Hysteresis effects in Bose-Einstein condensates

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Here, we consider damped two-component Bose-Einstein condensates with many-body interactions. We show that, when the external trapping potential has a double-well shape and when the nonlinear coupling factors are modulated in time, hysteresis effects may appear under some circumstances. Such hysteresis phenomena are a result of the joint contribution of the appearance of saddle node bifurcations and the damping effect.

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Atomic Bose-Einstein condensates (BECs) at zero temperature are described by means of nonlinear Schrödinger equations of the type

$$i\hbar\frac{\partial\psi}{\partial t} = H_0\psi + g_2|\psi|^2\psi + g_3|\psi|^4\psi + \dots, \qquad (1)$$

where H_0 represents the Hamiltonian of a single trapped atom and the nonlinear term $|\psi|^{2r}$, $r = 1,2,\dots$, is the $(r+1)$-body contact potential with coupling factor g_{r+1} [1]. In fact, BECs strongly depend on interatomic forces and the binary coupling term $g_2|\psi|^2\psi$ usually represents the dominant nonlinear term; when the higher nonlinear terms are neglected, then Eq. (1) takes the form of the well-known Gross-Pitaevskii equation [2]. The coupling factor of the binary nonlinear term is given by $g_2 = \mathcal{N}2\pi\hbar^2 a/m$, where m is the mass of the atoms, $\mathcal{N}$ is the total number of particles of the condensate, and a is the scattering length; for higher nonlinearity terms some expressions of the coupling factor have been recently proposed [3]. Recent experiments [4] have shown that the scattering length a can be changed, and, in fact, the two-body coupling factor g_2 can be tuned to be zero in the case of polar molecules in optical lattices driven by microwave fields [5]. In such a case, the three-body and, in general, the $(r+1)$-body interaction become significant [6,7] and thus the Gross-Pitaevskii equation becomes inadequate to describe BECs.

The basic properties of BECs with many-body interactions described by Eq. (1), where many nonlinear terms are simultaneously considered, are far from being well understood. In order to understand some fundamental features of BECs with many-body interactions, in this article we follow the approach proposed by Ref. [8]; that is, we are interested in the $(r+1)$-body interaction on its own. Therefore, we restrict ourselves to the basic nonlinear Schrödinger equation representing the properties of exactly $(r+1)$-body contact interaction of BECs at zero temperature:

$$i\hbar\frac{\partial\psi}{\partial t} = H_0\psi + g_{r+1}|\psi|^{2r}\psi, \quad \|\psi\| = 1, \qquad (2)$$

which depends on the Hamiltonian,

$$H_0 = -\frac{\hbar^2}{2m}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} + V(x),$$

of a single atom in dimension n with trapping potential $V(x)$, as well as on the $(r+1)$-body coupling factor g_{r+1}.

It is worth mentioning also the fact that Eq. (2) with nonlinearity corresponding to the power-law $|\psi|^{2r}$, where the parameter r takes any positive real value, is used in other contexts, including semiconductors [9] and nonlinear optics [10,11]. Actually, for Kerr media the refractive index varies with the square of field amplitude; in such a case the nonlinear Helmholtz equation has the form of Eq. (2) with $r = 1$, and the generalized form with $r \neq 1$ is a model for saturable media [12]. Furthermore, even if in most of the applications the parameter r takes only integer and positive values, here we take that r can assume noninteger values too, as considered in Ref. [13].

Since the wave function ψ is assumed to be normalized to 1, then the coupling factor g_{r+1} depends, in addition to the physical parameters of the problem such as the scattering length and the mass of the particles of the condensate, on the total number $\mathcal{N}$ of the particles of the condensate.

Recent experiments have shown that the total number $\mathcal{N}$ of the particles of the condensate can be adiabatically modulated in time by means of a suitable time-dependent combination of optical and magnetic forces [14,15]. Thus, we can consider the case where the nonlinear coupling factor in Eq. (2) is a given function which slowly depends on the time t:

$$g_{r+1} := g_{r+1}(t).$$

Another way to produce a time-dependent coupling factor consists of tuning the scattering length [16]. In a previous theoretical article by Pelinovsky, Kevrekidis, and Frantzeskakis [17], the Gross-Pitaevskii equation with a periodically varying nonlinearity coupling factor has been considered and it has shown a good agreement between solutions of the averaged and full equations.

In this article we show that the modulation of the nonlinearity coupling factor may give rise to hysteresis phenomenon for two-component BECs, where the the external trapping potential $V(x)$ has a double-well shape [18]. In fact, hysteresis effects have already been seen in rotating BECs; in particular, the number of vortices appearing in a rotating BEC depends on the rotation history of the trap, in addition to the number of vortices initially present in the condensate [19] (see also the theoretical analysis in Ref. [20]).

In particular, here we show that, in the semiclassical limit of sufficiently small $\hbar$, for the BEC equation, Eq. (2), with power law $|\psi|^{2r}$ in a double-well trapping potential and with a slow

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modulation, with respect to the beating period between the two wells, of the nonlinear coupling factor $g_{r+1}(t)$, a hysteresis effect appears provided that r is bigger than the critical value:

$$r_{\text{threshold}} = (3 + \sqrt{13})/2. \tag{3}$$

It is worth mentioning the fact that this result holds true for both attractive (e.g., $g_{r+1} < 0$) and repulsive (e.g., $g_{r+1} > 0$) nonlinearities; however, just for argument's sake, we restrict ourselves to the attractive case [21].

The hysteresis effect is strictly close to the appearance of the spontaneous symmetry breaking phenomenon (SSBP) related to saddle point nodes. In fact, for BECs with the $(r+1)$-body interaction governed by Eq. (2) it has been recently seen [22] that the SSBP appears when the nonlinearity power r is bigger than $r_{\text{threshold}}$. It is worth mentioning that the SSBP is a rather important effect that arises in a wide range of physical systems modeled by nonlinear equations [23]. We would also mention that hysteresis effects associated with bifurcations of stationary solutions are theoretically discussed for BECs in optical lattices under the effect of a Stark-like external field [24].

The n-dimensional linear Schrödinger equation with a symmetric double-well potential has stationary states of a definite even-parity φ_+ and a definite odd-parity φ_-, with associated nondegenerate eigenvalues $\lambda_+ < \lambda_-$. However, the introduction of a nonlinear term, which in quantum mechanics usually models an interacting many-particle system, may give rise to asymmetrical states related to the SSBP.

In the semiclassical limit, it has been proven that the symmetric stable stationary state bifurcates when the adimensional nonlinear parameter η takes an absolute value equal to the critical value:

$$\eta^* = 2^r/r. \tag{4}$$

The parameter η is associated with the coupling factor of the nonlinear perturbation by

$$\eta := \eta(t) = c g_{r+1}(t)/\omega, \tag{5}$$

and it is the adimensional effective nonlinear coupling factor, where ω is the (half of the) splitting between the two levels,

$$\omega = \tfrac{1}{2}(\lambda_- - \lambda_+), \tag{6}$$

and c is the constant given by

$$c = \langle \varphi_R, |\varphi_R|^{2r} \varphi_R \rangle = \langle \varphi_L, |\varphi_L|^{2r} \varphi_L \rangle \ .$$

Here, φ_R and φ_L are the normalized right- and left-hand-side vectors

$$\varphi_R = (\varphi_+ + \varphi_-)/\sqrt{2}$$

and

$$\varphi_L = (\varphi_+ - \varphi_-)/\sqrt{2},$$

usually named *single-well states* because they are localized on only one well. In fact, in the semiclassical limit (or also for large distance between the two wells) the splitting ω is exponentially small, as $\hbar$ goes to zero, and the supports of the two vectors φ_R and φ_L don't overlap up to an exponentially small term.

By adopting the two-level approximation, the wave function $\psi(x,t)$ is a linear combination of the right- and left-hand-side vectors,

$$\psi(x,t) = a_R(t)\varphi_R(x) + a_L(t)\varphi_L(x),$$

where we set

$$a_R = p e^{i\alpha}, \quad a_L = q e^{i\beta}, \quad p^2 + q^2 = 1.$$

Defining the relative phase difference $\theta = \alpha - \beta$ and the adimensional imbalance function $z = p^2 - q^2$, and rescaling the time as $\tau = \omega t/\hbar$ (hence, the linear beating period takes the value π), Eq. (2) can be written in the Hamiltonian form

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial \mathcal{H}}{\partial z} \quad \text{and} \quad \frac{\partial z}{\partial \tau} = -\frac{\partial \mathcal{H}}{\partial \theta},$$

with the Hamiltonian function

$$\mathcal{H} = 2\sqrt{1-z^2}\cos\theta - \eta \frac{(1+z)^{r+1} + (1-z)^{r+1}}{2^r(r+1)}.$$

The energy functional $\mathcal{E}$ associated with the nonlinear Schrödinger equation, Eq. (2), and written in the two-level approximation takes the form $\mathcal{E} = \Omega - \frac{1}{2}\omega\mathcal{H}$, where $\Omega = \frac{1}{2}(\lambda_- + \lambda_+)$ is the mean value between the two energy levels.

We consider at first the case $r \leqslant r_{\text{threshold}}$. Since η takes negative values, then the nonlinear ground state is a stable symmetric state for any $|\eta| < \eta^*$. At $|\eta| = \eta^*$ it bifurcates and we observe also an exchange of the stability properties: for $|\eta|$ larger than η^* the symmetric stationary state becomes unstable and the new asymmetrical states are stable [see Fig. 1(a)].

On the other side, for $r > r_{\text{threshold}}$, then a couple of saddle-node bifurcations, associated with the new asymmetrical stationary states, sharply appear when $|\eta|$ is equal to a given value η^+ such that $\eta^+ < \eta^*$ [25]; then, for increasing values of $|\eta|$, the two unstable solutions disappear at $|\eta| = \eta^*$ showing a subcritical pitch-fork bifurcation [see Fig. 2(a)].

In such a scenario, that is, the sharp appearance of new asymmetrical stationary solutions fully localized on a single well when $r > r_{\text{threshold}}$, a new relevant effect occurs: namely, we expect to observe a hysteresis effect when we adiabatically change the effective coupling factor η such that its absolute value moves from values less than η^* to values bigger than η^* and then it goes back to its initial value. In particular, we assume that the variation of $\eta(t)$ is slow with respect to the beating period $\pi\hbar/\omega$, which is the period of the functions $z(t)$ and $\theta(t)$ when the nonlinear term is absent. To this end we consider a state that, in the (z,θ) representation, is initially close to the symmetric stationary state: that is, its initial condition corresponds to $z_0 \approx 0$ and $\theta_0 \approx 0$.

In the case $r \leqslant r_{\text{threshold}}$, as $|\eta|$ increases, the state remains close to the symmetric stationary state for any $|\eta| < \eta^*$; at $|\eta| = \eta^*$ it experiences a bifurcation and it follows one of two branches for $|\eta| > \eta^*$. When $|\eta|$ decreases from values bigger than η^* to values less than η^* such a path is reversed and the state returns close to the initial symmetric stationary state when $|\eta|$ returns to its initial value such that $|\eta| < \eta^*$ (see the path indicated by the arrows in Fig. 1(a)).

On the other side, if $r > r_{\text{threshold}}$, the state still remains close to the initial stable stationary state (z_0,θ_0) for any

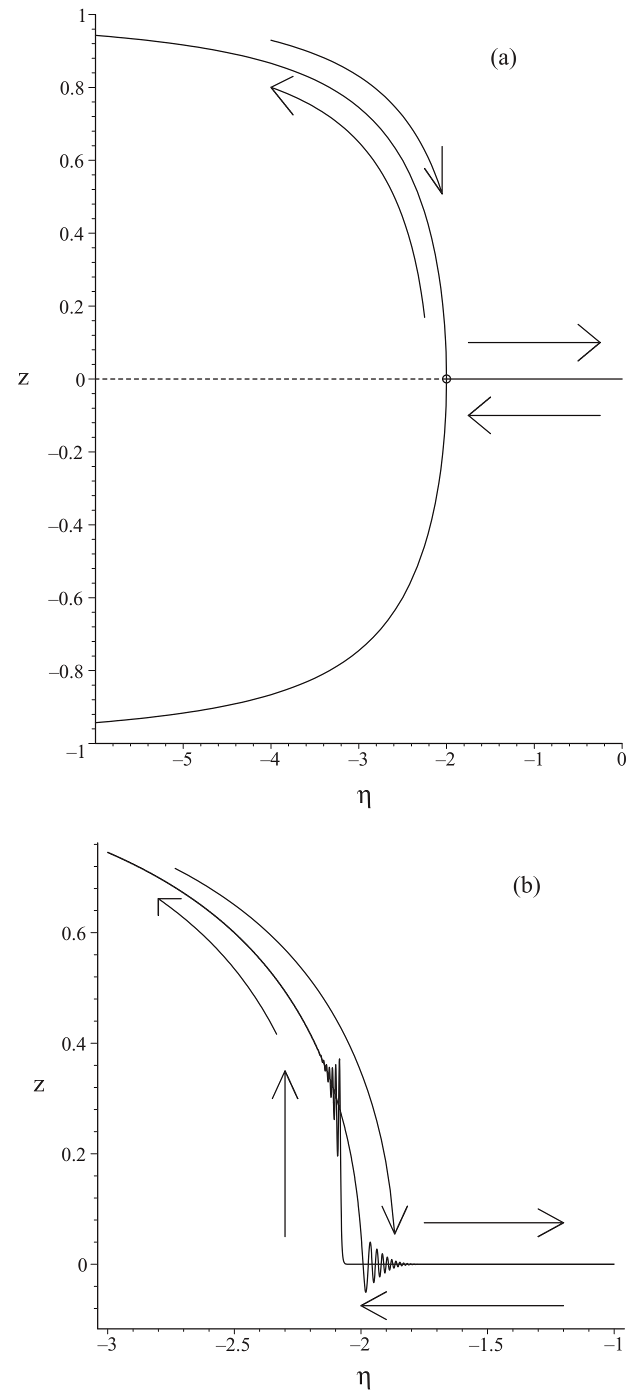

FIG. 1. In this figure we consider the case $r = 1 < r_{\mathrm{threshold}}$. In panel (a) we plot the bifurcation diagram where solid lines represents stable stationary solutions and dotted lines represent unstable states. The arrows represent the "path" of a solution initially close to the symmetric ground state. A state initially close to the stable symmetric stationary solution experiences a bifurcation effect at $|\eta| = \eta^* = 2$ and it follows the new asymmetric stable stationary solution on one of the branches for $|\eta| > \eta^*$. When $|\eta|$ moves from values bigger than η^* to values less than η^*, the state returns to close to the stationary symmetric state without exhibiting the hysteresis phenomenon. In panel (b) we show the numerical solution of Eqs. (7) and (8) for $r = 1$, $\nu = 0.5$, $z_0 = 0.01$, and $\theta_0 = 0$. Here, η is the adimensional effective nonlinear coupling factor and z is the adimensional imbalance function.

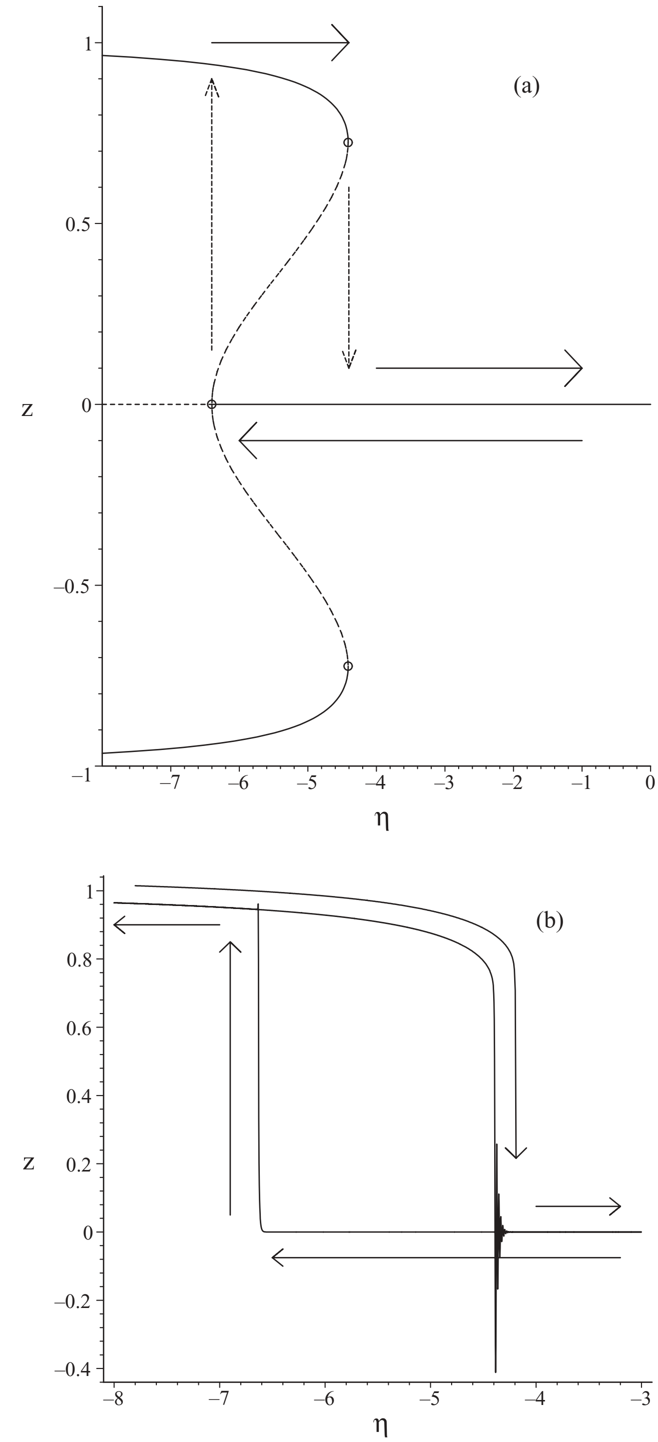

FIG. 2. In this figure we consider the case $r = 5 > r_{\mathrm{threshold}}$. A state initially close to the stable symmetric stationary solution jumps to the new asymmetric stable stationary solution on one of the branches for $|\eta| > \eta^* = 6.4$. When $|\eta|$ moves from values bigger than η^* to values less than η^* the state does not return close to the stationary state, but it exhibits a hysteresis phenomena for $|\eta|$ between η^* and η^+. Broken arrows are associated with the damping effect which forces the state to collapse on the ground state after some damped oscillations. In panel (b) we show the numerical solution of Eqs. (7) and (8) for $r = 5$, $\nu = 0.5$, $z_0 = 0.01$, and $\theta_0 = 0$. Here, η is the adimensional effective nonlinear coupling factor and z is the adimensional imbalance function.

$|\eta| < \eta^*$, but at $|\eta| = \eta^*$ we don't have a smooth bifurcation and for $|\eta| > \eta^*$ the state starts to oscillate around the stable asymmetric stationary solution localized on only one of the two wells. As $|\eta|$ decreases from values bigger than η^* to values between η^+ and η^*, the previous path is not reversed. In fact, the state continues to oscillate around the stable asymmetric stationary solution until $|\eta|$ reaches the value η^+. Then, while η is returning to its initial value, it takes the value $|\eta| = \eta^+$, for which the asymmetrical stable stationary states disappear, and the wave function starts to exhibit a wide oscillating motion around the symmetrical stationary solution corresponding to $z = 0$. If we introduce a small damping effect, then such oscillating motions are damped and the state will stay close to the symmetric stationary solution. In conclusion we can see in such a scenario that a hysteresis effect appears for values of $|\eta|$ between η^+ and η^* (see the path indicated by arrows in Fig. 2(a)).

In fact, such a hysteresis effect becomes more evident by adding a damping term which forces the state to collapse to the ground state. Actually, in physical systems we should expect to take into account a certain amount of damping due to the incoherent exchange of normal atoms. In particular, an accepted model for damped two-component BECs has been introduced in Ref. [26] and it reads as

$$\frac{\partial z}{\partial \tau} = -\frac{\partial \mathcal{H}}{\partial \theta} - \nu \frac{\partial \theta}{\partial \tau} := -\sqrt{1 - z^2}\sin\theta - \nu\frac{\partial \theta}{\partial \tau}, \qquad (7)$$

where $\nu > 0$ is the damping constant and

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial \mathcal{H}}{\partial z} := -\frac{2z\cos\theta}{\sqrt{1-z^2}} - \frac{\eta}{2^r}[(1+z)^r - (1-z)^r]. \qquad (8)$$

In Fig. 1(b) (for $r = 1$) and Fig. 2(b) (for $r = 5$), we plot the numerical solutions of this dynamical system, where we assume the initial condition $\theta_0 = 0$ and $z_0 = 0.01$ closed to the symmetric stationary solution, and for times $\tau \in [0,T]$, where $T = 4000$. The damping factor is chosen to be $\nu = 0.5$ and the time-dependent function η has the following forms:

$$\eta(\tau) = -1 - 2[1 - |2\tau/T - 1|], \quad \text{if} \quad r = 1,$$

and

$$\eta(\tau) = -3 - 5[1 - |2\tau/T - 1|], \quad \text{if} \quad r = 5.$$

As predicted by means of the previous analysis on the bifurcation of the stationary solutions, it appears that for $r = 1$ no hysteresis effect occurs; while for $r = 5$ the hysteresis effect occurs for $|\eta|$ between η^+ and η^*. Oscillations of the state that occurs when $|\eta|$ becomes less than η^* are damped because of the damping factor. It is worth mentioning also that the delay observed in the case $r = 1$, when the absolute value of η becomes larger than the branch point η^*, is not a consequence of some hidden physical effects but rather it comes from the singularity associated with the branch point and from the variation of $\eta(t)$. In fact, for larger values of T, η is almost constant around the branch point and this delay disappears.

In summary, we have shown that in a damped BEC with a double-well trapping potential the nonlinear term coming from an $(r+1)$-body interaction, for r bigger than 4, may give rise to a hysteresis effect when the corresponding coupling factor adiabatically changes. The modulation of the coupling factor can be performed by tuning the condensate population by means of suitable external fields. Such a hysteresis effect cannot be theoretically predicted by means of the well-known Gross-Pitaevskii equation, in which only binary contact potentials are considered.

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