Semiclassical estimates of electromagnetic Casimir self-energies of spherical and cylindrical metallic shells

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The leading semiclassical estimates of the electromagnetic Casimir stresses on a spherical and a cylindrical metallic shell are within 1% of the field theoretical values. The electromagnetic Casimir energy for both geometries is given by two decoupled massless scalars that satisfy conformally covariant boundary conditions. Surface contributions vanish for smooth metallic boundaries, and the finite electromagnetic Casimir energy in leading semiclassical approximation is due to quadratic fluctuations about periodic rays in the interior of the cavity only. Semiclassically, the nonvanishing Casimir energy of a metallic cylindrical shell is almost entirely due to Fresnel diffraction.

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I. INTRODUCTION

Casimir obtained his now famous attractive force between two neutral metallic plates [1] by considering the boundary conditions (BCs) these impose on the electromagnetic field. Half a century later, his prediction was verified experimentally [2] to better than 1%.

Twenty years after Casimir's work, Boyer calculated the zero-point energy of an ideal conducting spherical shell [3]. Contrary to the perception from the attraction between two parallel plates, the sphere tends to expand. Boyer's result has since been improved in accuracy and has been verified by a number of field-theoretic methods [4–8]—although there may be little hope of observing this effect experimentally in the near future [9].

Since field-theoretic methods require explicit or implicit knowledge of cavity *frequencies*, they have predominantly been used to obtain the Casimir energies of classically *integrable* systems. In addition to that of a spherical cavity, the electromagnetic Casimir energy of dielectric slabs [10–12], metallic parallelepipeds [13–15], and long cylinders [16–20] has been computed in this manner.

However, most systems are not integrable. It therefore is necessary to develop reliable methods for estimating Casimir energies of classically nonintegrable and even chaotic systems. Balian and co-workers calculate Casimir energies by a multiple-scattering expansion of the Green's function [21,22]. This approach does not require knowledge of the quantum-mechanical spectrum. In principle, the multiple scattering expansion is exact for sufficiently smooth and ideally metallic cavities. However, except for a few integrable systems, it is difficult to compute more than the first terms of this multiple-scattering expansion in practice. Also, the relative importance of orders in this expansion is hard to assess *a priori.* Reference [23] proposed a semiclassical method based on Gutzwiller's trace formula [24] for the response function to estimate (finite) Casimir energies. It is suitable for estimating Casimir energies of hyperbolic and chaotic systems [23,25,26] with isolated classical periodic orbits.

Although in general not exact, this semiclassical approximation associates the finite (Casimir) part of the vacuum energy with optical properties of the system. It captures aspects of Casimir energies that have been puzzling for some time [27]. Sophisticated path-integral methods [28–32], allow one to obtain Casimir forces to arbitrary precision by numerical computation, but tend not to provide much qualitative insight. Due to unresolved renormalization problems, these methods have not yet been used to study the self-stress on cavities. Below we use semiclassical methods for classically integrable systems to estimate and to analyze the Casimir self-stress of a spherical and a cylindrical shell.

The simplicity, transparency, and surprising accuracy of this approximation is first demonstrated on Boyer's problem [3–6,8], the electromagnetic Casimir energy of a spherical cavity with an (ideal) metallic boundary. The semiclassical analysis of this problem is an order of magnitude simpler than any given previously, and the positive sign of the Casimir energy is related to caustic surfaces of second order. In Sec. IV, the semiclassical estimate for the electromagnetic Casimir self-energy of a perfectly conducting cylindrical shell is reexamined by converting the sum of WKB estimates for the eigenfrequencies to the dual sum over periodic orbits. It was previously [33,34] found that the Casimir self-energy of a cylindrical metallic shell vanishes to leading semiclassical order. It vanishes only when the upper bound of a particular Fresnel integral is ignored. Requiring this physical bound in the longitudinal momentum fraction, the semiclassical estimate also is rather accurate for the self-stress of a metallic cylindrical shell. It is within 0.25% of the field-theoretic value [16]. The sign of the self-stress again is determined by the presence of caustics and their associated Maslov-Keller indices [35,36]. However, contrary to the spherical case, the self-stress of a cylindrical cavity is semiclassically primarily due to Fresnel diffraction. A discussion of the results with a critical assessment of difficulties that remain for a semiclassical interpretation of some Casimir self-energies concludes this paper.

II. THE DUAL PICTURE: ELECTROMAGNETIC CASIMIR ENERGIES OF INTEGRABLE SYSTEMS AND PERIODIC RAYS

Integrable systems may be semiclassically quantized in terms of periodic paths on invariant tori [37]—in much the same manner as Bohr first quantized the hydrogen atom. Although generally not an exact transformation, classical periodic orbits on the invariant tori are *dual* to the mode frequencies in a semiclassical sense. Applying the Poisson resummation formula, the semiclassical Casimir energy (SCE) due to a massless scalar may be written in terms of classical periodic orbits [24,27,38,39],

$$\mathcal{E}_c = \frac{1}{2}\sum_{\mathbf{n}} \hbar\omega_{\mathbf{n}} - \text{(UV subtractions)}$$
$$\sim \frac{1}{2\hbar^d}\sum_{\mathbf{m}}{}' e^{(-i\pi/2)\beta_{\mathbf{m}}} \int_{sp} \mathbf{dI}\, H(\mathbf{I}) e^{2\pi i \mathbf{m}\cdot\mathbf{I}/\hbar}. \qquad (1)$$

The components of the d-dimensional vector $\mathbf{I}$ in Eq. (1) are the actions of a set of properly normalized action-angle variables that describe the integrable system. The exponent of the integrand in Eq. (1) is the classical action (in units of $\hbar$) of a periodic orbit that winds m_i times about the ith cycle of the invariant torus. $H(\mathbf{I})$ is the associated classical energy, and $\beta_{\mathbf{m}}$ is the Keller-Maslov index [35,36] of a class of periodic orbits identified by $\mathbf{m}$. The latter is a topological quantity that does not depend on the actions $\mathbf{I}$. To leading semiclassical order, the (primed) sum extends only over sectors $\mathbf{m}$ with nontrivial stationary points (the classical periodic paths of finite action). The correspondence in Eq. (1) can only be argued semiclassically [24,38,39], and the integrals in this expression should be evaluated in stationary phase approximation (sp) only.

The semiclassical spectrum of a massless scalar is exact for a number of manifolds without boundary [40], and the definition of the SCE by the right-hand side of Eq. (1) coincides with the Casimir energy of ζ-function regularization in these cases. It is exact for massless scalar fields that satisfy periodic, Neumann, or Dirichlet BCs on parallelepipeds [13,15,27] as well as for some tessellations of spheres [27,41,42]. To physically interpret the finite SCE of a system, one has to consider the implicit subtractions in the spectral density [22,27,43].

Semiclassical contributions to the spectral density arise due to small fluctuations about closed classical paths. In the dual picture, local ultraviolet (UV) divergences are associated with fluctuations about arbitrarily short and contractible classical paths. If all local UV divergences can be subtracted unambiguously [27,43,44], the dependence of the remaining finite Casimir energy on macroscopic deformations of the system semiclassically arises from fluctuations about classical closed paths of finite action.

In disjoint systems, the Casimir interaction at small separations usually is dominated by fluctuations about periodic orbits [27]. If there are no stationary classical periodic orbits, the leading interaction typically arises from periodic orbits of extremal—rather than stationary—finite length [45]. These correspond to diffractive effects that lead to relatively weak but sometimes rather interesting interactions. Examples of systems without stationary periodic orbits include the Casimir pendulum of Ref. [46], perpendicular plates [30], a wedge above a plate and, most recently, a rotational ellipsoid above a plate with a hole [47]. Diffractive contributions also become important when the separation between two systems is comparable to, or larger than, the smaller system. The original Casimir-Polder interaction between two atoms [48] or between an atom and a metallic plate [49] fall into this category.

However, periodic (stationary or extremal) orbits need not dominate the Casimir energy when some nonperiodic closed classical paths of finite action are much shorter than any periodic ones. The Casimir force due to a massless scalar field, which satisfies Dirichlet BCs on the plate of a hemispherical Casimir piston [50], is an example.

The following dimensional argument suggests that the local UV divergence due to scalar fields that satisfy Dirichlet or Neumann conditions on both sides of a smooth and *infinitesimally thin* even-dimensional surface vanishes in a dimensionless regularization scheme. Barring other scales, the local contribution from a small $(d-1)$-dimensional surface element dA to the divergence for dimensional reasons is of the form $\hbar c f_\varepsilon(R_i/R_j)\, dA/R^d$, where R is the principal (local) radius of curvature of the surface and f_ε is a dimensionless function of the (dimensionless) regularization parameter ε and of ratios of the local curvatures only. The interior and exterior radii of curvature at the same point on the infinitesimally thin surface are of equal magnitude but are of opposite sign. Local divergent surface contributions from the interior and the exterior of the infinitesimally thin surface [with the same scaleless BCs on *both* of its sides] therefore cancel precisely when d is odd. For a spherical surface, this cancellation has been explicitly observed in Ref. [51]. In $d=3$, finite Casimir energies have also been calculated for scalar fields and an infinitesimally thin cylindrical shell [17,52,53]. The preceding argument suggests that surface divergences cancel locally for any *infinitesimally thin* (and sufficiently smooth) even-dimensional boundary, regardless of its shape or whether Dirichlet or Neumann conditions are imposed.

However, with scalar fields this cancellation occurs only for vanishingly thin even-dimensional smooth boundaries and in a dimensionless regularization scheme. In the presence of a physical cutoff, the previous argument implies only the absence of logarithmic divergences, and surface divergences do occur even for ideal boundaries. They can be unambiguously subtracted [54–57] or, equivalently, absorbed in parameters that describe physical properties of the surface. Although these subtractions are not ambiguous for even-dimensional ideal interfaces, the remaining finite contribution to the Casimir energy due to closed nonperiodic classical paths could be significant. In this case Eq. (1) can give an inadequate estimate of the Casimir self-stress due to scalar fields [see Refs. [39,50] and Sec. IV for some formally subleading semiclassical surface contributions that have been omitted in Eq. (1)].

Fortunately, the finite *electromagnetic* self-stress of a smooth and infinitesimally thin perfectly conducting boundary *is* predominantly due to the stationary points of the integrand in Eq. (1). The electromagnetic Casimir self-energy of a closed smooth and perfectly metallic shell may be decomposed into the contributions from two massless scalar fields—one that satisfies Dirichlet's, and one that satisfies Neumann's BC[1]

[1]For a spherical shell, the usual Robin and Dirichlet conditions on solutions $B_\ell(x)$ of the spherical Bessel equation are dictated [58] by the conformal covariance of free electromagnetic fields. They may be thought of as simple Neumann and Dirichlet BCs for radial functions $\phi_\ell(x) = x B_\ell(x)$ of vanishing conformal dimension that are solutions of $[\partial_x^2 + 1 - \ell(\ell+1)/x^2]\phi_\ell(x) = 0$.

on the surface [22]. Semiclassically, these BCs are enforced by a phase lag of $0(\pi)$ for the Neumann (Dirichlet) scalar at each specular reflection of the classical path. Since both scalar fields satisfy the same wave equation, only fluctuations about classical trajectories with an *even* number of reflections contribute to the electromagnetic spectral density of an ideal metallic cavity [22]. Potentially divergent contributions of the Neumann and Dirichlet scalars that arise from closed contractible classical paths that reflect just once off the metallic boundary in this case cancel each other *exactly*. The length of any closed classical ray with an *even* number of reflections off a sufficiently smooth cavity is bounded below by geometrical characteristics of the cavity—such as its smallest radius of curvature. The closed classical rays of minimal length with an even number of reflections off such cavities are periodic and correspond to either a stationary point or an extremum of the classical action. For the cylindrical and spherical shells we will consider, the extremal periodic rays creep about the exterior of the cavity a number of times. They lead to rather small diffractive corrections [39,45,59] that will be ignored. The stationary points of the action in Eq. (1) thus give the main contribution to the Casimir self-stress of smooth cavities in the electromagnetic case. They correspond to periodic trajectories inside the cavity characterized by their winding number and (even) number of reflections, such as those shown in Fig. 1(a).

The previous argument applies equally well to the contribution of any pair of decoupled scalar fields that satisfy Dirichlet and Neumann BCs, and it may not be apparent why the SCE should give a particularly good approximation for the self-stress of a metallic shell due to the *electromagnetic* field. The exact field-theoretic Casimir stress caused by two massless scalar fields that solve the same Helmholtz equation as the transverse electromagnetic fields but satisfy Neumann (instead of Robin) and Dirichlet BCs on the spherical shell not only differs in sign, but also is an order of magnitude larger than the electromagnetic one [7,17,51]: $-0.220\,967\cdots\hbar c/R$ instead of $+0.046\,17\cdots\hbar c/R$ in the electromagnetic case [3,6]. The reason for this difference is that the Neumann BC on a spherical shell is not conformally covariant [58] for a scalar that satisfies the Helmholtz equation, whereas the Robin BC is.[2] On an intrinsically flat boundary, such as a cylindrical shell, the electromagnetic Casimir energy of a metallic shell indeed decomposes into contributions from two scalar fields that satisfy Dirichlet and Neumann BCs [17].

The classical action of a massless scalar particle is conformally invariant, and the semiclassical approximation indeed reproduces [27] the Casimir energy of a conformal scalar field on curved manifolds without boundary, such as that of a three-dimensional sphere S_3. Since specular reflection and phase lag do not depend on the curvature of a surface, we conjecture that the SCE approximates the Casimir energy of massless scalars that satisfy *conformally covariant* Dirichlet and Neumann conditions on a smooth boundary. For boundaries with nonvanishing curvature, the latter correspond to Robin-like conditions.

[2] $\partial_r[r\phi(r)] = 0$ is conformally covariant if ϕ has conformal mass dimension 1.

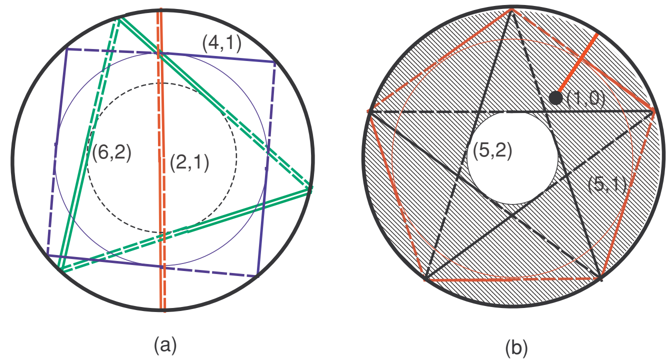

FIG. 1. (Color online) Classical periodic rays of a spherical and a cylindrical cavity. (a) The shortest primitive rays in sectors $(n,w) \in \{(2,1),(4,1),(6,2)\}$ that contribute to the electromagnetic SCE. (b) Closed paths in sectors $(n,w) \in \{(1,0),(5,1),(5,2)\}$ that reflect an odd number of times off the surface and whose contribution to the electromagnetic SCE vanishes. Caustic surfaces are indicated as thin circles. The dashed part of any trajectory is on one sheet, and its solid part is on the other sheet of a two-sheeted covering space. The phase space of the (5,2) sector is indicated by the hatched area. Note that the caustics are of second order for a spherical cavity but are of first order for a cylindrical one.

III. SELF-STRESS OF A SPHERICAL METALLIC SHELL

A massless particle in a spherical cavity is a classically integrable system, but the semiclassical spectrum is only asymptotically correct. The SCE therefore is not expected to exactly reproduce the field theoretic Casimir energy of a spherical cavity. It nevertheless will be surprisingly accurate. The SCE is obtained by performing the integrals of Eq. (1) in stationary phase and has a very transparent interpretation in terms of periodic orbits *within* the cavity. The sign of the SCE of a spherical cavity will be quite trivially established, and the discrepancy of 1% with the field-theoretic results may very well largely be due to the neglect of diffractive corrections from creeping orbits that wind about the exterior of the sphere. As argued earlier, there are no potentially divergent local contributions to the Casimir energy of such an idealized surface in the electromagnetic case. Its local surface tension, in fact, vanishes [22,60]. The only subtraction in the spectral density required for a finite Casimir energy with ideal metallic BCs is the Weyl contribution proportional to the volume of the sphere. This subtraction corresponds to ignoring the $\mathbf{m} = (0,0,0)$ term in the sum of Eq. (1). The remaining difficulty in calculating the SCE is a convenient choice of action-angle variables. For a massless scalar in three dimensions that satisfies BCs with spherical symmetry, an obvious set of actions is the magnitude of angular momentum $I_2 = L$, one of the components of angular momentum $I_3 = L_z$, and an action I_1 associated with the radial degree of freedom.

Since the azimuthal angle of any classical orbit is constant, the energy $E = H(I_1, I_2)$ of a particle in a spherical cavity of radius R does not depend on $I_3 = L_z$. In terms of this choice of actions, the classical energy is implicitly given by

$$\pi I_1 + I_2 \arccos\left(\frac{cI_2}{ER}\right) = \frac{ER}{c}\sqrt{1-\left(\frac{cI_2}{ER}\right)^2}. \qquad (2)$$

The branches of the square root and inverse cosine in Eq. (2) are chosen so that I_1 is positive. It is convenient to introduce dimensionless variables,

$$\lambda = 2ER/(\hbar c) \quad \text{and} \quad z = cI_2/(ER) \tag{3}$$

for the total energy [in units of $\hbar c/(2R)$] and the angular momentum (in units of ER/c) of an orbit. Note that $z \in [0,1]$ and that the semiclassical regime formally corresponds to $\lambda \gg 1$ (i.e., to wavelengths that are much shorter than the dimensions of the cavity). Using Eq. (2) and the definitions in Eq. (3), the angular frequency of the radial motion is given by $\omega^{-1} = (\partial E/\partial I_1)^{-1} = R\sqrt{1-[cI_2/(ER)]^2}/(\pi c) = (R/\pi c)\sqrt{1-z^2}$.

With the help of Eq. (2) and the definitions of Eq. (3), the semiclassical expression in Eq. (1) for the Casimir energy of a massless scalar field that satisfies Neumann or Dirichlet BCs on a spherical surface becomes

$$\mathcal{E}^{\text{sph}} = \frac{\hbar c}{4\pi R} \sum_{n,w\geqslant 0}{}' \operatorname{Re}\left[e^{-i\frac{\pi}{2}\beta(n,w)} \int_0^\infty d\lambda\, \lambda^3 \right.$$
$$\left. \times \int_0^1 dz\, z\sqrt{1-z^2}\, e^{i\lambda\{n[\sqrt{1-z^2}-z\arccos(z)]+w\pi z\}} \right]. \tag{4}$$

Here, the integral over I_3 has already been performed in stationary phase approximation. Because the Hamiltonian does not depend on I_3, only periodic orbits with $m_3 = 0$ contribute [27] significantly in stationary phase. Since $-I_2 \leqslant I_3 \leqslant I_2$, one has that $\int dI_3 = 2I_2 = \hbar\lambda z$. The factor $2I_2$ accounts for the $2(l+1/2)$ degeneracy of a state with angular momentum $L = \hbar(l+1/2) = I_2$. By taking (four times) the real part in Eq. (4), one can restrict the summations to nonnegative integers and choose the principal branches of the square root and inverse cosine functions in the exponent. The primed sum signifies that the summand is weighted by half if one of the integers vanishes and the $n = w = 0$ term is absent. The Keller-Maslov index $\beta(n,w)$ of a classical sector depends on whether Neumann or Dirichlet BCs are satisfied on the spherical shell. It is obtained as follows.

For positive integers w and n, the phase of the integrand in Eq. (4) is stationary at $z = \bar{z}(n,w) \in [0,1]$ where

$$0 = -n\arccos(\bar{z}) + w\pi$$
$$\Rightarrow \bar{z}(n,w) = \cos(w\pi/n), \quad n \geqslant 2w > 1. \tag{5}$$

Restrictions on the values of w and n arise because $\arccos(\bar{z}) \in [0,\pi/2]$ on the chosen branch. The phase is stationary at classically allowed points only for sectors with $n \geqslant 2w > 1$. Semiclassical contributions to the integrals of other sectors arise at the end points of the z integration only. These diffractive corrections are of subleading order in an asymptotic expansion of the spectral density for large λ and will be ignored. Note that $w \to w + n$ just amounts to choosing another branch of the inverse cosine function.

The classical action in sectors with stationary points is

$$S_{cl}(n,w) = \hbar\lambda \sin(w\pi/n)$$
$$= (E/c)2nR\sin(w\pi/n) = (E/c)L(n,w), \tag{6}$$

where $L(n,w)$ is the total length of the classical orbit. Some of these classical periodic orbits are shown in Fig. 1. The integer w in Eq. (6) gives the number of times an orbit circles or winds about the origin. The integer $n > 1$ gives the number of reflections off the spherical surface (windings of the radial motion). As indicated in Fig. 1, the envelope of the set of classical periodic orbits in the (n,w) sector is a caustic surface, and a double covering is required for a unique phase-space description [35]. The two sheets are joined at the inner caustic [indicated by a dashed circle in Fig. 1] and at the outer spherical shell of radius R. Every orbit that reflects off the spherical shell n times also passes through the caustic n times. The cross section of a bundle of rays is reduced to a point at the spherical caustic surface. Thus, the caustic is of second order and is associated with a phase loss of π every time it is crossed. At each specular reflection off the outer shell, Dirichlet BCs require an additional phase loss of π, whereas there is no phase change for Neumann BCs. Altogether, the Keller-Maslov index of the sector (n,w) depends on n only and is given by

$$\beta(n,w) = \begin{cases} 0, & \text{for Dirichlet BC} \\ 2n, & \text{for Neumann BC} \end{cases}. \tag{7}$$

As noted in Sec. 2, the electromagnetic field with (ideal) metallic BCs on a spherical shell may be viewed as two massless scalar fields of vanishing conformal dimension, one satisfying Dirichlet BCs, and one satisfying Neumann BCs [22]. Due to the Keller-Maslov phases of Eq. (7), only sectors (n,w) with *even* $n = 2k \geqslant 2w \geqslant 2$ contribute [22] to the SCE in the electromagnetic case.

The classical action vanishes for sectors with $w = 0$ or $n = 0$, and these sectors do not contribute to the SCE in leading approximation. The $n = 0$ sector corresponds to paths that do not reflect off the shell and gives rise to volume contributions that are subtracted. Equation (5) implies that extremal paths in the $n > 0, w = 0$ sectors have maximal angular momentum $1 = \bar{z} = Lc/(ER)$. These are great circles that lie wholly within the spherical shell. These paths are not stationary and lead to diffractive contributions that we will ignore here. For $n > 0, w > 0$, the curvature of the exponent at $\bar{z}(n,w)$ is finite,

$$\frac{\partial^2}{\partial z^2}\{n[\sqrt{1-z^2} - z\arccos(z)] + w\pi z\}\Big|_{\bar{z}(n,w)} = \frac{n}{\sin\left(\frac{w\pi}{n}\right)}, \tag{8}$$

whereas it diverges in sectors with $w = 0$. The behavior of the exponent for $z \sim 1$ in this case is

$$\sqrt{1-z^2} - z\arccos(z) = \frac{2\sqrt{2}}{3}(1-z)^{3/2} + O[(1-z)^{5/2}]. \tag{9}$$

Quadratic fluctuations about a classical orbit with $w = 0$ are of vanishing width, and these sectors do not contribute in stationary phase approximation.

To leading semiclassical accuracy, the Casimir energy of a spherical cavity with an ideal metallic boundary is

$$\mathcal{E}_{\rm EM}^{\rm sph} \sim \frac{\hbar c}{4\pi R}\,{\rm Re}\sum_{n=1}^{\infty}[1^n+(-1)^n]\sum_{w=1}^{n/2}\int_0^\infty d\lambda\,\lambda^3 e^{in\lambda\,\sin(w\pi/n)} \times \int_0^1 dz\,z\sqrt{1-z^2}\,e^{in\lambda[z-\bar z(n,w)]^2/[2\sin(w\pi/n)]}$$
$$\sim \frac{\hbar c}{R}\left[\sum_{n=1}^{\infty}\frac{1}{16\pi n^4}+\sum_{n=2}^{\infty}\frac{15\sqrt{2}}{256 n^4}\sum_{w=1}^{n-1}\frac{\cos\left(\frac{w\pi}{2n}\right)}{\sin^2\left(\frac{w\pi}{2n}\right)}\right] \sim 0.046\,68\cdots\frac{\hbar c}{R}. \qquad (10)$$

This semiclassical estimate is only about 1% larger than the best numerical value [6] $0.046\,17\cdots\hbar c/R$ for the electromagnetic Casimir energy of a spherical cavity with an infinitesimally thin metallic surface. Note that the contribution from the $(2w,w)$ sectors had to be considered separately in Eq. (10), since the measure $dz\,z$ vanishes at the stationary point $\bar z(2w,w)=\cos(\pi/2)=0$ of the integrand. As can be seen in Fig. 1(a), the classical rays of $(2w,w)$ sectors go back and forth between antipodes of the cavity and pass through its center—they have angular momentum $\mathbf{L}=0$.

The shortest primitive orbits give somewhat less than half $[1/(16\pi)\sim 0.02]$ of the total SCE of the spherical cavity—much less than the 92% they contribute to the Casimir energy of parallel plates. The reason is that contributions only drop off as $1/n^2$ rather than $1/n^4$ as for parallel plates. The orbit in the (4,1) sector [the inscribed square in Fig. 1(a)], furthermore, is just a factor of $\sqrt{2}$ longer than the (2,1) orbit [which, in turn, is a factor of $1/\sqrt{2}$ shorter than the (4,2) orbit]. To estimate the magnitude of the contribution from any particular sector, one has to take the available "phase space" as well as the ray's length into account. Thus, although the length of a $(2n,1)$ orbit tends to $2\pi R$ for $n\to\infty$, the associated "phase space" (essentially the volume of the shell between the boundary of the cavity and the inner caustic) decreases like $1/n^2$. This accounts for the relatively slow convergence of the sum in Eq. (10). To achieve a numerical accuracy of 10^{-5}, the first 50 terms of the sum were evaluated explicitly and the remainder estimated by the Richardson extrapolation method. However, note that the semiclassical two-reflection coefficient of $1/(16\pi)$ is just 2/5 of the leading field-theoretic coefficient for a dilute dielectric-diamagnetic spherical shell [61]. We will encounter a similar discrepancy in the case of a cylindrical shell and will discuss it further in Sec. V.

IV. SELF-STRESS OF A CYLINDRICAL METALLIC SHELL

The example of a spherical cavity suggests that one may obtain electromagnetic Casimir self-energies rather accurately by considering only small fluctuations about the classical periodic orbits. Since the classical periodic rays of finite length for a long cylindrical shell are the same as for a sphere, one would expect that a semiclassical calculation of the self-stress is just as straightforward for a cylindrical cavity.

This is not the case. The electromagnetic SCE of an ideal metallic cylindrical shell vanishes in stationary phase approximation [33,34]. The contribution from any periodic orbit to the electromagnetic SCE vanishes in this approximation for the same reason that it is positive for a spherical cavity—due to optical phases. However, going beyond stationary phase approximation and including certain Fresnel diffraction effects, we again obtain a very good approximation to the Casimir self-stress of a metallic cylinder from quadratic fluctuations about classical periodic paths.

The cylinder appears particularly suited for a semiclassical analysis in terms of massless scalars because the transverse electric and magnetic modes satisfy Dirichlet and Neumann BCs on the cylindrical surface. The classical system is integrable, and we could directly employ the formalism of Berry and Tabor [27,38] or an extended Gutzwiller approach [33,39] to obtain the SCE in much the same way as we did for the sphere. However, to better understand why the SCE vanishes in this approximation and to improve upon it, we first obtain the dual expression for the Casimir self-energy of a cylindrical cavity directly from the semiclassical (WKB) estimate of the eigenvalues of the scalar fields. The Casimir energy of an infinitesimally thin metallic cylindrical shell of radius R within a much larger cylinder of fixed radius $R_>\sim\infty$ is given by the R-dependent part of the zero-point energy,

$$\mathcal{E}_{\rm EM}^{\rm cyl}(R)=\lim_{R_>\to\infty}\frac{\hbar c L}{2\pi}\sum_{\substack{D,N\\ n}}\int_0^\infty dq\,\Big[\sqrt{q^2+\kappa_n^2(0,R)} +\sqrt{q^2+\kappa_n^2(R,R_>)}-\sqrt{q^2+\kappa_n^2(0,R_>)}\Big]_{D,N}, \qquad (11)$$

where $[\kappa_n(R_<,R_>)]_{D,N}$ is the spectrum of wave numbers for a scalar field that satisfies Dirichlet (D) or Neumann (N) BCs on a two-dimensional annulus with inner radius $R_<$ and outer radius $R_>$.

The asymptotic heat-kernel expansion [62] implies that the subtracted expression in Eq. (11) is finite: The potentially logarithmic divergence proportional to the average of the third power of the (extrinsic) curvature of the cylindrical surface is canceled. DeRaad and Milton obtained this finite self-energy some time ago [16]. Since they uniformly approach the exact wave numbers sufficiently rapidly, the SCE obtained by replacing the exact eigenvalues in Eq. (11) by their WKB estimates also is finite. For $R_>\sim\infty$, all periodic orbits in the annulus have a length of $O(R_>)$, and the only finite contribution to the SCE from the annulus is due to (diffractive) creeping orbits that wrap about the inner cylinder. We will neglect this small contribution to the SCE and will consider only radius-dependent semiclassical contributions from the inner cylinder. In WKB approximation, the transverse wave numbers $\kappa_{n\ell}(0,R)=x_{n\ell}/R$ of *interior* modes of the cylinder are positive solutions [39,63] of,

$$f_\ell(x_{n\ell})=\pi\left(n+\tfrac12\pm\tfrac14\right)\quad \text{for}\quad n=0,1,\ldots;\ \ell=0,1,\ldots, \qquad (12)$$

where the (+) and (−) signs correspond to Dirichlet and Neumann BCs, respectively, and

$$f_\ell(x)=\sqrt{x^2-\ell^2}-\ell\,\arccos(\ell/x). \qquad (13)$$

This semiclassical (Debye) approximation generally gives the zeros of Bessel functions of the first kind and their derivatives to better than 1%. There is but one notable exception: The zero of $J_0'(x) = J_1(x)$ at $x = 0$ corresponds to a WKB value of $x_{00} = \pi/4$. Note that $f_\ell(x = \ell) = 0$, and all semiclassical wave numbers satisfy $x_{n\ell} > \ell$.

Using the approximation in Eqs. (12) and (13) for the eigenvalues, the R-dependent contribution of interior modes to the SCE of a conducting cylindrical shell is the finite R-dependent part of

$$\begin{aligned}\mathcal{E}_{\rm EM}^{\rm cyl} &\sim \frac{\hbar c L}{2\pi^2 R^2}\sum_{n=-\infty}^{\infty}[i^n + (-i)^n]\sum_{\ell=0}^{\infty}\int_0^\infty dy \\ &\quad\times \int_\ell^\infty dx\,\sqrt{y^2+x^2}\,f_\ell'(x)\,e^{2inf_\ell(x)} \\ &\sim -\frac{\hbar c L}{2\pi^2 R^2}\,\mathrm{Im}\sum_{\ell=0}^{\infty}\int_0^\infty dy\int_\ell^\infty dx\,\sqrt{y^2+x^2} \\ &\quad\times \partial_x \ln\{\cos[2f_\ell(x)]\}. \end{aligned} \tag{14}$$

The summation over n has been performed in the last expression. It shows the equivalence of the present approach with one based on the generalized argument principle [64] with the function $\cos[2f_\ell(x)] = \frac{1}{2}(e^{-if_\ell(x)} + i e^{if_\ell(x)})(e^{-if_\ell(x)} - i e^{if_\ell(x)})$, whose zeros are the WKB estimates for the mode frequencies in each partial wave and a contour that runs from $x=\infty$ to $x=\ell$ just below the real axis and returns to $x=\infty$ just above it. Although divergent because it only includes the contribution from the interior, it is reassuring that the expression in Eq. (14) is not logarithmically divergent. The divergence can be subtracted unambiguously (it, in fact, is canceled by exterior contributions we do not consider), and a semiclassical evaluation becomes possible.

One arrives at Eq. (14) by applying the Poisson resummation formula,

$$\sum_{n=-\infty}^{\infty}\delta(f-n) = \sum_{n=-\infty}^{\infty} e^{2\pi i n f}, \tag{15}$$

to the dimensionless (and scaled) semiclassical spectral densities,

$$\begin{aligned}\rho_{D\,\mathrm{or}\,N}(x) &= \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}\delta(x-x_{n\ell}) \sim \sum_{\ell=0}^{\infty}\theta(x-\ell)f_\ell'(x) \\ &\quad\times \sum_{n=-\infty}^{\infty}\delta\left(f_\ell(x) - \left(n+\frac{1}{2}\pm\frac{1}{4}\right)\pi\right). \end{aligned}\tag{16}$$

The Heaviside function $\theta(x-\ell)$ in Eq. (16) ensures that f_ℓ is real. Employing the Poisson relation once more in the form

$$\sum_{\ell=0}^{\infty} g(\ell) = \frac{1}{2}g(0) + \sum_{w=-\infty}^{\infty}\int_0^\infty d\ell\, e^{2\pi i w \ell} g(\ell), \tag{17}$$

the sum over partial waves is converted to one over the winding number of classical paths. In terms of the dimensionless wave number $\lambda = \sqrt{x^2+y^2} = ER/(\hbar c)$ and the longitudinal- and angular-momentum fractions $\alpha = y/\lambda$ and $z = \ell/x$, one obtains:

$$\begin{aligned}\mathcal{E}_{\rm EM}^{\rm cyl} &\sim \frac{\hbar c L}{\pi^2 R^2}\sum_{n=-\infty}^{\infty}(-1)^n\sum_{w=-\infty}^{\infty}\int_0^\infty \lambda^3\,d\lambda\int_0^1 d\alpha \\ &\quad\times\int_0^1\sqrt{1-z^2}\,e^{2i\lambda\sqrt{1-\alpha^2}[4n(\sqrt{1-z^2}-z\arccos z)+w\pi z]}\,dz, \end{aligned}\tag{18}$$

where we used that only even n contribute in Eq. (14). Only the principal branch of the inverse cosine is to be considered here, and the integrals are formal in the sense that they are to be evaluated to leading (nonvanishing) order in stationary phase approximation only. The integrals are finite in this restricted sense, and their divergent part has been implicitly subtracted. They otherwise diverge (in all sectors) due to the behavior of the integrand at large λ near $\alpha \sim 1$ (that is for large longitudinal momentum fractions). As explained in Sec. II, we are assured that all surface divergences cancel for metallic BCs on a cylinder. We are interested only in the finite contributions that arise from quadratic fluctuations about periodic classical trajectories [stationary points of the integrand in the $n \neq 0, w \neq 0$ sectors].

The $\frac{1}{2}g(0)$ term in Eq. (17) gives a surface correction [39] to Eq. (18). As mentioned in Sec. II, such surface contributions cancel in the electromagnetic case. One may explicitly verify this by subtracting the $n = 0$ sector and observing that

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\mathrm{Re}\sum_{n=1}^{\infty}(-1)^n\int_0^\infty d\lambda\,\lambda^2\int_0^{\pi/2-\varepsilon}d\phi\,e^{4in\lambda\cos\phi} \\ &\quad= \frac{3\zeta(3)}{128}\lim_{\varepsilon\to 0^+}\mathrm{Re}\left(i\int_0^{\pi/2-\varepsilon}\frac{d\phi}{\cos^3\phi}\right) = 0. \end{aligned}\tag{19}$$

To quadratic order, fluctuations about the stationary point at $\alpha = 0$ in Eq. (18) give rise to Fresnel-like integrals of the type (for generic $\gamma > 0$):

$$\begin{aligned}\int_0^1 d\alpha\, e^{i\pi\gamma^2\sqrt{1-\alpha^2}} &\sim e^{i\pi\gamma^2}\int_0^1 d\alpha\, e^{-i\pi\gamma^2\alpha^2/2} \\ &= e^{i\pi\gamma^2}[C(\gamma) - iS(\gamma)]/\gamma. \end{aligned}\tag{20}$$

Extending the upper bound of the fluctuation integral in Eq. (20) to ∞ and thereby replacing the cosine- and sine-Fresnel integrals of Eq. (20) by their mean value of 1/2 gives a vanishing value for the SCE of a metallic cylindrical shell [33,34]. The leading nonvanishing contribution to the SCE arises from the finite upper bound of the fluctuation integral in Eq. (20). The Casimir self-stress of a metallic cylindrical shell in a semiclassical sense is almost entirely due to Fresnel diffraction effects.

The semiclassical evaluation of the integrals in Eq. (18) again is facilitated by noting that one can take (four times) the real part of the $(n > 0, w > 0)$-sector contributions and that the spectral density is analytic in the first quadrant. The $\lambda = i\xi$ integral in Eq. (18) may thus be performed along the positive imaginary axis and the contour is closed in the complex λ plane by a large quarter circle on which the integrand, or better, a regularized version of it, vanishes sufficiently rapidly. The integrand is real on the imaginary energy axis. For $0 < w \leqslant n$, the stationary points of the integrand in Eq. (18) are at

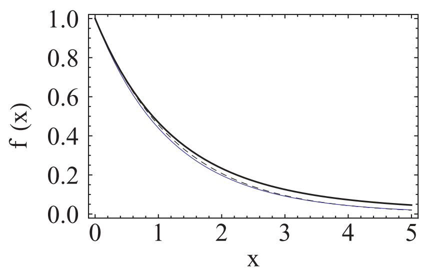

FIG. 2. (Color online) Integration over the longitudinal momentum fraction. The upper solid curve shows the exact integral in Eq. (22) over the longitudinal momentum fraction α; the lower solid line is its semiclassical approximation to quadratic order in the fluctuations. The dashed curve gives the simple exponential approximation $e^{-\pi x/4}$.

$z = \cos\frac{\pi w}{2n}$ and $\alpha = 0$. They correspond to planar periodic rays of a cylindrical cavity like those in Fig. 1(a) with winding number w and $2n$ vertices. Expanding to quadratic order about these stationary points, we first integrate out fluctuations in z. By rescaling $\xi \to \xi/(1-\alpha^2/2)$, the remaining ξ and α integrations may then be carried out independently.[3] Note that the analytic continuation of the integrals is possible only if we assume an upper bound at $\alpha = 1-\varepsilon$. By starting with the expression of Eq. (18), this procedure gives the following semiclassical estimate of $\mathcal{E}^{\rm cyl}_{\rm EM}$ [with $c_{nw} := \cos(\frac{\pi w}{2n})$ and $s_{nw} := \sin(\frac{\pi w}{2n})$]:

$$\begin{aligned}\mathcal{E}^{\rm cyl}_{\rm EM} &\sim \frac{4\hbar c L}{\pi^2 R^2}\sum_{n=1}^{\infty}(-1)^n\sum_{w=1}^{n}\int_0^\infty \xi^3\, d\xi\int_0^1 d\alpha \\ &\quad\times\int_0^1 dz\, s_{nw}\,\exp\left\{-4n\xi s_{nw}\left[1+\frac{(z-c_{nw})^2}{2s_{nw}^2}-\frac{\alpha^2}{2}\right]\right\} \\ &\sim \frac{\hbar c L 15\sqrt{2}}{R^2 512\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\left(-\frac{1}{2}+\sum_{w=1}^{n}s_{nw}^{-2}\right) \\ &\quad\times\int_0^1 d\alpha\left(1-\frac{\alpha^2}{2}\right)^{-7/2} \\ &= \frac{7\pi(7\pi^2-240)}{276\,480}\frac{\hbar c L}{R^2} = -0.013\,594\cdots\frac{\hbar c L}{R^2}.\end{aligned} \tag{21}$$

The SCE of a conducting cylindrical shell differs by less than 0.25% from the field-theoretic value [16] of $-0.013\,5613\cdots\hbar c L/R^2$ once a certain kind of Fresnel diffraction is included. It was necessary to go beyond the formally leading semiclassical approximation because the latter vanishes. Enforcing the bound on longitudinal momenta, the semiclassical approximation also closely reproduces the Casimir self-stress on a metallic cylindrical shell.

To better understand the approximation made, compare the following estimates to the Fresnel-like α integral in Eq. (18):

$$\begin{aligned}\int_0^1 d\alpha\, e^{-x\sqrt{1-\alpha^2}} &= \frac{\pi}{2}[H_{-1}(ix) - I_1(x)] \sim e^{-\pi x/4} \\ \sim \int_0^1 d\alpha\, e^{-x(1-\alpha^2/2)} &= -ie^{-x}\sqrt{\frac{\pi}{2x}}\,\mathrm{erf}\left(i\sqrt{\frac{x}{2}}\right).\end{aligned} \tag{22}$$

The first line expresses the α integral that occurs in Eq. (18) for values of the energy on the positive imaginary axis in terms of Struve and Bessel functions. The function $e^{-\pi x/4}$ gives the best exponential fit for small $x \sim 0$. The second line of Eq. (22) is the semiclassical approximation to second order in the fluctuations. It also is a uniform approximation to the integral that decays exponentially for $x \sim \infty$ (but as $e^{-x/2}/x$). As shown in Fig. 2, both approximations reproduce the integral well for small values of x—where the result is sizable—but cut down its slow decay ($\sim 1/x^2$) at large $x \sim \infty$. As noted before, the powerlike decay of the original integral gives rise to divergent contributions to the Casimir energy in every (n,w) sector. By assuming (and heat-kernel considerations show this is possible for an ideal metallic shell [54,62,65]) all divergences are subtracted unambiguously, the leading nonvanishing semiclassical approximation apparently is quite a reasonable estimate of the finite remainder. One could improve on the representation of the integral by including higher orders in the expansion of $\sqrt{1-\alpha^2}$ about $\alpha = 0$, but this would be inconsistent with retaining only the leading order WKB approximation to the eigenvalues. We can assess the sensitivity of the SCE to the precise manner in which the powerlike tail is cut off by comparing with the exponential fit $e^{-\pi x/4}$ for the α integral: In this approximation, the Casimir energy for a metallic cylinder becomes $(7\pi^2-240)\hbar c L/(288\pi^3\sqrt{2}R^2) = -0.013\,533\cdots\hbar c L/R^2$, which is well within the error of the semiclassical estimate.

V. DISCUSSION AND CONCLUSIONS

We obtained the electromagnetic Casimir self-stress of perfectly conducting spherical and cylindrical shells in semiclassical approximation. This approach reproduces the field-theoretic values [3,6] to better than 1%. The semiclassical description by two massless scalars, in general, gives the *electromagnetic* Casimir stress of a metallic spherical shell rather than the sum of the Casimir stresses due to scalar fields that satisfy Neumann and Dirichlet BCs. For a spherical shell, the latter has the opposite sign and is an order of magnitude larger [7,17,51]. The semiclassical description is inherently conformal and best suited to describe scalar fields that satisfy conformally covariant BCs [58]. It was previously observed [27] that the dispersion for massless scalar particles leads to a semiclassical description of conformally coupled scalar fields on curved spaces such as S_3. They also appear to couple conformally to curved boundaries. Neumann BCs are semiclassically imposed by requiring no phase change and specular reflection. They do not change under conformal rescaling of the boundary and, in this sense, are conformally covariant. For spherical boundaries,

[3]Note that the powerlike decay arises from the integration region $\alpha \sim 1$, where the longitudinal wave number is much larger than the transverse one. It may be regularized by imposing $\alpha < 1-\varepsilon$ with finite $\varepsilon > 0$.

they describe (massless) scalar fields that satisfy conformally covariant [58] Robin conditions. This conjecture is supported by the fact that the low-lying semiclassical eigenfrequencies implied[4] by Eq. (10) are closer to those of two scalar fields that satisfy Dirichlet and Robin rather than Dirichlet and Neumann conditions. Robin BCs approach Neumann BCs at high frequencies, and both are similarly implemented in geometrical optics (i.e., with specular reflection and no phase loss).

On a cylindrical surface in flat space, ordinary Neumann and Dirichlet conditions already are conformally covariant. We estimated the electromagnetic Casimir self-stress of a metallic cylindrical shell by explicitly summing WKB approximations to the eigenfrequencies of the two scalars in the dual picture using the Poisson resummation formulas. The dual of the principal quantum number is the number of interactions with the shell. Summing over these, one arrives at the general argument principle [see Eq. (14)] often used [64] to evaluate Casimir energies when the spectrum is only implicitly known. The difference is that in semiclassical approximation the roots of the characteristic function are WKB estimates of the eigenfrequencies. The dual to the sum over partial waves is the sum over windings about the origin. The stationary points of minimal action in each sector are classical periodic trajectories. We evaluate the integrals in each nontrivial sector to quadratic order in the fluctuations [38,39]. For a cylindrical shell, it is essential to enforce the upper bound on the longitudinal momentum fraction $0 \leqslant \alpha < 1$. Ignoring it and integrating without restriction over quadratic fluctuations about the stationary point at $\alpha = 0$, the SCE of a cylindrical shell would vanish [33,34]. Restricting the quadratic fluctuations in the longitudinal momentum fraction to $\alpha < 1$ essentially reproduces the field-theoretic value for the Casimir energy of a metallic cylinder. This restriction is required by causality: The spectral density otherwise is not analytic in the first quadrant. This violation of causality does not occur for any other fluctuation integral, which, to leading order, are evaluated without restriction.

The Casimir energy of a cylindrical metallic shell in a semiclassical sense is almost entirely due to Fresnel diffraction. From a semiclassical point of view, the metallic cylindrical shell therefore is a rather interesting geometry and conceptually more rewarding than the spherical one. Because a straightforward semiclassical evaluation to leading order resulted in a vanishing Casimir stress, it was previously believed [34] that this might explain the *exact* vanishing of the Casimir stress on a dielectric-diamagnetic cylindrical shell (with equal speed of light on either side) to first order in the reflection coefficients [5,18,66]. The nonvanishing Casimir energy of a metallic cylindrical shell was attributed to subleading diffraction effects. The latter conjecture has been verified in this investigation, but the diffractive contribution due to classical orbits with just two reflections, the $n = 1$ contribution[5] to Eq. (21), is $-0.017\,4076\cdots\hbar c L/R^2$. It does not vanish and is the largest contribution in magnitude, larger than the total self-stress of the metallic cylinder. We thus still lack a semiclassical understanding of a weakly reflecting dielectric-diamagnetic cylindrical shell, and it should be pointed out that the self-stress of a dielectric-diamagnetic spherical shell [61] also is underestimated by a factor of 2.5 in this approximation.

In the limit of very small reflection coefficients, it probably is impossible to ignore contributions from the exterior. It may be necessary to include diffractive effects from paths that creep about the cylinder to describe weakly reflective interfaces semiclassically. Mathematically, the exact cancellation in the cylindrical case is an addition theorem for Bessel functions that explicitly requires contributions from the exterior [57]. That perturbation in the reflection coefficients is quite delicate becomes evident for dielectric cylinders [67] and spheres [68]: To second order in the reflection coefficients, the Casimir stress, in the dielectric case, is finite and comparable to the situation where the speed of light is continuous across the boundary; the total Casimir self-stress, however, diverges logarithmically [54,65] once the speed of light in the interior and the exterior do not match exactly.

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[4] One can extract the semiclassical estimate of the eigenfrequencies from the spectral density of Eq. (10) by reversing the procedure used to obtain the semiclassical spectral density in Eq. (14) from the WKB estimate of the eigenfrequencies in the case of a cylindrical shell.

[5] The factor $(-\frac{1}{2} + \sum_{w=1}^{n} s_{nw}^{-2}) = (4n^2 - 1)/6$ in Eq. (21) vanishes for $n = 1/2$, but not for $n = 1$!

[1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948).

[2] S. K. Lamoreaux, Phys. Rev. Lett. **78**, 5 (1997); U. Mohideen and A. Roy, *ibid.* **81**, 4549 (1998); T. Ederth, Phys. Rev. A **62**, 062104 (2000); G. Bressi, G. Carugno, R. Onofrio, and G. Ruoso, Phys. Rev. Lett. **88**, 041804 (2002); R. S. Decca, D. Lopez, E. Fishbach, G. L. Klimchitskaya, D. E. Krause, and V. M. Mostepanenko, Ann. Phys. **318**, 37 (2005).

[3] T. H. Boyer, Phys. Rev. **174**, 1764 (1968).

[4] B. Davies, J. Math. Phys. **13**, 1324 (1972).

[5] R. Balian and B. D. Duplantier, Ann. Phys. (NY) **112**, 165 (1978).

[6] K. A. Milton, L. L. DeRaad, and J. S. Schwinger, Ann. Phys. (NY) **115**, 388 (1978).

[7] V. V. Nesterenko and I. G. Pirozhenko, Phys. Rev. D **57**, 1284 (1998).

[8] G. Esposito, A. Y. Kamenshchik, and K. Kirsten, Phys. Rev. D **62**, 085027 (2000).
[9] G. Barton, J. Phys. A **37**, 1011 (2004).
[10] E. M. Lifshitz, Zh. Eksp. Teor. Fiz. **29**, 894 (1955) [Sov. Phys. JETP **2**, 73 (1956)].
[11] P. W. Milonni, *The Quantum Vacuum* (Academic, New York, 1993).
[12] K. A. Milton, J. Phys. A **37**, R209 (2004).
[13] W. Lukosz, Physica **56**, 109 (1971).
[14] J. S. Dowker and R. Critchley, J. Phys. A **9**, 535 (1976).
[15] J. Ambjörn and S. Wolfram, Ann. Phys. **147**, 1 (1983).
[16] L. L. DeRaad, Jr. and K. A. Milton, Ann. Phys. (NY) **136**, 229 (1981).
[17] P. Gosdzinsky and A. Romeo, Phys. Lett. B **441**, 265 (1998).
[18] K. A. Milton, A. V. Nesterenko, and V. V. Nesterenko, Phys. Rev. D **59**, 105009 (1999).
[19] G. Lambiase, V. V. Nesterenko, and M. Bordag, J. Math. Phys. **40**, 6254 (1999).
[20] V. V. Nesterenko and I. G. Pirozhenko, J. Math. Phys. **41**, 4521 (2000).
[21] R. B. Balian and C. Bloch, Ann. Phys. (NY) **60**, 401 (1970); **63**, 592 (1971); **64**, 271 (1971); **84**, 559(E) (1974); **69**, 76 (1972); **85**, 514 (1974).
[22] R. Balian and B. Duplantier, Ann. Phys. **104**, 300 (1977); **112**, 165 (1978); *IOP Conference Series 176: Recent Developments in Gravitational Physics*, p. 73, edited by I. Ciufolini, E. Coccia, M. Colpi, V. Gorini, and R. Peron (Taylor & Francis, Boca Raton, 2006) [quant-ph/0408124v1]
[23] M. Schaden and L. Spruch, Phys. Rev. A **58**, 935 (1998).
[24] M. C. Gutzwiller, *Chaos in Classical Quantum Mechanics* (Springer, New York, 1990).
[25] O. Schröder, A. Scardicchio, and R. L. Jaffe, Phys. Rev. A **72**, 012105 (2005).
[26] A. Bulgac, P. Magierski, and A. Wirzba, Phys. Rev. D **73**, 025007 (2006).
[27] M. Schaden, Phys. Rev. A **73**, 042102 (2006).
[28] C. Schubert, Phys. Rep. **355**, 73 (2002).
[29] H. Gies and K. Langfeld, Int. J. Mod. Phys. A **17**, 966 (2002); B. Ponsot, V. Schomerus, and J. Teschner, J. High Energy Phys. 02 (2002) 016; H. Gies and K. Klingmüller, Phys. Rev. Lett. **96**, 220401 (2006); A. Weber and H. Gies, *ibid.* **105**, 040403 (2010).
[30] H. Gies and K. Klingmüller, Phys. Rev. Lett. **97**, 220405 (2006); N. Graham, A. Shpunt, T. Emig, S. J. Rahi, R. L. Jaffe, and M. Kardar, Phys. Rev. D **81**, 061701 (2010); D. Kabat, D. Karabali, and V. P. Nair, *ibid.* **81**, 125013 (2010).
[31] T. Emig, A. Hanke, R. Golestanian, and M. Kardar, Phys. Rev. Lett. **87**, 260402 (2001); R. Büscher and T. Emig, *ibid.* **94**, 133901 (2005); T. Emig, N. Graham, R. L. Jaffe, and M. Kardar, *ibid.* **99**, 170403 (2007); M. T. H. Reid, A. W. Rodriguez, J. White, and S. G. Johnson, *ibid.* **103**, 040401 (2009).
[32] M. Schaden, Phys. Rev. A **79**, 052105 (2009).
[33] F. D. Mazzitelli, M. J. Sanchez, N. N. Scoccola, and J. von Stecher, Phys. Rev. A **67**, 013807 (2003).
[34] M. Schaden, e-print arXiv:hep-th/0604119v2.
[35] J. B. Keller, Ann. Phys. (NY) **4**, 180 (1958); *Calculus of Variations and its Applications*, *Proceedings of Symposia in Applied Math.*, edited by L. M. Graves (McGraw-Hill, NY and Am. Math. Soc., Providence, 1958), p. 27.
[36] V. P. Maslov and V. S. Buslaev, *Théorie des Perturbations et Méthodes Asymptotiques* (Dunod, Paris, 1972); V. P. Maslov and M. V. Fedoriuk, *Semi-Classical Approximations in Quantum Mechanics* (Reidel, Dordrecht, 1981).
[37] A. Einstein, Verh. Dtsch. Phys. Ges. **19**, 82 (1917).
[38] M. V. Berry and M. Tabor, Proc. R. Soc. London A **349**, 101 (1976); J. Phys. A **10**, 371 (1977); Proc. R. Soc. London A **356**, 375 (1977).
[39] M. Brack and R. Bhaduri, *Semiclassical Physics* (Addison-Wesley, Reading, MA, 1997).
[40] J. S. Dowker, Ann. Phys. (NY) **62**, 361 (1971).
[41] P. Chang and J. S. Dowker, Nucl. Phys. B **395**, 407 (1993).
[42] J. S. Dowker, Class. Quant. Grav. **23**, 2771 (2006); J. Geom. Phys. **57**, 1505 (2007).
[43] G. Barton, J. Phys. A **34**, 4083 (2001).
[44] D. Deutsch and P. Candelas, Phys. Rev. D **20**, 3063 (1979); P. Candelas, Ann. Phys. (NY) **143**, 241 (1982).
[45] M. Schaden and L. Spruch, Ann. Phys. (NY) **313**, 37 (2004).
[46] A. Scardicchio and R. L. Jaffe, Nucl. Phys. B **704**, 552 (2005); **743**, 249 (2006).
[47] M. Levin, A. P. McCauley, A. W. Rodriguez, M. T. H. Reid, and S. G. Johnson, e-print arXiv:1003.3487.
[48] H. B. G. Casimir and D. Polder, Phys. Rev. **73**, 360 (1948).
[49] L. Spruch, in *Long-Range Casimir Forces: Theory and Recent Experiments in Atomic Systems*, edited by F. S. Levin and D. A. Micha (Plenum, New York, 1993).
[50] M. Schaden, Phys. Rev. A **79**, 052105 (2009).
[51] C. M. Bender and K. A. Milton, Phys. Rev. D **50**, 6547 (1994); K. A. Milton, *ibid.* **55**, 4940 (1997).
[52] A. A. Saharian, e-print IC/2000/14, hep-th/0002239.
[53] A. A. Saharian and A. S. Tarloyan, J. Phys. A **39**, 13371 (2006).
[54] M. Bordag, K. Kirsten, and D. Vassilevich, Phys. Rev. D **59**, 085011 (1999).
[55] N. Graham, R. L. Jaffe, V. Klemani, M. Quandt, M. Scandurra, and H. Weigel, Nucl. Phys. B **645**, 49 (2002); R. L. Jaffe, AIP Conf. Proc. **687**, 3 (2003).
[56] K. A. Milton, Phys. Rev. D **68**, 065020 (2003); J. Phys. A **37**, R209 (2004).
[57] I. Cavero-Pelaez, K. A. Milton, K. Kirsten, J. Phys. A **40**, 3607 (2007).
[58] T. P. Branson and P. B. Gilkey, Trans. Am. Math. Soc. **344**, 479 (1994).
[59] J. B. Keller, J. Opt. Soc. Am. **52**, 116 (1962); J. B. Keller and R. M. Lewis, *Asymptotic Theory of Wave Propagation and Diffraction* (unpublished); J. B. Keller, J. Appl. Phys. **30**, 1452 (1952); B. R. Levy and J. Keller, Commun. Pure Appl. Math. **12**, 159 (1959); H. M. Nussenzveig, Ann. Phys. **34**, 23 (1965); G. Vattay, A. Wirzba, and P. E. Rosenqvist, Phys. Rev. Lett. **73**, 2304 (1994); H. Primack, H. Schranz, U. Smilansky, and I. Ussishkin, J. Phys. A **30**, 6693 (1997).
[60] F. Bernasconi, G. M. Graf, and D. Hasler, Ann. Inst. H. Poincaré **4**, 1001 (2003); e-print arXiv:math-ph/0302035v1.
[61] I. Klich, Phys. Rev. D **61**, 025004 (1999).
[62] K. Kirsten, *Spectral Functions in Mathematics and Physics* (Chapman & Hall/CRC, Boca Raton, FL, 2002).
[63] J. B. Keller and S. I. Rubinow, Ann. Phys. (NY) **9**, 24 (1960).

[64] N. G. Van Kampen, B. R. A. Nijboer, and K. Schram, Phys. Lett. A **26**, 307 (1968).
[65] M. Bordag and I. G. Pirozhenko, Phys. Rev. D **64**, 025019 (2001).
[66] V. V. Nesterenko and I. G. Pirozhenko, Phys. Rev. D **60**, 125007 (1999).
[67] I. Cavero-Pelaez and K. A. Milton, Ann. Phys. (NY) **320**, 108 (2005); A. Romeo and K. A. Milton, Phys. Lett. B **621**, 309 (2005).
[68] I. Brevik, V. N. Marachevsky, and K. A. Milton, Phys. Rev. Lett. **82**, 3948 (1999).