# Atomic-number fluctuations in a mixture of condensates

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We study particle number fluctuations in the quantum ground states of a mixture of two spin-1 atomic condensates when the interspecies spin-exchange coupling interaction $c_{12}\beta$ is adjusted. The two spin-1 condensates forming the mixture are, respectively, ferromagnetic and polar in the absence of an external magnetic (B) field. We categorize all possible ground states using the angular momentum algebra and compute their characteristic atom number fluctuations, focusing especially on the the AA phase (when $c_{12}\beta > 0$), where the ground state becomes fragmented and atomic-number fluctuations exhibit drastically different features from a single standalone spin-1 polar condensate. Our results are further supported by numerical simulations of the full quantum many-body system.

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## I. INTRODUCTION

Since the first production of an atomic $^{23}$Na condensate in an optical trap [1], spin degrees of freedom for condensed atoms become accessible, which has since given rise to a rich variety of phenomena such as domain formations [2], spin-mixing dynamics [3], topological defects [4], etc. The properties of a three-component ($F = 1$) spinor condensate are first studied by Ho [5] and Ohmi [6]. Many predictions are experimentally verified [2]; the most fundamental property concerns the existence of two different phases: the so-called polar and ferromagnetic states, respectively, corresponding to the $F = 1$ state of $^{23}$Na and $^{87}$Rb atomic condensates.

In contrast to a scalar condensate, both the spatial and internal spin part of the wave functions are required to discuss a spinor condensate. For both $F = 1$ states of $^{23}$Na and $^{87}$Rb atoms, density-density interactions are significantly larger than the spin-exchange interactions. The single spatial-mode approximation (SMA) [5–9] is often adopted, whereby one adopts mean-field approximation (MFT) to determine the condensate spatial wave function neglecting their spin dependence. The spin degrees of freedom is then considered assuming the spatial wave function is identical. The spin-related properties can be investigated using either a "mean-field" theory or "semiclassical treatment" [5,6], where the spinor is described by a vector formed by three $c$ numbers, $\zeta_\alpha(\zeta_\alpha^*)$ $(\alpha = 1,0,-1)$, or using many-body theory [7,8] treating the spinor from the bosonic mode as operators $\hat{a}_\alpha$, satisfying $[\hat{a}_\alpha,\hat{a}_\beta^\dagger] = \delta_{\alpha\beta}$. The quantum ground states for a spin-1 condensate have been studied extensively.

It was initially predicted that the ground state of $^{23}$Na Bose-Einstein condensate (BEC) ($c_2 > 0$) is either polar ($n_0 = N$) or antiferromagnetic ($n_1 = n_{-1} = N/2$) in the mean-field theory [5,6]. A quantum treatment based on the SMA by Law *et al.* and Pu *et al.* [7,8] revealed, however, the ground state for $^{23}$Na atoms is actually a spin singlet with properties ($n_1 = n_0 = n_{-1} = N/3$), drastically different from the polar state predicted within the mean-field theory. Further studies pointed out that this spin-singlet state is a fragmented condensate with anomalously large number fluctuations and thus has fragile stability [10,11]. The remarkable nature of this fragmentation is characterized by three macroscopic eigenvalues (see previously mentioned) for the single-particle reduced density matrix, which is capable of exhibiting anomalously large atom number fluctuations $\Delta n_{1,0,-1} \sim N$.

The interests in spinor condensates extend to higher spins [12,13] and quantum ground states are already well known and categorized for spin-2 condensates [14]. Atomic Feshbach resonance was implemented in a double condensate, enabling tunable interactions, whose effects on superfluid dynamics and controlled phase separations are observed [15]. The spin-exchange interaction between individual atoms can be precisely tuned through *optical Feshbach resonances* [16] by adjusting the two s-wave scattering lengths $a_0$ and $a_2$. This inspired several recent theoretical studies on mixtures of spinor condensates [17–20] and tunable or controlled dynamics [21].

In this paper we report anomalous fluctuations for the numbers of atoms in a binary mixture of spin-1 condensates. We hope to stimulate experiments, using the most relevant experimental case, the mixture of $^{23}$Na (polar) and $^{87}$Rb (ferromagnetic) condensates in their $F = 1$ manifold, as an example. The quantum spin properties in the special ground state of the AA phase, where the interspecies antiferromagnetic spin exchange is large enough to polarize both species but forming a maximally entangled state between two species, are studied and we give the exact number-fluctuation distributions. Then we resort to numerical diagonalizations to show that particle numbers and number fluctuations undergo dramatic changes as interspecies coupling $c_{12}\beta$ varies.

## II. THE MODEL HAMILTONIAN FOR THE MIXTURE

Intracondensate atomic interaction takes the form $V_j(\mathbf{r}) = (\alpha_j + \beta_j \mathbf{F}_j \cdot \mathbf{F}_j)\delta(\mathbf{r})$ with $j = 1,2$ for the ferromagnetic ($\beta_1 < 0$) and polar ($\beta_2 > 0$) atoms. The interspecies interaction between the ferromagnetic and polar atoms is described as $V_{12}(\mathbf{r}) = \frac{1}{2}(\alpha + \beta \mathbf{F}_1 \cdot \mathbf{F}_2 + \gamma P_0)\delta(\mathbf{r})$ [17], which is more complicated because collision can occur in the total spin $F_{\text{tot}} = 1$ channel between different atoms, in contrast to intracondensate interactions between identical atoms [17,22]. The parameters $\alpha, \beta$, and $\gamma$ are related to the $s$-wave scattering

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lengths in the the total spin channels [17], analogous to spin-1 condensates [5,6]. $P_0$ projects an interspecies pair into the spin-singlet state and $\mu = M_1M_2/(M_1+M_2)$ denotes the reduced mass for the pair of atoms, one each from the two different species with masses $M_1$ and $M_2$, respectively.

Our model Hamiltonian is given by

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_{12}, \tag{1}$$

$$\hat{H}_1 = \int d\mathbf{r}\left\{\hat{\Psi}_i^\dagger\left(\frac{\hbar^2}{2M_1}\nabla^2 + U_1\right)\hat{\Psi}_i + \frac{\alpha_1}{2}\hat{\Psi}_i^\dagger\hat{\Psi}_j^\dagger\hat{\Psi}_j\hat{\Psi}_i + \frac{\beta_1}{2}\hat{\Psi}_i^\dagger\hat{\Psi}_j^\dagger\mathbf{F}_{1il}\cdot\mathbf{F}_{1jk}\hat{\Psi}_k\hat{\Psi}_l\right\},$$

$$\hat{H}_{12} = \frac{1}{2}\int d\mathbf{r}\left\{\alpha\hat{\Psi}_i^\dagger\hat{\Phi}_j^\dagger\hat{\Phi}_j\hat{\Psi}_i + \beta\hat{\Psi}_i^\dagger\hat{\Phi}_j^\dagger\mathbf{F}_{1il}\cdot\mathbf{F}_{2jk}\hat{\Phi}_k\hat{\Psi}_l + \frac{\gamma}{3}\hat{O}^\dagger\hat{O}\right\}. \tag{2}$$

$H_2$ is identical to $H_1$ except for the substitution of subscript 1 by 2 and $\hat{\Psi}_i$ by $\hat{\Phi}_i$. The latter two are atomic field operators for the spin state $|1,i\rangle$. $\hat{O} = \hat{\Psi}_1\hat{\Phi}_{-1} - \hat{\Psi}_0\hat{\Phi}_0 + \hat{\Psi}_{-1}\hat{\Phi}_1$.

We adopt the SMA [7–9] for each of the two spinor condensates with modes $\Psi(\mathbf{r})$ and $\Phi(\mathbf{r})$, that is, setting

$$\hat{\Psi}_i = \hat{a}_i\Psi, \qquad \hat{\Phi}_i = \hat{b}_i\Phi,$$

with $\hat{a}_i$ ($\hat{b}_i$) the annihilation operator for the ferromagnetic (polar) atoms satisfying $[\hat{a}_i,\hat{a}_j] = 0$ and $[\hat{a}_i,\hat{a}_j^\dagger] = \delta_{ij}$ (and the same form of commutations for $\hat{b}_i$). The spin-dependent Hamiltonian for our mixture mode then reads

$$\hat{H} = \frac{c_1\beta_1}{2}\hat{\mathbf{F}}_1^2 + \frac{c_2\beta_2}{2}\hat{\mathbf{F}}_2^2 + \frac{c_{12}\beta}{2}\hat{\mathbf{F}}_1\cdot\hat{\mathbf{F}}_2 + \frac{c_{12}\gamma}{6}\hat{\Theta}_{12}^\dagger\hat{\Theta}_{12}, \tag{3}$$

where $\hat{\mathbf{F}}_1 = \hat{a}_i^\dagger\mathbf{F}_{1ij}\hat{a}_j(\hat{\mathbf{F}}_2 = \hat{b}_i^\dagger\mathbf{F}_{2ij}\hat{b}_j)$ are defined in terms of the $3\times 3$ spin-1 matrices $\mathbf{F}_{1ij}$ ($\mathbf{F}_{2ij}$), and

$$\hat{\Theta}_{12}^\dagger = \hat{a}_0^\dagger\hat{b}_0^\dagger - \hat{a}_1^\dagger\hat{b}_{-1}^\dagger - \hat{a}_{-1}^\dagger\hat{b}_1^\dagger$$

creates a singlet pair with one atom each from the two species, similar to the following two:

$$\hat{A}^\dagger = (\hat{a}_0^\dagger)^2 - 2\hat{a}_1^\dagger\hat{a}_{-1}^\dagger, \quad \hat{B}^\dagger = (\hat{b}_0^\dagger)^2 - 2\hat{b}_1^\dagger\hat{b}_{-1}^\dagger,$$

for intraspecies spin-singlet pairs [10]. The interaction coefficients are $c_1 = \int d\mathbf{r}|\Psi(r)|^4$, $c_2 = \int d\mathbf{r}|\Phi(r)|^4$, and $c_{12} = \int d\mathbf{r}|\Psi(r)|^2|\Phi(r)|^2$, which can be tuned through the control of the trapping frequency. Here we focus on the spin-dependent part when we assume that the two species are sufficiently overlapped. The scattering properties between any pairs of atoms in specific Zeeman hyperfine component states are determined from their corresponding interaction potentials observing the symmetries of the two states. We notice that under the so-called degenerate internal-state approximation (DIA) the $\gamma$ term vanishes [23]. Within the DIA, the Zeeman hyperfine state of an alkali-metal atom is expanded in terms of the electronic (valence electron) and its corresponding nuclear spin states. Between any two atoms, the corresponding interaction potential is constructed from the appropriate weighted spin-singlet or triplet potentials when expanded out correspondingly. When the spin-singlet and triplet potentials are further approximated by low-energy scattering pseudopotentials, the total scattering properties are determined completely by two scattering lengths of the singlet and triplet potentials. The parameter $\gamma$ is therefore no longer needed and the Hamiltonian (8) finally reduces to

$$\hat{H}_A = \frac{c_1\beta_1}{2}\hat{\mathbf{F}}_1^2 + \frac{c_2\beta_2}{2}\hat{\mathbf{F}}_2^2 + \frac{c_{12}\beta}{2}\hat{\mathbf{F}}_1\cdot\hat{\mathbf{F}}_2. \tag{4}$$

## III. THE GROUND STATES

We rewrite the Hamiltonian (4) as

$$\hat{H} = a\hat{\mathbf{F}}_1^2 + b\hat{\mathbf{F}}_2^2 + c\hat{\mathbf{F}}^2, \tag{5}$$

where $a = c_1\beta_1/2 - c_{12}\beta/4$, $b = c_2\beta_2/2 - c_{12}\beta/4$, and $c = c_{12}\beta/4$, $\hat{\mathbf{F}} = \hat{\mathbf{F}}_1 + \hat{\mathbf{F}}_2$ is the total spin operator. The eigenstates of (5) are the common eigenstates for the commuting operators $\hat{\mathbf{F}}_1^2, \hat{\mathbf{F}}_2^2, \hat{\mathbf{F}}^2$, and $\hat{F}_z$, given by

$$|F_1,F_2,F,m\rangle = \sum_{m_1m_2} C_{F_1,m_1;F_2,m_2}^{F,m}|F_1,m_1\rangle|F_2,m_2\rangle,$$

with the uncoupled basis states,

$$|F_1,m_1\rangle = Z_1^{-\frac{1}{2}}(\hat{F}_{1-})^{F_1-m_1}(\hat{a}_1^\dagger)^{F_1}(\hat{A}^\dagger)^{(N_1-F_1)/2}|0\rangle,$$

and analogously for $|F_2,m_2\rangle$ which span a Hilbert space of dimension $(N_j+1)(N_j+2)/2$ [14]. $C$ is the Clebsch-Gordon coefficient, $Z_j$ is a normalization constant, and $\hat{F}_{j-}$ is the lowering operator for $m_j$. The corresponding eigenenergy is

$$E = aF_1(F_1+1) + bF_2(F_2+1) + cF(F+1). \tag{6}$$

Given $N_j$, the allowed values of $F_j$ are $F_j = 0,2,4,\ldots,N_j$ if $N_j$ is even and $F_j = 1,3,5,\ldots,N_j$ if $N_j$ is odd, satisfying $|F_1 - F_2| \leqslant F \leqslant F_1 + F_2$.

We next consider the special case of $N_1 = N_2 = N$ and for $N$ even. Minimizing the energy (6), we can get the ground-state phases determined by different parameters $a$, $b$, and $c$, or equivalently $c_1\beta_1$, $c_2\beta_2$, and $c_{12}\beta$. Taking $N = 100$, for example, Fig. 1 presents the results for the order parameters $\langle\hat{\mathbf{F}}_1^2\rangle$, $\langle\hat{\mathbf{F}}_2^2\rangle$, and $\langle\hat{\mathbf{F}}_1\cdot\hat{\mathbf{F}}_2\rangle = (\langle\hat{\mathbf{F}}^2\rangle - \langle\hat{\mathbf{F}}_1^2\rangle - \langle\hat{\mathbf{F}}_2^2\rangle)/2$ [17] as $c_{12}\beta$ changes but fixed $c_1\beta_1 = -1$, $c_2\beta_2 = 2$ (in units of $|c_1\beta_1|$), which are found to agree very well with the mean-field ones obtained from simulated annealing [17].

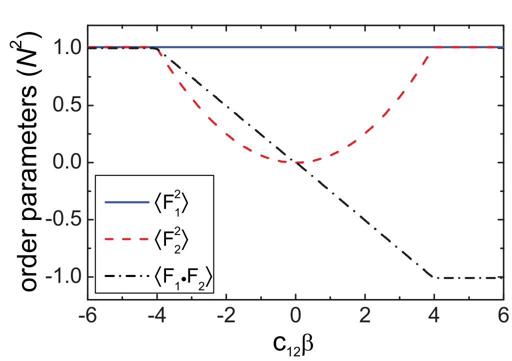


FIG. 1. (Color online) The dependence of ground-state phases on $\beta$ at fixed values of $c_1\beta_1 = -1$, $c_2\beta_2 = 2$. All interaction parameters are in units of $|c_1\beta_1|$.

TABLE I. Quantum phases for the ground state.

| Phases | Parameterrange | Groundstates$\|F_1,F_2,F,F\rangle$ |
|---|---|---|
| $FF$ | $-\infty < c_{12}\beta < \frac{-(2N-1)}{N}c_2\beta_2$ | $C^{2N,2N}_{N,N;N,N}\|N,N\rangle\|N,N\rangle$ |
| $MM_-$ | $\frac{-(2N-1)}{N}c_2\beta_2 < c_{12}\beta < 0$ | $C^{N+F_2,N+F_2}_{N,N;F_2,F_2}\|N,N\rangle\|F_2,F_2\rangle$ |
| $MM_+$ | $0 < c_{12}\beta < \frac{2N-1}{N+1}c_2\beta_2$ | $\sum_{m_1,m_2} C^{N-F_2,N-F_2}_{N,m_1;F_2,m_2}\|N,m_1\rangle\|F_2,m_2\rangle$ |
| $AA$ | $\frac{2N-1}{N+1}c_2\beta_2 < c_{12}\beta < +\infty$ | $\sum_{m_1} C^{0,0}_{N,m_1;N,-m_1}\|N,m_1\rangle\|N,-m_1\rangle$ |

Overall, there are four different phases in the general case, separated by three critical points: $-(2N-1)c_2\beta_2/N$, 0, and $(2N-1)c_2\beta_2/(N+1)$ (corresponding to $-4$, 0, and 4 in Fig. 1). The extreme states $|F_1,F_2,F,F\rangle$ are classified into FF, MM$_-$, MM$_+$, and AA phases as in Table I. Other degenerate states are found by repeated applications of $\hat{F}_{1-}+\hat{F}_{2-}$,

$$|F_1,F_2,F,m\rangle = (\hat{F}_{1-}+\hat{F}_{2-})^{F-m}\,|F_1,F_2,F,F\rangle, \qquad (7)$$

with $m = 0, \pm 1, \ldots \pm F$.

The ground state for a spin-1 polar condensate is fragmented [10], described by a spin-singlet state of the form $(\hat{B}^\dagger)^{N/2}|0\rangle$. $\hat{B}^\dagger$ ($\hat{B}$) is invariant under rotations and commutes with $\hat{F}_2$ and $\hat{F}_{2z}$. For ferromagnetic condensate, the condensate ground state favors all atoms aligned along the same direction [i.e., takes the form $(\hat{a}_1^\dagger)^N|0\rangle$], and is more stable. When mixing the two together, we expect the polar atoms are strongly affected, while the back action on to the more stable ferromagnetic atoms is negligible.

The FF phase may be simply described as $Z^{1/2}(\hat{a}_1^\dagger)^N(\hat{b}_1^\dagger)^N|0\rangle$, with all polar atoms slaved into the same direction as the ferromagnetic ones. For $-(2N-1)c_2\beta_2/N < c_{12}\beta < 0$, the MM$_-$ phase arises when polar atoms are partly polarized in the same direction with the ferromagnetic ones, as if there were a finite B field. Increase of the coupling interaction ($|c_{12}\beta|$) breaks singlet pairs in polar atoms one by one with $\Delta F_2 = 2$, and gives rise to stepwise increases of the total spin. Similar to the FF phase, the MM$_-$ phase has a simpler form, $Z^{1/2}(\hat{a}_1^\dagger)^N(\hat{b}_1^\dagger)^{F_2}(\hat{B}^\dagger)^{(N-F_2)/2}|0\rangle$, with its energy $E = c_1\beta_1 N(N+1)/2 + c_2\beta_2 F_2(F_2+1)/2 + c_{12}\beta N F_2/2$. This phase corresponds to the exact ground state for $F_2 - 1 < -c_{12}\beta N/2c_2\beta_2 - 1/2 < F_2+1$ [14], and we find the order parameters,

$$\begin{aligned}\langle \hat{F}_1^2\rangle &= N(N+1),\\ \langle \hat{F}_2^2\rangle &= \left(\frac{N}{2c_2\beta_2}\right)^2 (c_{12}\beta)^2 - \frac{1}{4},\\ \langle \hat{F}_1\cdot\hat{F}_2\rangle &= -\frac{N^2}{2c_2\beta_2}c_{12}\beta - \frac{N}{2},\end{aligned} \qquad (8)$$

agree with the mean-field results [17] to terms $\sim 1/N$.

For $0 < c_{12}\beta < (2N-1)c_2\beta_2/(N+1)$, however, the MM$_+$ phase favors polar atoms polarized opposite to the ferromagnetic atoms resulting in a decreased total spin. This situation is more complicated because all states satisfying $m_1 + m_2 = N - F_2$ are involved. The ground-state energy is $E = c_1\beta_1 N(N+1)/2 + c_2\beta_2 F_2(F_2+1)/2 - c_{12}\beta(N+1)F_2/2$, and we find that

$$\begin{aligned}\langle \hat{F}_1^2\rangle &= N(N+1),\\ \langle \hat{F}_2^2\rangle &= \left(\frac{N+1}{2c_2\beta_2}\right)^2 (c_{12}\beta)^2 - \frac{1}{4},\\ \langle \hat{F}_1\cdot\hat{F}_2\rangle &= -\frac{(N+1)^2}{2c_2\beta_2}c_{12}\beta - \frac{N+1}{2}.\end{aligned} \qquad (9)$$

The stepwise fine structure of the order parameters is not included in Eqs. (8) and (9). When $c_{12}\beta > (2N-1)c_2\beta_2/(N+1)$, again all states satisfying $m_1+m_2 = 0$ are included and the AA phase is described by a singlet state [18],

$$|N,N,0,0\rangle = \sum_{m_1=-N}^{N} C^{0,0}_{N,m_1;N,-m_1}|N,m_1\rangle|N,-m_1\rangle. \qquad (10)$$

We find interestingly that the total spin vanishes, while the species spins satisfy $\langle \hat{F}_1^2\rangle = \langle \hat{F}_2^2\rangle = N(N+1)$ (see Fig. 1).

### IV. ATOMIC-NUMBER FLUCTUATIONS

As a special case, the AA phase is a singlet state, which enables us to derive analytically the average particle numbers and their associated fluctuations. With the help of the methods developed in a related cavity QED problem [24], we expand the eigenvectors $|N,m_1\rangle$ in terms of the Fock states which are defined as $\hat{n}_\alpha^{(j)}|n_1^{(j)},n_0^{(j)},n_{-1}^{(j)}\rangle = n_\alpha^{(j)}|n_1^{(j)},n_0^{(j)},n_{-1}^{(j)}\rangle, \alpha = 0, \pm 1$. Explicitly, for even and odd $m_1$, we find, respectively,

$$\begin{aligned}|N,m_1\rangle &= \sum_{k=0}^{(N-|m_1|)/2} B^{(e)}_{m_1k}\left|\frac{N+m_1}{2}-k,2k,\frac{N-m_1}{2}-k\right\rangle,\\ |N,m_1\rangle &= \sum_{k=0}^{(N-|m_1|-1)/2} \times B^{(o)}_{m_1k}\left|\frac{N+m_1-1}{2}-k,2k+1,\frac{N-m_1-1}{2}-k\right\rangle,\end{aligned} \qquad (11)$$

where,

$$\begin{aligned}B^{(e)}_{m_1k} &= \frac{(\sqrt{2})^{2k}\left(\frac{(N+m_1)!(N-m_1)!}{2N!}\right)^{1/2}\sqrt{N!}}{\left[(2k)!\left(\frac{N-m_1}{2}-k\right)!\left(\frac{N+m_1}{2}-k\right)!\right]^{1/2}},\\ B^{(o)}_{m_1k} &= \frac{(\sqrt{2})^{2k+1}\left(\frac{(N+m_1)!(N-m_1)!}{2N!}\right)^{1/2}\sqrt{N!}}{\left[(2k+1)!\left(\frac{N-m_1-1}{2}-k\right)!\left(\frac{N+m_1-1}{2}-k\right)!\right]^{1/2}}.\end{aligned} \qquad (12)$$

We calculate the particle numbers and number fluctuations in the AA phase and find that the average numbers of atoms in the six components are exactly all equal,

$$\langle n_1^{(j)}\rangle = \langle n_0^{(j)}\rangle = \langle n_{-1}^{(j)}\rangle = N/3, \tag{13}$$

that is, the condensate is fragmented [10]. The fluctuations are given explicitly,

$$\begin{aligned}\langle \Delta n_0^{(j)}\rangle &= \frac{\sqrt{N^2+9N}}{3\sqrt{5}},\\ \langle \Delta n_{\pm 1}^{(j)}\rangle &= \frac{2\sqrt{N^2+3N/2}}{3\sqrt{5}},\end{aligned} \tag{14}$$

which approximately satisfy $\langle \Delta n_1^{(j)}\rangle = 2\langle \Delta n_0^{(j)}\rangle = \langle \Delta n_{-1}^{(j)}\rangle$ for large $N$, as opposed to $2\langle \Delta n_1\rangle = \langle \Delta n_0\rangle = 2\langle \Delta n_{-1}\rangle$ for the single-species singlet state ($c_2 > 0$) [10].

For a comprehensive understanding of the fluctuations in the entire parameter region, we consider the direct product of the Fock states of the two species $|n_1^{(1)},n_0^{(1)},n_{-1}^{(1)}\rangle \otimes |n_1^{(2)},n_0^{(2)},n_{-1}^{(2)}\rangle$, which may be equivalently defined as

$$\begin{aligned}&\hat{n}_\alpha^{(1,2)}|n_0^{(1)},m_1,n_0^{(2)},m_2;m\rangle\\ &\quad = n_\alpha^{(1,2)}|n_0^{(1)},m_1,n_0^{(2)},m_2;m\rangle.\end{aligned} \tag{15}$$

Here $m_1$ and $m_2$ are the corresponding magnetization specified as $m_j = n_1^{(j)} - n_{-1}^{(j)}$ and $m = m_1 + m_2$ is the total magnetization. For simplification, we restrict ourselves into the subspace that the total magnetization is conserved $m = 0$, in which case all states are nondegenerate. Using the full quantum approach of exact diagonalization, we simulate the distribution in this subspace with $N_1 = N_2 = 100$, and illustrate the dependence of the particle numbers and fluctuations for the six components on $c_{12}\beta$ in Fig. 2.

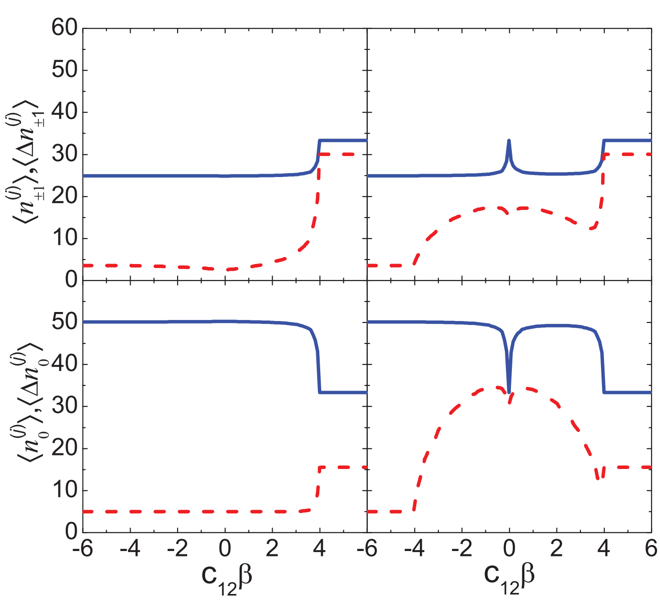


FIG. 2. (Color online) The dependence of atom numbers and fluctuations on $c_{12}\beta$ at fixed values of $c_1\beta_1 = -1$ and $c_2\beta_2 = 2$. Blue solid (red dashed) lines denote $\langle n_\alpha^{(j)}\rangle$ ($\langle \Delta n_\alpha^{(j)}\rangle$). Note $\langle n_{+1}^{(j)}\rangle = \langle n_{-1}^{(j)}\rangle$ and $\langle \Delta n_{+1}^{(j)}\rangle = \langle \Delta n_{-1}^{(j)}\rangle$. Left (right) column denotes the ferromagnetic (polar) condensate.

For very large values of $|c_{12}\beta|$, the distribution of atoms and fluctuations behave uniformly, indicating typical ferromagnetic (FF) or antiferromagnetic (AA) spin-exchange features. The number distributions for all components are essentially the same over the entire region, except for the case of $c_{12}\beta = 0$. A tiny $c_{12}\beta \neq 0$ brings the system into the ferromagnetic-like distribution, that is, $2\langle n_1^{(j)}\rangle = \langle n_0^{(j)}\rangle = 2\langle n_{-1}^{(j)}\rangle = N/2$, consistent with the prediction that the ferromagnetic condensate is more stable. The antiferromagnetic distributions for the six components are the same $\langle n_1^{(j)}\rangle = \langle n_0^{(j)}\rangle = \langle n_{-1}^{(j)}\rangle$, in agreement with the analytical results (14). The fluctuations are, on the other hand, quite different for positive or negative $c_{12}\beta$. In the FF phase ($c_{12}\beta < -4$) the fluctuations for both species are small ($\sim\sqrt{N}$). In the AA phase ($c_{12}\beta > 4$) the fluctuations are large and approximatively satisfy $\langle \Delta n_1^{(j)}\rangle = 2\langle \Delta n_0^{(j)}\rangle = \langle \Delta n_{-1}^{(j)}\rangle$ (with $\langle \Delta n_{\pm1}^{(j)}\rangle \approx 30.03$ and $\langle \Delta n_0^{(j)}\rangle \approx 15.56$). In the regions of $MM_-$ and $MM_+$ ($-4 < c_{12}\beta < 4$), polar atoms exhibit larger fluctuations than the ferromagnetic ones, which change quadratically with $c_{12}\beta$. When $c_{12}\beta > 0$, as the interspecies coupling increases, the fluctuation of polar atoms dramatically changes from $2\langle \Delta n_1^{(2)}\rangle = \langle \Delta n_0^{(2)}\rangle = 2\langle \Delta n_{-1}^{(2)}\rangle$ to $\langle \Delta n_1^{(2)}\rangle = 2\langle \Delta n_0^{(2)}\rangle = \langle \Delta n_{-1}^{(2)}\rangle$ (approximatively).

Before concluding, we show the difference of the singlet state $|N,N,0,0\rangle$ and the fully paired state $Z^{1/2}(\hat{\Theta}_{12}^\dagger)^N|0\rangle$ from the viewpoint of quantum fluctuation [18,20]. Using the method of generating function [18], the difference between the two states can be shown by taking $N_1 = N_2 = 2$ as an example. The result remains the same from the angular momentum theory, that is,

$$\begin{aligned}|2,2,0,0\rangle &= \sum_{m_1,m_2} C_{F_1,m_1;F_2,m_2}^{F=0,m=0}\,|2,m_1\rangle\,|2,m_2\rangle\\ &= \frac{1}{2\sqrt{5}}\left(\hat{\Theta}_{12}^{\dagger 2} - \frac{1}{3}\hat{A}^\dagger\hat{B}^\dagger\right)|0\rangle\,.\end{aligned} \tag{16}$$

More generally, according to the multinomial theorem,

$$(x_1+x_2+x_3)^n = \sum_{k=0}^{n}\sum_{l=0}^{k} c_{nlk}\,x_1^{n-k}x_2^{k-l}x_3^{l}, \tag{17}$$

with $c_{nlk} = n!/[l!(k-l)!(n-k)!]$, we find that the state $(\hat{\Theta}_{12}^\dagger)^N|0\rangle$ can be described by the Fock state $|n_1^{(1)},n_0^{(1)},n_{-1}^{(1)}\rangle \otimes |n_1^{(2)},n_0^{(2)},n_{-1}^{(2)}\rangle$ as

$$\begin{aligned}(\hat{\Theta}_{12}^\dagger)^N|0\rangle &= (\hat{a}_0^\dagger\hat{b}_0^\dagger - \hat{a}_1^\dagger\hat{b}_{-1}^\dagger - \hat{a}_{-1}^\dagger\hat{b}_1^\dagger)^N|0\rangle\\ &= \sum_{k=0}^{N}\sum_{l=0}^{k} c_{Nlk}(\hat{a}_0^\dagger\hat{b}_0^\dagger)^{N-k}(-\hat{a}_1^\dagger\hat{b}_{-1}^\dagger)^{k-l}(-\hat{a}_{-1}^\dagger\hat{b}_1^\dagger)^l|0\rangle\\ &= \sum_{k=0}^{N}\sum_{l=0}^{k}(-1)^k N!\,|k-l,N-k,l\rangle\otimes|l,N-k,k-l\rangle,\end{aligned} \tag{18}$$

where we have used the property $(\hat{a}^\dagger)^N|0\rangle = \sqrt{N!}|N\rangle$. We calculate the atom numbers and fluctuations for the two states $|N,N,0,0\rangle$ and $Z^{1/2}(\hat{\Theta}_{12}^\dagger)^N|0\rangle$, and find that one cannot distinguish them from the atom number distributions,

$$\langle n_1^{(j)}\rangle = \langle n_0^{(j)}\rangle = \langle n_{-1}^{(j)}\rangle = N/3, \tag{19}$$

which are the same. The fluctuations, however, reveal the secret. For state $Z^{1/2}(\hat{\Theta}_{12}^{\dagger})^{N}|0\rangle$, the number fluctuations are equally distributed, that is,

$$\langle\Delta n_1^{(j)}\rangle = \langle\Delta n_0^{(j)}\rangle = \langle\Delta n_{-1}^{(j)}\rangle = \sqrt{N(N+1)/6 - N^2/9}, \tag{20}$$

different from the results for the state $|N,N,0,0\rangle$ we obtained in Eq. (14).

## V. CONCLUSION

To conclude, we study atom number distributions and fluctuations for the ground state of a mixture of two spin-1 condensates, one being ferromagnetic and the other being polar, in the absence of a B field. For all possible interspecies coupling parameter $c_{12}\beta$, the exact quantum states are constructed from angular momentum algebra which are further expanded into the six-component Fock states of the mixture. The ground-state quantum phases are classified into four types according to $c_{12}\beta$. The most interesting AA phase of a singlet for the total spin, where spins of each species are polarized in opposite directions, is fragmented with six components equally distributed $\langle n_1^{(j)}\rangle = \langle n_0^{(j)}\rangle = \langle n_{-1}^{(j)}\rangle = N/3$ and exhibits anomalous number fluctuations. We find that the fluctuations satisfy $\langle\Delta n_1^{(j)}\rangle = 2\langle\Delta n_0^{(j)}\rangle = \langle\Delta n_{-1}^{(j)}\rangle$ for large $N$, different from the result of single-species polar state ($c_2 > 0$) $2\langle\Delta n_1\rangle = \langle\Delta n_0\rangle = 2\langle\Delta n_{-1}\rangle$. Our results highlight the significant promises for experimental work on Na and Rb atomic condensate mixtures since optical Feshbach resonances make it possible to tune the spin-exchange interaction.

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