many possible applications in atomic and molecular physics.

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I. INTRODUCTION

The factorization method due to Hull and Infeld [1] has been widely exploited in quantum mechanics to determine the spectra and wave functions of exactly solvable potentials. This approach has been formalized in supersymmetric quantum mechanics (SUSY QM) [2], which has been used to find many new isospectral potentials. The usual procedure is to find a factorization of a quantum-mechanical Hamiltonian and the methods of SUSY QM then guarantee that a supersymmetric partner potential is isospectral to the original Hamiltonian. As verified in the following, this procedure yields a pair of potentials with the same spectra (possibly apart from the ground state) and related wave functions. Throughout this article we work in $\hbar = 2m = 1$ units.

Let us consider a one-dimensional Hamiltonian

$$H_-^{(0)} = -\partial_x^2 + V_-^{(0)}(x),$$

where $V_-^{(0)}(x)$ is an arbitrary nonsingular potential with at least one bound state and zero ground-state energy [given the Hamiltonian $H = -\partial_x^2 + V(x)$ one simply subtracts the zero point energy to obtain $H_-^{(0)}$]. It is a second-order linear operator and it can be factored into a product of first-order linear operators as follows:

$$H_-^{(0)} = [-\partial_x + W_0(x)][\partial_x + W_0(x)] \equiv A_0^\dagger A_0,$$

once the ground-state wave function $\psi_0(x)$ is specified. The function $W_0(x) = -\partial_x \ln \psi_0(x)$ is called superpotential generating the potential

$$V_-^{(0)}(x) = W_0^2(x) - W_0'(x).$$

Fortunately, the factorization does not commute $A_0^\dagger A_0 \neq A_0 A_0^\dagger$ unless the superpotential is constant. In other words, an inverted product $A_0 A_0^\dagger$ is a certain new Hamiltonian $H_+^{(0)} = A_0 A_0^\dagger = -\partial_x^2 + V_+^{(0)}(x)$ where

$$V_+^{(0)}(x) = W_0^2(x) + W_0'(x),$$

is also free of singularities. It turns out that the eigenfunctions and eigenvalues of these partner Hamiltonians are related. Indeed, we have the following first-order intertwining relations

$$H_-^{(0)} A_0^\dagger = A_0^\dagger H_+^{(0)} \text{ and } H_+^{(0)} A_0 = A_0 H_-^{(0)}, \tag{1}$$

from which one observes that since $A_0\psi_0(x) = 0$, the spectra of $H_+^{(0)}$ and $H_-^{(0)}$ are connected by $\tilde{E}_n = E_{n+1}$ $(n = 0,1,\ldots)$ where $\tilde{E}_n$ and E_n denote the eigenvalues of the Hamiltonians $H_+^{(0)}$ and $H_-^{(0)}$, respectively, with eigenfunctions $\tilde{\psi}_n$ and ψ_n. Thus, the Hamiltonians have identical energy spectrum except for the ground state of $H_-^{(0)}$. The wave functions satisfy $\tilde{\psi}_n(x) \propto A_0\psi_{n+1}(x)$, $\psi_{n+1}(x) \propto A_0^\dagger \tilde{\psi}_n(x)$ and if $\psi_{n+1}(x)$ is normalizable, then $\tilde{\psi}_n(x)$ is also normalizable and vice versa because

$$\begin{aligned}\langle \tilde{\psi}_n(x), \tilde{\psi}_n(x)\rangle &= \langle \psi_{n+1}(x), A_0^\dagger A_0 \psi_{n+1}(x)\rangle \\ &= E_{n+1}\langle \psi_{n+1}(x), \psi_{n+1}(x)\rangle.\end{aligned}$$

Note that for singular potentials (for instance, with a $1/x^2$ singularity) some of the wave functions $\tilde{\psi}_n(x)$ are not acceptable as they may not be normalizable [3]. That is, for singular potentials the degeneracy of energy levels is only partially valid or invalid at all. The upshot of all this is that one can generate new isospectral potentials from existing exactly solvable potentials.

Luckily, the previously discussed factorization is not unique. For example, we have

$$(-\partial_x + 1)(\partial_x + 1) = [-\partial_x + \tanh(x)][\partial_x + \tanh(x)],$$

that is, two different superpotentials can give rise to the same potential (in this particular example with no bound states). One can try to construct new isospectral potentials exploiting the nonuniqueness of factorization and obtain a one-parameter family of potentials with the parameter arising as an integration constant [4,5].

Suppose the Hamiltonian $H_+^{(0)}$ can be factorized by the operators different than A_0 and $A_0^\dagger$, namely,

$$B = \partial_x + f(x) \text{ and } B^\dagger = -\partial_x + f(x),$$

where $f(x)$ is the temporarily undetermined function

$$H_+^{(0)} = BB^\dagger = -\partial_x^2 + f^2(x) + f'(x).$$

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Now demanding that this Hamiltonian involve the potential $V_+^{(0)}(x)$ results in a differential equation that must be satisfied

$$f'(x) + f^2(x) - V_+^{(0)}(x) = 0.$$

This is a Riccati equation in its canonical form. The explicit closed-form solution of this equation is not known typically, but one understands that the superpotential $W_0(x)$ is a particular solution. This is enough to construct the general solution $f(x)$ which depends on an arbitrary integration constant that can be considered as a free parameter in the partner Hamiltonian

$$H = B^\dagger B = -\partial_x^2 + V_+^{(0)}(x) - 2f'(x) = -\partial_x^2 + V(x).$$

According to SUSY QM the potentials $V_+^{(0)}(x)$ and $V(x)$ are isospectral [except for the lowest state of $V(x)$] provided that $f(x)$ is nonsingular. In addition, since $BB^\dagger = A_0 A_0^\dagger$, it follows that the potentials $V_-^{(0)}(x)$ and $V(x)$ have strictly identical spectra.

In Ref. [4] Mielnik performed factorization of the harmonic oscillator potential in this manner. Mielnik obtained a one-parameter family of potentials with the oscillator spectrum, but as we have just seen the procedure is straight forwardly generalized to any potential $V_+^{(0)}(x)$.

In the standard [i.e., based on the first-order intertwining relation (1)] unbroken SUSY QM it is impossible to use an excited state of the original potential and at the same time avoid creating singularities in the partner potential [6]. There is no guarantee that the resulting wave functions are normalizable and energy levels degenerate. The purpose of the present article is to modify the operators B and $B^\dagger$ in such a way as to determine new strictly isospectral potentials without being forced to solve Riccati equations (by reducing the Riccati equation whose appearance in the factorization problems is typical to the solvable Bernoulli equation), and more importantly, by applying the nonuniqueness of factorization to the superpotentials generated by the excited states of a potential since these also satisfy the Schrödinger equation.

II. MODIFIED FACTORIZATION

In this section we show the consequences of the nonuniqueness of the factorization method extended to the excited states of a potential, rather than just the ground state. In the literature the Hamiltonians $H_+^{(0)}$ and $H_-^{(0)}$ are called "bosonic" and "fermionic," respectively. We show that the degeneracy of energy levels of partner potentials depends on whether the bosonic or fermionic Hamiltonians admit nonunique factorization.

A. Bosonic Hamiltonian

Let there be given an analytically solvable nonsingular potential $V_-^{(0)}(x)$ whose energy eigenvalues E_n and wave functions $\psi_n(x)$ are known. Without loss of generality, let E_0 be zero, so that $V_-^{(0)}(x) = \psi_0''(x)/\psi_0(x) = W_0^2(x) - W_0'(x)$ and also define

$$V_-^{(n)}(x) = \psi_n''(x)/\psi_n(x) = W_n^2(x) - W_n'(x),$$

where $W_n(x) = -\partial_x \ln \psi_n(x)$ is taken to be the superpotential corresponding to $\psi_n(x)$. From the Schrödinger equation it follows that $V_-^{(n)}(x) = V_-^{(0)}(x) - E_n$, so that the potentials $V_-^{(n)}(x)$ are nonsingular, even though the superpotentials $W_n(x)$ are always singular for $n > 0$. Adjusting the energy scale seems appropriate: One simply subtracts from the potential the energy of the excited state so that the resulting potential can be factored.

Next we introduce the operators

$$B_n = \partial_x + f(x) + W_n(x) \text{ and } B_n^\dagger = -\partial_x + f(x) + W_n(x),$$

where $f(x)$ will be determined in the following. Notice when $n = 0$ these definitions reduce to the familiar case of standard unbroken SUSY QM if $f(x) = 0$ and to the Mielnik's factorization [4] if $f(x) \neq 0$.

The factorization of the Hamiltonian $\tilde{H}_-^{(n)} = B_n^\dagger B_n$ leads to

$$\tilde{H}_-^{(n)} = -\partial_x^2 + V_-^{(n)}(x) + f^2(x) + 2W_n(x)f(x) - f'(x).$$

If we require that $f^2(x) + 2W_n(x)f(x) - f'(x) = 0$ the Hamiltonian becomes trivial because the potential $V_-^{(n)}(x)$ is related to $V_-^{(0)}(x)$ by a constant shift. On the other hand, the partner Hamiltonian $\tilde{H}_+^{(n)} = B_n B_n^\dagger$ is less trivial

$$\tilde{H}_+^{(n)} = -\partial_x^2 + V_+^{(n)} + 2f'(x),$$

where $V_+^{(n)}(x) = W_n^2(x) + W_n'(x)$. The function $f(x)$ is not arbitrary—it is a solution of the Bernoulli equation (a specific example of the Riccati equation)

$$f'(x) = f^2(x) + 2W_n(x)f(x),$$

and reads

$$f_n(x) = \frac{\psi_n^{-2}(x)}{C - \int_{x_0}^x \psi_n^{-2}(s)ds},$$

where C, x_0 are constants. It follows that $\psi_n(x)$ must be inverse square integrable; however, in general, the wave functions do not possess this property.

There is yet another problem, namely, singularity of the potentials $V_+^{(n)}(x)$ for $n \neq 0$ corresponding to the zeros of the wave functions. Consequently, the breakdown of the degeneracy of energy levels of the Hamiltonians $\tilde{H}_-^{(n)}$ and $\tilde{H}_+^{(n)}$ occurs [in addition to $H_-^{(n)}$ and $H_+^{(n)}$].

B. Fermionic Hamiltonian

The difficulties of establishing the degeneracy theorem for bosonic Hamiltonians suggest to reverse the order of the operators B_n and $B_n^\dagger$ and start with the fermionic Hamiltonian $\tilde{H}_+^{(n)} = B_n B_n^\dagger$

$$\tilde{H}_+^{(n)} = -\partial_x^2 + V_+^{(n)}(x) + f^2(x) + 2W_n(x)f(x) + f'(x),$$

where $V_\pm^{(n)}(x)$ are defined as usual. We again obtain the Bernoulli equation

$$f'(x) + f^2(x) + 2f(x)W_n(x) = 0,$$

whose general solution is

$$f_n(x) = \frac{\psi_n^2(x)}{C + \int_{x_0}^{x} \psi_n^2(s)ds}, \tag{2}$$

where C, x_0 are constants and $\psi_n(x)$ is assumed to be square-integrable.

If it is possible to restrict the domain of the parameter C and make $f_n(x)$ free of singularities, then the potential $\tilde{V}_-^{(n)}(x)$ in

$$\tilde{H}_-^{(n)} = B_n^\dagger B_n = -\partial_x^2 + \tilde{V}_-^{(n)} = -\partial_x^2 + V_-^{(n)} - 2f_n'(x),$$

constitute a one-parameter family of potentials isospectral to the potential $V_-^{(n)}(x)$.

To see this note that the Schrödinger equation $H_-^{(n)}\psi_k = (E_k - E_n)\psi_k$ implies

$$\begin{aligned}\tilde{H}_-^{(n)}[B_n^\dagger A_n \psi_k] &= B_n^\dagger B_n B_n^\dagger A_n \psi_k \\ &= B_n^\dagger A_n A_n^\dagger A_n \psi_k \\ &= (E_k - E_n)[B_n^\dagger A_n \psi_k],\end{aligned}$$

where we have used the nonuniqueness of factorization of the Hamiltonian $H_+^{(n)} = A_n A_n^\dagger = B_n B_n^\dagger$. So if $\psi_k(x)$ is an eigenfunction of the Hamiltonian $H_-^{(n)}$ with energy eigenvalue $E_k - E_n$, then $B_n^\dagger A_n \psi_k$ is an eigenfunction of $\tilde{H}_-^{(n)}$ with the same energy. Similarly, from the Schrödinger equation $\tilde{H}_-^{(n)}\tilde{\psi}_k^{(n)} = \tilde{E}_k^{(n)}\tilde{\psi}_k^{(n)}$ [where in $\tilde{E}_k^{(n)}$, k denotes the energy level and (n) refers to the nth eigenfunction of the Hamiltonian $H_-^{(n)}$] it follows that

$$H_-^{(n)}\left[A_n^\dagger B_n \tilde{\psi}_k^{(n)}\right] = \tilde{E}_k^{(n)}\left[A_n^\dagger B_n \tilde{\psi}_k^{(n)}\right].$$

Hence, the normalized eigenfunctions of the Hamiltonians $H_-^{(n)}$ and $\tilde{H}_-^{(n)}$ are related by

$$\tilde{\psi}_k^{(n)}(x) = (E_k - E_n)^{-1}[B_n^\dagger A_n \psi_k(x)], \tag{3}$$

and

$$\psi_k(x) = (E_k - E_n)^{-1}\left[A_n^\dagger B_n \tilde{\psi}_k^{(n)}(x)\right],$$

where $k \neq n$. The operators A_n or B_n destroy a node in the eigenfunctions, but they are followed, respectively, by the operators $B_n^\dagger$ or $A_n^\dagger$ that create an extra node. Thus, the overall number of the nodes does not change. In addition, the normalization does not require positive semidefiniteness of the energy eigenvalues, as in the standard case. This is good because negative energy states appear when $n > 0$.

For any n there is always one missing state $k = n$, which can be obtained by solving the first-order differential equation $B_n\tilde{\psi}_n^{(n)} = 0$ [by construction the state $\tilde{\psi}_n^{(n)}$ has to be annihilated by the operator B_n]

$$\begin{aligned}\frac{d\tilde{\psi}_n^{(n)}(x)}{dx} &= -\left[W_n(x) + \frac{\psi_n^2(x)}{C + \int_{x_0}^{x}\psi_n^2(s)ds}\right]\tilde{\psi}_n^{(n)}(x) \\ &= \frac{d}{dx}\left[\ln\frac{\psi_n}{C + \int_{x_0}^{x}\psi_n^2(s)ds}\right]\tilde{\psi}_n^{(n)}(x).\end{aligned}$$

Therefore,

$$\tilde{\psi}_n^{(n)}(x) = N(C) \times \frac{\psi_n}{C + \int_{x_0}^{x}\psi_n^2(s)ds}, \tag{4}$$

with the corresponding energy $\tilde{E}_n^{(n)} = 0$. All other energy eigenvalues satisfy $\tilde{E}_k^{(n)} = E_k - E_n$. The normalization constant $N(C)$ depends on the parameter C and other parameters of the potential such as width, depth, and so on. It is a constraint that allows one to determine the values of C for which the potentials $\tilde{V}_-^{(n)}(x)$ are nonsingular and eigenfunctions $\tilde{\psi}_k^{(n)}(x)$ are well-defined.

One observes that the intertwining relationship between the Hamiltonians $H_-^{(n)}$ and $\tilde{H}_-^{(n)}$ is of the second order

$$\tilde{H}_-^{(n)}B_n^\dagger A_n = B_n^\dagger A_n H_-^{(n)} \text{ and } H_-^{(n)}A_n^\dagger B_n = A_n^\dagger B_n \tilde{H}_-^{(n)}.$$

In the second-order SUSY QM [7] two different Hamiltonians are intertwined by an operator of the second order in derivatives, say, $A = \partial_x^2 + \eta(x)\partial_x + \gamma(x)$. If A can be written as a product of two first-order differential operators with real superpotentials, then we call it reducible (otherwise one refers to it as irreducible). Thus, our construction is equivalent to the second-order SUSY QM with the reducible operator $A = -B_n^\dagger A_n$. Performing an explicit factorization one finds that $-\eta(x) = f_n(x)$ and $-\gamma(x) = V_-^{(n)}(x) + f_n(x)W_n(x)$. The pros and cons of these related approaches are discussed in detail in the concluding section.

From now on we will discuss the degeneracy of energy levels of the Hamiltonians $H_-^{(n)}$ and $\tilde{H}_-^{(n)}$ only, leaving aside the Hamiltonian $H_+^{(n)}$ which plays an intermediate role in this construction.

III. EXAMPLES

Here we illustrate the results developed in the preceding section by providing examples that arise from well-known potentials and obtain some previously unreported potentials which might be of interest in various fields of physics and chemistry. One can also consult the Ref. [3] where factorizations of the harmonic oscillator potential were performed.

A. Morse potential

Let us first consider the Morse potential

$$V_-^{(0)}(x) = A^2 - B(2A + \alpha)e^{-\alpha x} + B^2 e^{-2\alpha x}, \tag{5}$$

where the constants A, B, and α are nonnegative. There is a finite number of energy levels $E_k = k\alpha(2A - k\alpha)$ where k takes integer values from zero to the greatest value for which $k\alpha < A$. For concreteness let us take $A = 2$ and $\alpha = B = 1$. The partner potential $\tilde{V}_-^{(0)}(x)$ is obtained from the ground-state wave function $\psi_0(x) = e^{-2x-e^{-x}}$ of the potential $V_-^{(0)}(x) = 4 - 5e^{-x} + e^{-2x}$

$$\begin{aligned}&\tilde{V}_-^{(0)}(x) \\ &\quad = 4 - 5e^{-x} + e^{-2x} \\ &\qquad -16\frac{d}{dx}\left[\frac{e^{-4x-2e^{-x}}}{C + e^{-2e^{-x}}(3 + 6e^{-x} + 6e^{-2x} + 4e^{-3x})}\right].\end{aligned}$$

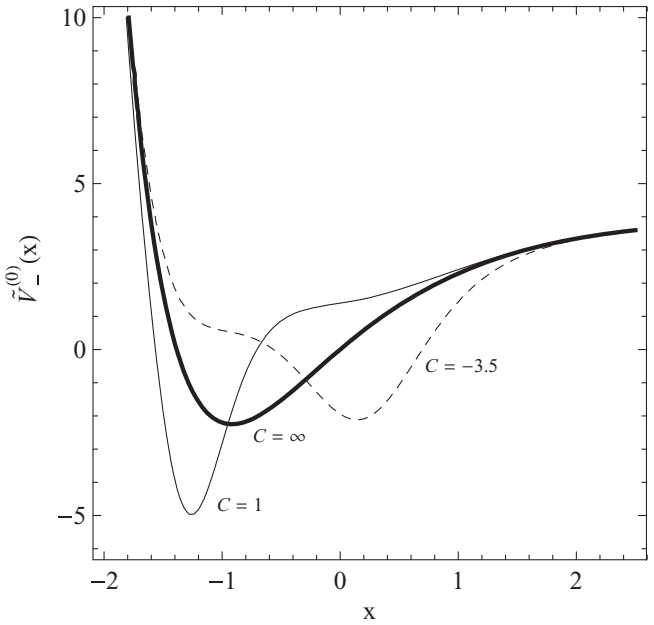

FIG. 1. A few members of the one-parameter family of potentials $\tilde{V}_-^{(0)}(x)$ isospectral to the Morse potential $V_-^{(0)}(x)$ with $A = 2$ and $\alpha = B = 1$ (thick solid line).

As the potential $V_-^{(0)}(x)$ it has only two bound states with eigenvalues $\tilde{E}_0^{(0)} = 0$ and $\tilde{E}_1^{(0)} = 3$. The normalized ground-state wave function is

$$\tilde{\psi}_0^{(0)}(x) = \frac{\sqrt{\frac{8C(C+3)}{3}}e^{-2x-e^{-x}}}{C + e^{-2e^{-x}}(3 + 6e^{-x} + 6e^{-2x} + 4e^{-3x})}.$$

Hence, the potential $\tilde{V}_-^{(0)}(x)$ is nonsingular as long as $C \notin [-3,0]$ (see Fig. 1).

The normalized wave function $\tilde{\psi}_1^{(0)}(x)$ is determined by applying the operator $B_0^\dagger A_0$ to the first (and only) normalized excited state $\psi_1(x) = 2/\sqrt{3}e^{-x-e^{-x}}(3 - 2e^{-x})$ of the potential $V_-^{(0)}(x)$

$$\tilde{\psi}_1^{(0)}(x) = \frac{2e^{-e^{-x}}(6 + 12e^{x} + 9e^{2x}) + Ce^{e^{-x}}(3e^{2x} - 2e^{x})}{\sqrt{3}(4 + 6e^{x} + 6e^{2x} + 3e^{3x} + Ce^{2e^{-x}+3x})}.$$

We would like to remind the ladder operators for the wave functions of the Morse potential given in (5) and explicitly derive them for the wave functions of the isospectral partner potential. Let us denote $s = A/\alpha$ and $y = 2B/\alpha e^{-\alpha x}$, which is the common choice in the SUSY QM literature. Then for the creation K_+ and annihilation K_- operators we have [8]

$$K_+ = \left[\partial_y + \frac{s-n}{y} - \frac{s+1/2}{2s-2n-1}\right],$$

and

$$K_- = -\left[\partial_y - \frac{s-n}{y} + \frac{s+1/2}{2s-2n+1}\right],$$

(we note that $K_- \neq K_+^\dagger$) with the following effect $K_+\psi_k(y) \propto \psi_{k+1}(y)$ and $K_-\psi_{k+1}(y) \propto \psi_k(y)$. The proportionality factors can be calculated after normalizing the eigenfunctions $\psi_k(y) = y^{s-k}e^{-y/2}L_n^{2s-2k}(y)$, where $L_k^{2s-2k}(y)$ are associated Laguerre polynomials.

Equation (3) enables us to deduce the ladder operators for the eigenvectors $\tilde{\psi}_k^{(n)}(y)$ of the potential $\tilde{V}_-^{(n)}(x)$ whose energy spectrum is identical to that of the Morse potential $V_-^{(n)}(x)$. The corresponding raising and lowering operators for $\tilde{\psi}_k^{(n)}(y)$ with $k \neq n$ are $(B_n^\dagger A_n)K_+(A_n^\dagger B_n)$ and $(B_n^\dagger A_n)K_-(A_n^\dagger B_n)$. Exploration of the higher-order ladder operators is the direct consequence of extending the first-order SUSY QM.

B. CPRS potential

In Ref. [9] Cariñena, Perelomov, Rañada, and Santander (CPRS) have studied the following one-dimensional nonpolynomial exactly solvable potential [we define our Hamiltonian to be $H_-^{(0)} = 2H_{\rm CRPS} + 3$]

$$V_-^{(0)}(x) = x^2 + 3 + 8\frac{2x^2-1}{(2x^2+1)^2}.$$

This potential asymptotically behaves like a simple harmonic oscillator, but its minimum at the origin is much deeper than in the case of the harmonic oscillator. Using SUSY QM techniques it was shown by Fellows and Smith [10] that $V_-^{(0)}(x)$ is a partner potential of the harmonic oscillator $x^2 + 5$ and therefore their energy levels are the same. Here we further analyze the CPRS potential and find new potentials with the oscillator spectrum (see also Ref. [3]).

The ground-state energy $E_0 = 0$ and wave function

$$\psi_0(x) = \frac{e^{-x^2/2}}{2x^2+1},$$

of the potential $V_-^{(0)}(x)$ allows one to find its isospectral partner

$$\tilde{V}_-^{(0)}(x) = x^2 + 3 + 8\frac{2x^2-1}{(2x^2+1)^2} - 8\frac{d}{dx}\left[\frac{e^{-x^2}}{2x(2x^2+1)e^{-x^2} + (2x^2+1)^2(C+\sqrt{\pi}\,\mathrm{erf}x)}\right],$$

which has no singularities when $|C| > \sqrt{\pi}$ (see Fig. 2) as follows from normalizing the ground-state wave function $\tilde{\psi}_0^{(0)}(x)$.

Its eigenvalues are the same as that of the potential $V_-^{(0)}(x)$ and given by $\tilde{E}_k^{(0)} = 2k + 4$ for $k = 1,2,\ldots$. The normalized ground-state wave function

$$\tilde{\psi}_0^{(0)}(x) = \frac{\sqrt{2(C^2-\pi)/\sqrt{\pi}}e^{-x^2/2}}{2xe^{x^2} + (2x^2+1)[C+\sqrt{\pi}\,\mathrm{erf}(x)]},$$

corresponds to the energy eigenvalue $\tilde{E}_0^{(0)} = 0$. The rest of the eigenfunctions can be derived using Eq. (3).

Neither Cariñena *et al.*, nor Fellows and Smith provided the raising and lowering operators for the wave functions $\psi_k(x)$ of the CPRS potential. Here we address the question of finding ladder operators for the CPRS potential and its isospectral partner. Taking into account that the CPRS potential itself is a partner of the harmonic oscillator, we obtain its raising $A^\dagger a^\dagger A$ and lowering $A^\dagger a A$ operators where

$$A = \partial_x + x + \frac{4x}{2x^2+1},$$

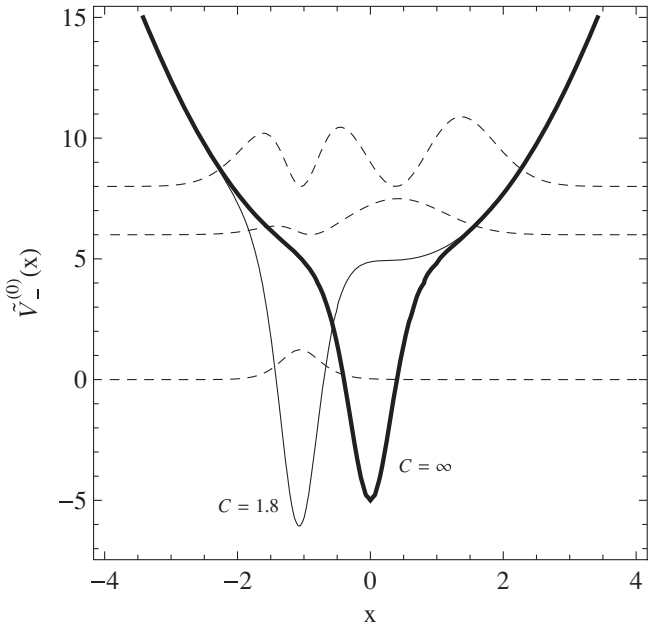

FIG. 2. Plot of the potential $\tilde{V}_-^{(0)}(x)$ with $C = 1.8$ (close to $\sqrt{\pi}$) and the unnormalized probability densities (dashed line at the corresponding level position) for its three lowest-energy levels. The limit $C \to \infty$ corresponds to the CPRS potential (thick solid line).

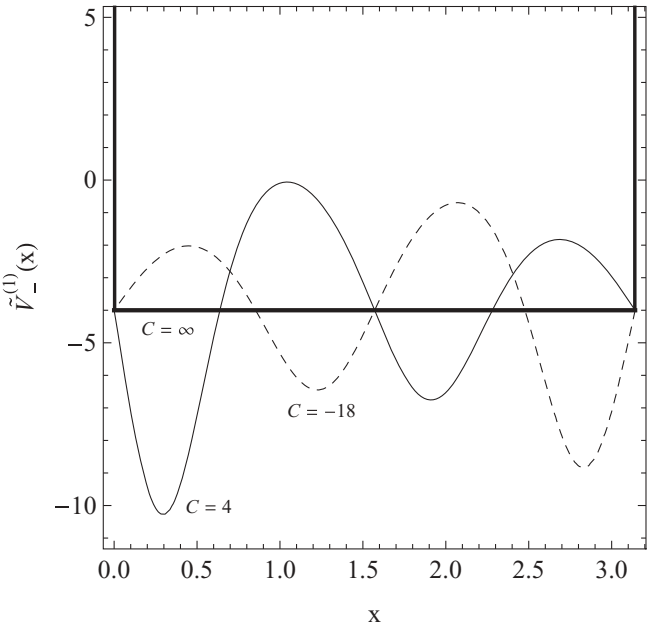

FIG. 3. Selected members of the family of one-parameter potentials $\tilde{V}_-^{(1)}(x)$. The limit $C \to \infty$ corresponds to the infinite square well $V_-^{(0)}(x) = -4$ of width $L = \pi$ (thick solid line).

is needed to move between the CPRS potential and harmonic oscillator whose creation and annihilation operators are $a^\dagger$ and a, respectively. Thus, the ladder operators for the wave functions $\tilde{\psi}_k^{(n)}(x)$ of the potential $\tilde{V}_-^{(n)}$ become $(B_n^\dagger A_n)A^\dagger a^\dagger A(A_n^\dagger B_n)$ and $(B_n^\dagger A_n)A^\dagger a A(A_n^\dagger B_n)$ for $k \neq n$.

C. Infinite square well potential

Despite its simplicity, the one-dimensional infinite square well potential with a deformed bottom requires some new techniques for obtaining solutions of the corresponding Schrödinger equation and usually one is unable to solve it exactly. In a recent article [11], the exact solution for the problem with a sinusoidal bottom has been deduced. In this section we explicitly find potentials with undulating bottom and energy spectrum coinciding with that of the infinite square well.

The wave functions and energy eigenfunctions of the infinite square well potential $V_-^{(0)}(x) = -\pi^2/L^2$ of width L are given by $\psi_k(x) = \sin[(k+1)\pi x/L]$ with $0 \leqslant x \leqslant L$ and $E_k = k(k+2)\pi^2/L^2$. Using this time for diverseness the first excited state wave function $\psi_1(x)$ we find a pair of partner potentials, namely, the infinite square well potential with flat bottom

$$V_-^{(1)}(x) = -4\pi^2/L^2,$$

and the infinite square well potential with nonflat bottom also defined in the region $0 \leqslant x \leqslant L$ (see Fig. 3)

$$\tilde{V}_-^{(1)}(x) = -\frac{4\pi^2}{L^2} - 16\frac{d}{dx}\left[\frac{\sin^2(2\pi x/L)}{C + 4x - L/\pi \sin(4\pi x/L)}\right].$$

Both of the potentials have identical energy spectra $\tilde{E}_k^{(1)} = (k-1)(k+3)\pi^2/L^2$. The normalized first excited state of the potential $\tilde{V}_-^{(1)}(x)$ is calculated from (4) and reads

$$\tilde{\psi}_1^{(1)}(x) = \sqrt{\frac{2C(C+4L)}{L}}\frac{\sin(2\pi x/L)}{C + 4x - L/\pi \sin(4\pi x/L)},$$

provided that $C \notin [-4L, 0]$. The wave functions $\tilde{\psi}_0^{(1)}(x)$, $\tilde{\psi}_2^{(1)}(x), \ldots$, can be found from (3). We only calculate the normalized lowest-state eigenfunction

$$\tilde{\psi}_0^{(1)}(x) = \frac{\sin\frac{\pi x}{L}\left[3\pi(C+4x) - 8L\sin\frac{2\pi x}{L} + L\sin\frac{4\pi x}{L}\right]}{3\sqrt{\frac{L}{2}}\left[L\sin\frac{4\pi x}{L} - \pi(C+4\pi)\right]}.$$

It corresponds to the negative energy $\tilde{E}_0^{(1)} = -3\pi^2/L^2$ as expected since the potential $\tilde{V}_-^{(1)}(x)$ is generated by the first excited state of the original potential. Note that the potential $\tilde{V}_-^{(1)}(x)$ satisfies

$$\tilde{V}_-^{(1)}(C,x) = \tilde{V}_-^{(1)}(C+2L, x+L/2).$$

It is known [8] that the eigenvectors $\psi_k(x)$ of the Hamiltonian $H_-^{(n)}$ admit the following creation and annihilation operators

$$M_+ = \cos\left(\frac{\pi x}{L}\right)\hat{k} + \frac{L}{\pi}\sin\left(\frac{\pi x}{L}\right)\partial_x,$$

and

$$M_- = \left[\cos\left(\frac{\pi x}{L}\right)\hat{k} - \frac{L}{\pi}\sin\left(\frac{\pi x}{L}\right)\partial_x\right]\hat{k}^{-1}(\hat{k}-1),$$

where one defines the "number" operator $\hat{k}$ and its inverse $\hat{k}^{-1}$ such that $\hat{k}\psi_k(x) = (k+1)\psi_k(x)$ and $\hat{k}^{-1}\psi_k(x) = (k+1)^{-1}\psi_k(x)$. The ladder operators $M_\pm$ obey

$$M_-\psi_k(x) = k\psi_{k-1}(x) \text{ and } M_+\psi_k(x) = (k+1)\psi_{k+1}(x).$$

It is not hard to convince yourself that the raising and lowering operators for the wave functions $\tilde{\psi}_k^{(n)}$ of the partner isospectral Hamiltonian $\tilde{H}_-^{(n)}$ are given by $(B_n^\dagger A_n)M_+(A_n^\dagger B_n)$ and $(B_n^\dagger A_n)M_-(A_n^\dagger B_n)$, respectively, for $k \neq n$ [when $k = n$ use Eq. (4)].

D. Two-parameter set of potentials isospectral to the harmonic oscillator

Given an eigenfunction $\psi_n(x)$ of the potential $V_-^{(0)}(x)$ one can find the wave function $\tilde{\psi}_k^{(n)}(x)$ of the one-parameter potential $\tilde{V}_-^{(n)}$ using Eq. (3). Now one can repeat this procedure and instead of the eigenfunction $\psi_n(x)$ in (2) and (3) use $\tilde{\psi}_k^{(n)}(x)$ to obtain a two-parameter potential $\tilde{V}_-^{(n,k)}(x)$ and its eigenfunctions. One can go on with this construction and obtain the well-defined multiparameter potentials strictly isospectral to the potential $V_-^{(k)}(x)$.

Let us focus on the harmonic oscillator $V_-^{(0)}(x) = x^2 - 1$ (with $\omega = 2$) whose ground-state wave function is $\psi_0(x) = e^{-x^2/2}$. The potential $\tilde{V}_-^{(0)}(x)$ is carefully discussed in Refs. [3,4,12] each using different approaches, so in the following we omit unnecessary calculations and only state its normalized first excited state wave function

$$\tilde{\psi}_1^{(0)}(x) = \sqrt{\frac{2}{\sqrt{\pi}}}\frac{e^{-3x^2/2}[1 + 2Cxe^{x^2} + \sqrt{\pi}xe^{x^2}\mathrm{erf}(x)]}{2C + \sqrt{\pi}\mathrm{erf}(x)},$$

where $|C| > \sqrt{\pi}/2$ to guarantee nonsingularity of the potential $\tilde{V}_-^{(0)}(x)$. Applying (2) to the wave function $\tilde{\psi}_1^{(0)}(x)$ we get the two-parameter potential (see Fig. 4)

$$\tilde{V}_-^{(0,1)}(x) = x^2 - 3 - 2\frac{d}{dx}\left\{\frac{e^{-x^2}}{C + \sqrt{\pi}/2\mathrm{erf}(x)} + \frac{\left(\tilde{\psi}_1^{(0)}(x)\right)^2}{\tilde{C} + \int_{x_0}^x \left[\tilde{\psi}_1^{(0)}(s)\right]^2 ds}\right\},$$

which is isospectral to the potential $\tilde{V}_-^{(0)}(x) - 2$, which is, in turn, isospectral to the harmonic oscillator $V_-^{(1)}(x) = x^2 - 3$ [i.e., its energy levels are $\tilde{E}_k^{(0,1)} = 2(k-1)$].

The potential $\tilde{V}_-^{(0,1)}(x)$ is nonsingular for any $C \neq 0$ and $|\tilde{C} + 1/(4C)| > \sqrt{\pi}/4$ as follows from normalizing its ground-state wave function. This family includes the oscillator potential $x^2 - 3$ in the limit $C, \tilde{C} \to \infty$; the potential $\tilde{V}_-^{(0)}(x)$ arises when $\tilde{C} \to \infty$; and finally $\tilde{V}_-^{(0,1)}(x)$ reduces to the potential $\tilde{V}_-^{(1)}(x)$ [3] in the limit $C \to \infty$.

Let us briefly mention how to obtain its eigenfunctions. There is an expression similar to (3) for $k = 2,3,\ldots$

$$\tilde{\psi}_k^{(0,1)}(x) \propto \tilde{B}_1^\dagger \tilde{A}_1 \tilde{\psi}_k^{(0)}(x) \propto \tilde{B}_1^\dagger \tilde{A}_1 B_0^\dagger A_0 \psi_k(x),$$

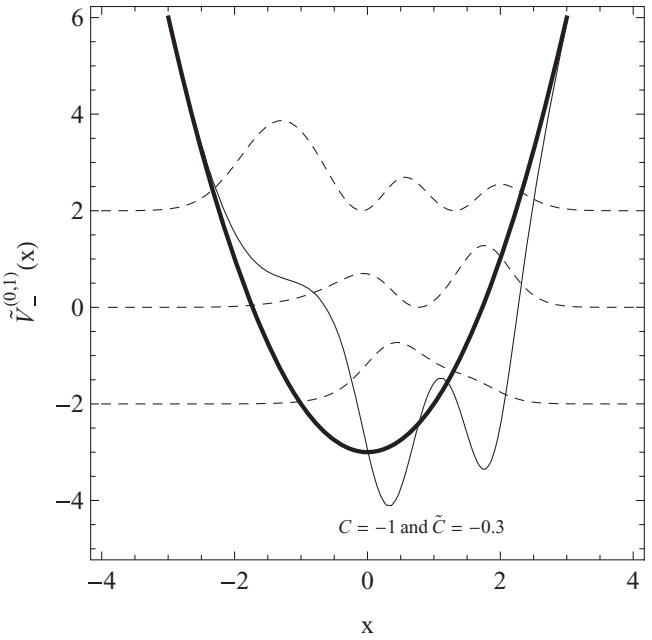

FIG. 4. Plot of the potentials $\tilde{V}_-^{(0,1)}(x)$, $V_-^{(1)}(x) = x^2 - 3$ and the nonnormalized probability densities (dashed line at the corresponding level position) for the three lowest energy levels of $\tilde{V}_-^{(0,1)}(x)$.

where $\tilde{\psi}_k^{(0)}(x)$ and $\psi_k(x)$ are the eigenfunctions of the potential $\tilde{V}_-^{(0)}(x)$ and the harmonic oscillator accordingly. The operators $\tilde{B}_1^\dagger$, $\tilde{A}_1$ are defined by

$$\tilde{A}_1 = \partial_x - \partial_x \ln \tilde{\psi}_1^{(0)}(x),$$

and

$$\tilde{B}_1^\dagger = -\partial_x + \partial_x \ln \frac{\left\{\tilde{C} + \int_{x_0}^x \left[\tilde{\psi}_1^{(0)}(s)\right]^2 ds\right\}}{\tilde{\psi}_1^{(0)}(x)}.$$

Lastly, the raising and lowering operators for the eigenvectors $\tilde{\psi}_k^{(0,1)}(x)$ are given by $\tilde{B}_1^\dagger \tilde{A}_1 B_0^\dagger A_0 A_0^\dagger A_0^\dagger B_0 \tilde{A}_1^\dagger \tilde{B}_1$ and $\tilde{B}_1^\dagger \tilde{A}_1 B_0^\dagger A_0 A_0 A_0^\dagger B_0 \tilde{A}_1^\dagger \tilde{B}_1$ with $A_0^\dagger$, A_0 being the creation and annihilation operators of the harmonic oscillator.

The two-parameter family of potentials with oscillator spectrum was also derived by the so-called second-order intertwining technique in Ref. [13]. The advantages of the presented technique of getting multiparameter sets of isospectral potentials are apparent.

IV. CONCLUSION

After the discovery of supersymmetry in string theory and then field theory, factorization was recognized as the application of supersymmetry to quantum mechanics. The nonuniqueness of factorization serves as an avenue for the construction of many isospectral potentials. In this article, we have explored the generality of this method by extending it to the excited states of a potential. Some nonsingular isospectral potentials that arise from the technique have been presented in this article. These include one-parameter extensions of the well-known infinite square well and Morse potentials as well as not so familiar CPRS potential and two-parameter extension of the harmonic oscillator. For some potentials the associated

wave functions and probability densities have been derived and plotted. The ladder operators were determined explicitly. The application of this technique may be of significant interest because it can be applied to any one-dimensional quantum mechanical potential.

The most general approach in the second-order SUSY QM is based on an arbitrary solution of the Schrödinger equation for the initial potential, rather than on its ground or excited state wave functions as discussed in the present article. However, there are certain advantages in such a presentation. For example, one can explicitly construct the ladder operators for both isospectral Hamiltonians. It is also possible to avoid some technical complexities of the most general approach by mimicking the traditional first-order SUSY QM. For instance, in the second-order SUSY QM none of the expressions $AA^\dagger$ or $A^\dagger A$ coincide with any of the isospectral partner Hamiltonians, but are quadratic forms in them. For comparison in our construction, which is based on the nonuniqueness of factorization, the appearance of the atypical Hamiltonian $H_+^{(n)}$ at the intermediate stage does not affect the isospectral partner Hamiltonians $\tilde{H}_-^{(n)}$ and $H_-^{(n)}$.

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