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Not all pure entangled states are useful for sub-shot-noise interferometry

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We investigate the connection between the shot-noise limit in linear interferometers and particle entanglement. In particular, we ask whether sub-shot-noise sensitivity can be reached with all pure entangled input states of N particles if they can be optimized with local operations. Results on the optimal local transformations allow us to show that for $N=2$ all pure entangled states can be made useful for sub-shot-noise interferometry while for $N>2$ this is not the case. We completely classify the useful entangled states available in a bosonic two-mode interferometer. We apply our results to several states, in particular to multiparticle singlet states and to cluster states. The latter turn out to be practically useless for sub-shot-noise interferometry. Our results are based on the Cramer-Rao bound and the Fisher information.

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I. INTRODUCTION

The field of quantum interferometry has received much attention recently due to the prospect of enabling phase sensitivities below the shot noise, with applications in various fields such as quantum frequency standards, quantum lithography, quantum positioning and clock synchronization, and quantum imaging [1]. Current research on linear interferometers is directed at the search for optimal input states and output measurements [2–12], adaptive phase measurement schemes [13–16], and the influence of particle losses [17–19]. Several proof-of-principle experiments reaching a sub-shot-noise sensitivity have been performed, for a fixed number of particles with photons [20–24] and ions [25], while squeezed states for interferometry with a nonfixed number of particles have been prepared with Bose-Einstein condensates [26–30], atoms at room temperature [31], and light [32,33]. Also, schemes for nonlinear interferometers are under investigation [34–38].

In this article, we are interested in the connection between particle entanglement and phase estimation for linear interferometers with input states of a fixed number of particles N. It has been shown recently that for a linear interferometer sequence and arbitrary mixed separable input states the phase sensitivity cannot surpass the shot-noise limit [39–41]:

$$\Delta\theta_{\rm SN}=\frac{1}{\sqrt{N}}. \tag{1}$$

Hence, if a quantum state allows for a phase estimation scheme with a sub-shot-noise (SSN) phase uncertainty, it is necessarily entangled. We will refer to such states as useful for SSN interferometry or simply as useful.

In particular, we consider the question of whether all pure entangled states of a fixed particle number N can be made useful if arbitrary local operations can be applied on them before they enter the interferometer. We allow for such operations as they correspond to a local change of basis, and hence cannot create entanglement. The related problem of finding the local transformation optimizing the interferometric performance of a given input state is of interest for experimental applications, where typically such operations are relatively easy to implement. We consider separately the cases where the particles can or cannot be addressed individually (cf. Fig. 1 for examples of these situations).

We start by introducing the general framework of parameter estimation with linear interferometers and basic facts about entanglement in Sec. II. General observations regarding the local transformations optimizing the phase sensitivity of an input state are made in Sec. III. The main results concerning the usefulness of pure entangled states under local transformations are presented Sec. IV. Here we also comment on the use of more general local transformations which are not unitary. Finally, we apply the results to two important families of states in Sec. V. We summarize our results in Sec. VI.

II. BASIC CONCEPTS

A. Linear interferometers and collective operators

In linear interferometers, such as the Mach-Zehnder interferometer [cf. Fig. 1(a)], the phase shift is due to the independent action of some external effect on each particle. We restrict ourselves to the situation that the interferometer is performed in a two-level subspace here. The two levels could be two momentum states as for the Mach-Zehnder interferometer, the two wells of a double-well, or two internal states of the particles (cf. Fig. 1). The corresponding phase transformation can be characterized in terms of collective spin operators $\hat{J}_i=\frac{1}{2}\sum_{k=1}^{N}\hat{\sigma}_i^{(k)}$, where $\hat{\sigma}_i^{(k)}$ is the ith Pauli matrix acting on particle k. Here and in the following, we label the three Pauli matrices with x,y,z or with $1,2,3$. The input state is transformed by $\exp[-i\hat{J}_{\vec{n}}\theta]$, where $\hat{J}_{\vec{n}}=\vec{n}\cdot\vec{\hat{J}}$ and θ is the phase shift. For a Mach-Zehnder interferometer consisting of a beam splitter $\exp[i\hat{J}_x\frac{\pi}{2}]$, a phase shift $\exp[-i\hat{J}_z\theta]$, and another beam splitter $\exp[-i\hat{J}_x\frac{\pi}{2}]$, the effective rotation is [3]

$$U_{\rm MZ}=e^{-i\hat{J}_x\frac{\pi}{2}}e^{-i\hat{J}_z\theta}e^{i\hat{J}_x\frac{\pi}{2}}=e^{-i\hat{J}_y\theta}; \tag{2}$$

hence, $\vec{n}=\hat{y}$. This transformation also describes other applications such as the Fabry-Perot interferometer, Ramsey spectroscopy, and the Michelson-Morley interferometer.

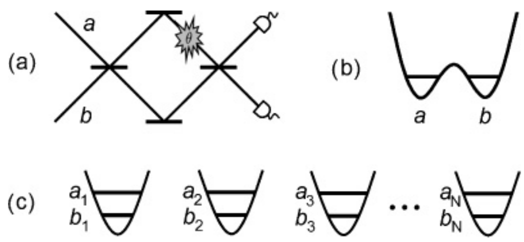

FIG. 1. Systems that can be used for linear two-state interferometry: (a) Archetypical optical Mach-Zehnder interferometer as in Refs. [22,23], (b) double-well system as implemented in recent experiments on squeezing in BECs [28–30], and (c) system of single wells as in ion traps [25,42]. In the first two cases, each of the N particles lives in the subspace of the two states labeled a and b, corresponding to momentum states in case (a) and to the left and the right well in case (b). In case (c), there is one particle per well, and particle k in trap k has the two internal degrees of freedom a_k and b_k (displayed are trap states, while in ion traps typically internal states of the ions are used [42]). The interferometer operations acts on the a-b subspace in the cases (a) and (b) and identically on the subspaces a_k-b_k in case (c). In the latter case, the particles are accessible individually via the different traps in principle. They can be treated as distinguishable particles labeled by the trap number k if the spacial wave functions of the particles in the different traps do not overlap [43].

Note that since the collective spin operators are just sums of single-particle operators, the transformation factorizes,

$$e^{-i\hat{J}_{\vec{n}}\theta} = e^{-i\hat{\sigma}_{\vec{n}}^{(1)}\frac{\theta}{2}} \otimes e^{-i\hat{\sigma}_{\vec{n}}^{(2)}\frac{\theta}{2}} \otimes \cdots \otimes e^{-i\hat{\sigma}_{\vec{n}}^{(N)}\frac{\theta}{2}}, \tag{3}$$

where $\hat{\sigma}_{\vec{n}} = \vec{\hat{\sigma}} \cdot \vec{n}$. Therefore, this operation acts only locally on the particles, and no entanglement can be created this way. Note that this is true in particular for the beam-splitter operation $\exp[-i\hat{J}_x\frac{\pi}{2}]$.

This is different if a mode picture is used. Let us consider situations (a) and (b) of Fig. 1. In this case, the beam splitter can turn a separable input state $|N_a\rangle \otimes |N_b\rangle$ (written in the Fock basis of the two modes a and b) into an entangled state, and the connection of entanglement and SSN interferometry is lost.

We call an operation of the form (3) a collective local unitary (CLU) operation, since each particle is acted on with the same unitary operator. A general local unitary (LU) operation is one which factorizes as well but where the unitary operations acting on two different particles can be different. Note that if the particles cannot be addressed individually, as in cases (a) and (b) of Fig. 1, then only CLU operations can be implemented, while LU operations are available if we can address the particles separately [cf. Fig. 1(c)].

We generally work with the particle picture and call the eigenstates of the $\hat{\sigma}_z$ operator $|0\rangle$ and $|1\rangle$ such that $\hat{\sigma}_z|0\rangle = |0\rangle$ and $\hat{\sigma}_z|1\rangle = -|1\rangle$. Note that from now on we label the two states 0 and 1 instead of a and b as done in Fig. 1. The eigenstates of the collective spin operator $\hat{J}_z$ will be denoted with $|j,m\rangle$, where $j = N/2$ and $2m$ is the difference of particles in the state $|0\rangle$ and particles in the state $|1\rangle$. These states are also known as Dicke states [44]. They fulfill $\hat{\vec{J}}^2|j,m\rangle = j(j+1)|j,m\rangle$ and $\hat{J}_z|j,m\rangle = m|j,m\rangle$. In general, the eigenvalue m is degenerate. However, the *symmetric* Dicke states $|N/2,m\rangle_S$ are uniquely defined. Here and in the following, pure symmetric states are those which are invariant under the interchange of any two particles [45]. Examples for two particles are $|1,-1\rangle_S = |1\rangle \otimes |1\rangle \equiv |11\rangle$, $|1,0\rangle_S = (|10\rangle + |01\rangle)/\sqrt{2}$, and $|1,1\rangle_S = |00\rangle$ and for three particles $|3/2,1/2\rangle_S = (|100\rangle + |010\rangle + |100\rangle)/\sqrt{3}$.

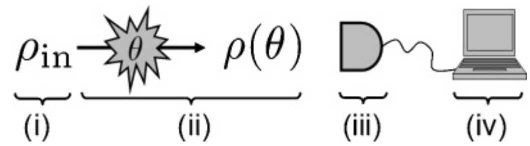

FIG. 2. A general phase estimation scheme consisting of (i) the input state, (ii) the phase transformation, (iii) the measurement, and (iv) the data-processing stage.

B. Phase estimation

In a general phase estimation scenario [46,47] (see also [48] for an introduction), the initial state $\rho_{\rm in}$ is transformed to $\rho(\theta)$ by some transformation depending only on the single parameter θ, and finally, a measurement is performed. The phase is then estimated from the results of this measurement. This scheme is schematically depicted in Fig. 2.

The phase transformation could be, for instance, the operator $\exp[-i\hat{J}_y\theta]$ for a Mach-Zehnder interferometer, as we have seen in the last section. A general measurement can be expressed by its positive operator valued measure (POVM) elements $\{\hat{E}(\xi)\}_\xi$ [49]. Depending on the possible outcomes ξ, θ can be estimated from the results of these measurements with an estimator $\theta_{\rm est}(\xi)$.

For so-called unbiased estimators, the relation $\bar{\theta}_{\rm est} = \theta$ holds, the estimated phase shift is on average equal to the true phase shift. The phase sensitivity is defined as the standard deviation of the estimator. If the estimator is unbiased, it is bounded by the Cramér-Rao theorem [46,47] as

$$\Delta\theta_{\rm est} \geqslant \frac{1}{\sqrt{m}}\frac{1}{\sqrt{F}}, \tag{4}$$

where m is the number of independent repetitions of the measurement and F is the so-called Fisher information. Fisher's theorem ensures that the bound (4) can be saturated in the central limit, typically for large m, with a maximum-likelihood estimator [50].

The Fisher information quantifies the statistical distinguishability of quantum states along a path described by a single parameter [51–53]. It is defined as

$$F[\rho(\theta);\{\hat{E}(\xi)\}] = \int d\xi\, P(\xi|\theta)[\partial_\theta \log P(\xi|\theta)]^2, \tag{5}$$

where the conditional probabilities are given by the quantum mechanical expectation values $P(\xi|\theta) = {\rm Tr}[\hat{E}(\xi)\rho(\theta)]$. This holds for general parameter-estimation protocols. In this article, we only consider estimation protocols for a dimensionless phase shift and linear interferometers.

The so-called quantum Fisher information F_Q is defined as the Fisher information maximized over all possible measurements,

$$F_Q[\rho(\theta)] = \max_{\{\hat{E}(\xi)\}} F[\rho(\theta);\{\hat{E}(\xi)\}]. \tag{6}$$

For pure input states and for a unitary phase transformation with the generator $\hat{H}$, where $\hat{H} = \hat{J}_{\vec{n}}$ for linear two-mode interferometers, the quantum Fisher information is [52,53]

$$F_Q[|\psi\rangle; \hat{H}] = 4\langle \Delta \hat{H}^2 \rangle_\psi = 4(\langle \hat{H}^2 \rangle_\psi - \langle \hat{H} \rangle_\psi^2). \quad (7)$$

For mixed input states, the quantum Fisher information is given by [52,53]

$$F_Q[\rho; \hat{H}] = 2 \sum_{j,k} (\lambda_j + \lambda_k) \left(\frac{\lambda_j - \lambda_k}{\lambda_j + \lambda_k} \right)^2 |\langle j|\hat{H}|k\rangle|^2, \quad (8)$$

where $\rho = \sum_k \lambda_k |k\rangle\langle k|$ is the spectral decomposition of the input state and the sum is over terms where $\lambda_j + \lambda_k \neq 0$ only.

A useful property of F, and consequently of F_Q, is that it is convex for mixed states; that is, if $\rho = p\rho_1 + (1-p)\rho_2$ with $0 \leqslant p \leqslant 1$, then $F(\rho) \leqslant pF(\rho_1) + (1-p)F(\rho_2)$ for fixed phase-transformation and output measurement [54] (see also Ref. [55]).

C. Entanglement vs shot-noise limit

A pure state of N particles is called fully separable if it can be written as a product state, $|\psi_{\text{fs}}\rangle = \otimes_{i=1}^N |\psi^{(i)}\rangle$, where $|\psi^{(i)}\rangle$ is a pure state of particle i. A mixed state is fully separable if it can be written as an incoherent mixture of such product states,

$$\rho_{\text{fs}} = \sum_k p_k |\psi_k^{(1)}\rangle\langle\psi_k^{(1)}| \otimes |\psi_k^{(2)}\rangle\langle\psi_k^{(2)}| \otimes \cdots \otimes |\psi_k^{(N)}\rangle\langle\psi_k^{(N)}|, \quad (9)$$

where $\{p_k\}$ is a probability distribution [56]. Any such state can be generated by local operations and classical communication [49,56]. Nonseparable states are entangled, and nonlocal operations are needed for their production.

Recently, it has been shown that for all fully separable input states and for any unitary generator $\hat{H} = \hat{J}_{\vec{n}}$, the Fisher information is bounded by the number of particles, $F[\rho_{\text{fs}}; \hat{J}_{\vec{n}}] \leqslant N$ [40]. By the Cramér-Rao bound (4), the phase sensitivity is then bounded by the shot-noise limit,

$$\Delta\theta_{\text{est}} \geqslant \frac{1}{\sqrt{N_{\text{tot}}}}, \quad (10)$$

where $N_{\text{tot}} = mN$ is the total number of particles used in the m runs. Therefore, only entangled input states can reach a SSN sensitivity.

The so-called Heisenberg limit, that is, the ultimate limit on the phase sensitivity, depends on the constraints on the resources used. If m and N are fixed separately, then the ultimate sensitivity allowed by quantum mechanics is given by [39]

$$\Delta\theta = \frac{1}{\sqrt{m}N}. \quad (11)$$

However, the total number of particles used in the protocol is N_{tot}, and therefore it is reasonable to consider the bound where only this number is fixed [9,57]. The corresponding limit is given by

$$\Delta\theta_{\text{HL}} = \frac{1}{N_{\text{tot}}}, \quad (12)$$

which can be saturated for $m = 1$ only.

We remark that if the interferometer transformation is not equal to $\exp[-i\hat{J}_{\vec{n}}\theta]$, then the shot-noise limit and the Heisenberg limit have to be redefined accordingly. Assume, for instance, that the unknown phase shift θ can be applied to a photon a number of times p at will. Then a protocol where single photons are passing through an interferometer one after the other, such that photon k experiences the phase shift $p_k\theta$, can reach a sensitivity scaling as $\Delta\theta \sim 1/N$ [14,15]. Here, N is the total number of resources, where not only a photon but also the application of the phase shift is counted as a resource.

D. Statement of the problem

Now we are ready to start the main investigation. We want to classify the pure entangled states with respect to their usefulness for interferometry. Since entanglement cannot be generated by local operations, we allow for such operations to be applied to the input state. The question we want to answer is: can we obtain a bound below the shot-noise limit for every pure entangled state in this scenario? In other words, can $F > N$ be achieved for every pure entangled state?

We consider two cases. (i) If the particles are indistinguishable bosons which cannot be individually addressed, then the input states have to be symmetric under interchange of the particles. The results relating separable states to the shot-noise limit is still valid here, but since the state space is reduced, the only admissible separable states are of the form (9), where all single-particle states are identical; that is, $|\psi_k^{(i)}\rangle = |\psi_k\rangle$ for all k [58]. A typical example is the state $|0\rangle^{\otimes N}$ of N particles all entering the Mach-Zehnder interferometer at the same input port. This situation is present in cases (a) and (b) depicted in Fig. 1. Only CLU operations can be implemented in this case. (ii) The particles (bosons or fermions) can be individually addressed, for instance because each particle is trapped in a different trap as in case (c) depicted in Fig. 1. Then, the particles can be effectively treated as being distinguishable [43] and any LU operation can be implemented.

Before we start, let us make two further remarks. First, optimizing the Fisher information minimizes the lower bound on the sensitivity (4). However, the smallest number m for which this bound is saturated depends on the input state. For fixed total resources $N_{\text{tot}} = mN$ and two input states $|\psi\rangle$ and $|\phi\rangle$, it may may therefore be possible to reach a better sensitivity $\Delta\theta$ with the state $|\phi\rangle$ even if $F(|\psi\rangle) > F(|\phi\rangle)$ [9,57]. Second, the problem we investigate can be viewed as one further step in the optimization of the Fisher information when the phase is generated unitarily:

$$F[\rho; \hat{H}; \{\hat{E}(\xi)\}_\xi] \leqslant F_Q[\rho; \hat{H}] \leqslant \max_{U_{\text{L}}} F_Q[U_{\text{L}}\rho U_{\text{L}}^\dagger; \hat{H}], \quad (13)$$

where U_{L} is a LU operation. Both steps preserve the fact that the shot-noise limit cannot be overcome with separable states.

III. OPTIMAL FISHER INFORMATION UNDER CLU AND LU OPERATIONS

In this section, we search for the optimal value of the quantum Fisher information that can be achieved if CLU or LU operations are applied on a pure input state. We first investigate their effect on F_Q before we find the optimal value of F_Q for CLU operations and an upper bound for LU operations. Finally, we show that even though we are considering pure states only in this article, similar results for the optimal values of F_Q hold for mixed states as well.

A. Effect of CLU and LU operations on F_Q

When the input state $|\psi_{\rm in}\rangle$ is transformed by a LU transformation $U_{\rm L} = U_1 \otimes U_2 \otimes \cdots \otimes U_N$, then the quantum Fisher information Eq. (7) changes as

$$F_Q' = 4\langle \Delta \hat{H}^2\rangle_{U_{\rm L}\psi_{\rm in}} = 4\langle \Delta \hat{H}'^2\rangle_{\psi_{\rm in}}, \tag{14}$$

where $\hat{H}' = U_{\rm L}^\dagger \hat{H} U_{\rm L}$. Hence, for $\hat{H} = \hat{J}_y$ as in the Mach-Zehnder interferometer, applying a LU operation to the initial state is equivalent to a local transformation of the interferometer operation according to $\hat{J}' = \frac{1}{2}\sum_{k=1}^N U_k^\dagger \hat{\sigma}_y U_k$. The relation $U^\dagger \hat{\vec{\sigma}} U = O\hat{\vec{\sigma}}$ holds, where O is an orthogonal matrix; hence, a unitary transformation of the vector of Pauli matrices corresponds to a rotation [61]. It follows that

$$\hat{J}' = \frac{1}{2}\sum_{k=1}^N \vec{n}^{(k)} \cdot \hat{\vec{\sigma}}. \tag{15}$$

Here $\vec{n}^{(k)} = O_k^T \hat{y}$, and $\hat{y}$ is the unit vector pointing in the y direction.

Therefore, changing the input state with a LU operation is equivalent to a change of the local directions of the spins. A collective spin operator is in general acting differently on the spins after this operation. If a CLU operation is applied, where $U_k = U$ for all k, then the collective operator remains collective; only its direction is changed.

B. Optimum under CLU operations

Given a general pure state $|\psi\rangle$, the optimal direction $\vec{n}_{\rm max}$ of the generator $\hat{J}_{\vec{n}}$ and the maximal F_Q can be determined directly.

Observation 1. The maximal F_Q that can be achieved for $\hat{H} = \hat{J}_{\vec{n}}$ when $\vec{n}$ can be optimized over is given by $4\lambda_{\rm max}[\gamma_C]$, where $\lambda_{\rm max}$ is the maximal eigenvalue of the real 3×3 covariance matrix γ_C with entries

$$[\gamma_C]_{ij} = \tfrac{1}{2}\langle \hat{J}_i\hat{J}_j + \hat{J}_j\hat{J}_i\rangle - \langle \hat{J}_i\rangle\langle \hat{J}_j\rangle, \tag{16}$$

and the optimal direction $\vec{n}_{\rm max}$ is the corresponding eigenvector [62]. We will also call γ_C the collective covariance matrix.

Proof. For $\hat{H} = \vec{n}\cdot\vec{J}$ we have $F_Q = 4\langle(\Delta \hat{J}_{\vec{n}})^2\rangle = 4\vec{n}^T\gamma_C\vec{n}$ since $\vec{n}$ is real. It is known from linear algebra that this expression is maximized by choosing $\vec{n} = \vec{n}_{\rm max}$ as the eigenvector corresponding to the maximal eigenvalue. ■

The matrix γ_C has appeared before in the context of interferometry [63] and in the derivation of the optimal spin squeezing inequalities for entanglement detection [64]. The results presented in the latter article allow for a different proof of the fact that separable states cannot beat the shot-noise limit [65].

Let us consider as examples of the usefulness of Observation 1 three prominent symmetric states which are known to provide SSN sensitivity. Their weights in the basis of symmetric Dicke states $|\frac{N}{2},m\rangle_S$ are depicted in Fig. 3.

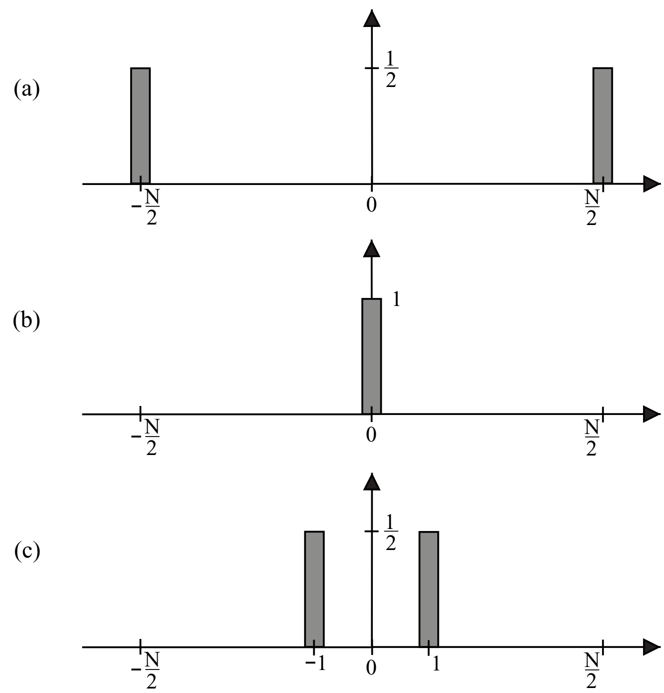

FIG. 3. Examples of symmetric states on which we apply Observation 1 given in the basis of symmetric Dicke states. (a) $N00N$ state, (b) twin-Fock state, (c) state considered in Ref. [9]. Plotted are the squared absolute values of the weights of the symmetric Dicke states in the superpositions.

(a) The so-called $N00N$ state (N-particle path-entangled state $|N,0\rangle + |0,N\rangle$) is given by [8,66]

$$|N00N\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}). \tag{17}$$

For this state, we find $4\gamma_C^{N00N} = \mathrm{diag}(N,N,N^2)$. The $N00N$ state achieves the maximal value of the Fisher information $F_Q = N^2$ [39] when the generator of the phase shift is $\hat{J}_z$, while it gives sensitivity at the shot-noise limit if a collective operator in the x-y plane is chosen instead. Hence, if a $N00N$ state is entering a normal Mach-Zehnder interferometer, it has to be rotated first by $\exp(\pm i\frac{\pi}{2}\hat{J}_x)$ in order to reach the optimal sensitivity. This happens because only the $\hat{J}_z$ operator leads to the maximal relative phase shift $\exp[-iN\phi]$ between the two states in the superposition of the $N00N$ state. Similar corrections have to be applied in the Ramsey scheme originally considered in Ref. [8].

(b) Another state promising SSN sensitivity is the Twin-Fock state [4]:

$$|\mathrm{TF}\rangle = \left|\frac{N}{2},0\right\rangle_S. \tag{18}$$

In this state, half of the particles are in the state $|0\rangle$ and the other half are in the state $|1\rangle$. Note that this state is a product state in a mode picture, $|N/2\rangle_0 \otimes |N/2\rangle_1$, while it is multipartite entangled in the particle picture we use. We find $4\gamma_C^{\mathrm{TF}} = (\frac{N^2}{2} + N)\mathrm{diag}(1,1,0)$; hence, the SSN sensitivity for interferometry with the generator $\hat{J}_{\vec{n}}$ is bounded identically by the Cramér-Rao lower bound for any $\vec{n}$ in the x-y plane, while the state is insensitive to the phase change if $\vec{n} = \hat{z}$.

(c) The third example is the following state of an even number of particles, which offers advantages when used in a Mach-Zehnder interferometer with a Bayesean estimation protocol [9]:

$$|\mathrm{PS}\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{N}{2},1\right\rangle_S + \left|\frac{N}{2},-1\right\rangle_S\right), \tag{19}$$

which yields $4\gamma_C^{\mathrm{PS}} = \mathrm{diag}(\frac{3}{4}N^2 + \frac{3}{2}N - 2, \frac{1}{4}N^2 + \frac{1}{2}N - 2, 4)$. We expressed the states with the symmetric Dicke states $|j,m\rangle_S$ introduced previously. While for both $\vec{n} = \hat{y}$ and $\vec{n} = \hat{x}$ the quantum Fisher information is larger than N, it is largest for $\vec{n} = \hat{x}$.

C. Optimum under LU operations

When general LU operations are applied, then the quantum Fisher information takes the form $F_Q[|\psi\rangle; \hat{J}'] = 4\langle(\Delta J')^2\rangle = \vec{m}^T\gamma_R\vec{m}$, where we introduced the real $3N \times 3N$ covariance matrix γ_R with entries

$$[\gamma_R]_{(k_1,i_1)(k_2,i_2)} = \frac{1}{2}\left(\left\langle\hat{\sigma}_{i_1}^{(k_1)}\hat{\sigma}_{i_2}^{(k_2)}\right\rangle + \left\langle\hat{\sigma}_{i_2}^{(k_2)}\hat{\sigma}_{i_1}^{(k_1)}\right\rangle\right) - \left\langle\hat{\sigma}_{i_1}^{(k_1)}\right\rangle\left\langle\hat{\sigma}_{i_2}^{(k_2)}\right\rangle \tag{20}$$

and the real vector $\vec{m}^T = ([\vec{n}^{(1)}]^T, [\vec{n}^{(2)}]^T, \ldots, [\vec{n}^{(N)}]^T)$. The matrix entries are parametrized by two double indices (k_1, i_1) and (k_2, i_2). The optimal value of F_Q is hence given by the solution of the problem

$$F_Q^{\max} = \max_{\vec{m}} \vec{m}^T\gamma_R\vec{m}\big|_{[\vec{n}^{(k)}]^T\vec{n}^{(k)}=1\ \forall k}. \tag{21}$$

A simple upper bound on $F_Q^{\max}$ can be obtained by relaxing the N constraints $[\vec{n}^{(k)}]^T\vec{n}^{(k)} = 1$ to the single constraint $\vec{m}^T\vec{m} = N$.

Observation 2. The maximal F_Q for $\hat{H} = \hat{J}_y$ that can be achieved when arbitrary LU operations can be applied on the input state is bounded by

$$F_Q^{\max} \leqslant \max_{\vec{m}} \vec{m}^T\gamma_R m\big|_{\vec{m}^T\vec{m}=N} = N\lambda_{\max}[\gamma_R]. \tag{22}$$

Equality holds in relation (22) if and only if there is a vector $\vec{m}^*$ optimizing problem (21) which is an eigenvector of γ_R corresponding to the maximal eigenvalue. If $\vec{m}^*$ is the eigenvector corresponding to the maximal eigenvalue of γ_R which fulfills all the N constraints $[\vec{n}^{(k)}]^T\vec{n}^{(k)} = 1$, then local directions can be converted into LU transformations as in Observation 1 [62].

This simple observation, which can be proven in the same way as Observation 1, will turn out to be very useful for the proof Proposition 2 and also for the examples discussed in Sec. V.

Here we obtained an upper bound on the maximal F_Q that can be obtained when arbitrary LU operations are available by making it possible to optimize over more general operations. In turn, a simple *lower* bound can be obtained from Observation 1, since in this case the operations are more restricted.

We remark that the covariance matrix γ_R with entries given in Eq. (20) has appeared previously in studies of macroscopic entanglement [67,68]. Given a pure state $|\psi\rangle$, the index $p \in [1,2]$ introduced in these references indicates the presence of macroscopic entanglement if $p = 2$. Similar to our case, its computation involves the maximization of the variance of a local operator $\hat{A} = \sum_{k=1}^N \alpha_k\vec{n}^{(k)} \cdot \vec{\hat{\sigma}}$, where the α_k fulfill $\sum_{k=1}^N |\alpha_k|^2 = N$. Due to the additional parameters $\{\alpha_k\}_k$, the problem is then of the form $\max_{\vec{m}} \vec{m}^\dagger\gamma_R\vec{m}\big|_{\vec{m}^\dagger\vec{m}=N}$, and the maximum can always be reached by the maximal eigenvector and the corresponding eigenvalue [68]. Here the dagger † appears instead of the transposition T since the α_k are not restricted to be real [68].

D. Optimum for mixed states

Even though in this article we are only considering pure states, we note that Observation 1 also holds in the case of mixed input states $\rho = \sum_k \lambda_k|k\rangle\langle k|$ when the collective covariance matrix γ_C is replaced with the matrix Γ_C with entries

$$[\Gamma_C]_{ij} = \frac{1}{2}\sum_{l,m}(\lambda_l + \lambda_m)\left(\frac{\lambda_l - \lambda_m}{\lambda_l + \lambda_m}\right)^2 \langle l|\hat{J}_i|m\rangle\langle m|\hat{J}_j|l\rangle. \tag{23}$$

In analogy, Observation 2 holds if the matrix γ_R is replaced by the matrix Γ_R with entries

$$[\Gamma_R]_{(k_1,i_1),(k_2,i_2)} = \frac{1}{2}\sum_{l,m}(\lambda_l + \lambda_m)\left(\frac{\lambda_l - \lambda_m}{\lambda_l + \lambda_m}\right)^2 \langle l|\hat{\sigma}_{i_1}^{(k_1)}|m\rangle\langle m|\hat{\sigma}_{i_2}^{(k_2)}|l\rangle. \tag{24}$$

This follows directly from the form of F_Q for mixed states when the phase comes from unitary evolution generated by $\hat{J}_{\vec{n}}$ and $\hat{J}'$, respectively [see Eq. (8)]. Note that the matrices Γ_C and Γ_R are symmetric because of the sums over l and m. For pure states, they reduce to γ_C and γ_R, respectively.

IV. USEFULNESS OF PURE ENTANGLED STATES

Now we are prepared to consider the general question: Are all pure entangled states useful for SSN interferometry under CLU and LU operations? In the first part of this section, we consider pure symmetric states and CLU operations, corresponding to the situation in a system of N bosons which cannot be individually addressed, as in cases (a) and (b) depicted in Fig. 1. If the input state is symmetric but general LU operations can be applied, we find that F_Q cannot be increased beyond the value obtained with the optimal CLU operation. The results allow us to draw conclusions on the usefulness of general states. In the final part of this section, we briefly comment on how these results change when more general local operations than CLU and LU are available.

A. Reduced states of pure symmetric states

We start by considering reduced density matrices of pure symmetric states. This will be very useful for the proofs presented later. The reduced density matrix for two particles of any state $|\psi\rangle$ can always be written as

$$\rho^{(r)} = \frac{1}{4}\sum_{i,j=0}^{3} \lambda_{ij}\hat{\sigma}_i \otimes \hat{\sigma}_j, \tag{25}$$

where $\lambda_{ij} = \langle\hat{\sigma}_i \otimes \hat{\sigma}_j\rangle_\psi$, $\hat{\sigma}_0 = \mathbb{1}$, and $\hat{\sigma}_{1,2,3} = \hat{\sigma}_{x,y,z}$. Normalization is ensured by $\lambda_{00} = 1$.

If $|\psi\rangle$ is symmetric under the interchange of particles, then the matrix λ is not only real, but also symmetric, and the diagonal elements fulfill [45]

$$\sum_{i=1}^{3}\lambda_{ii} = 1. \tag{26}$$

Note that this holds for the case $N = 2$, where $\rho^{(r)} = |\psi\rangle\langle\psi|$, and also for the case $N > 2$, since then the reduced density matrix also acts on the symmetric subspace only.

If we consider CLU transformations of $|\psi_S\rangle$, then $\rho^{(r)} \to U \otimes U\ \ \rho^{(r)}\ \ U^\dagger \otimes U^\dagger$. Since $U\hat{\vec{\sigma}}U^\dagger = O^T\hat{\vec{\sigma}}$ as mentioned before, λ transforms as

$$\lambda \equiv \begin{pmatrix} 1 & \vec{s}^T \\ \vec{s} & T \end{pmatrix} \to \begin{pmatrix} 1 & \vec{s}^T O \\ O^T\vec{s} & \bar{T} \end{pmatrix} \equiv \bar{\lambda}, \tag{27}$$

where $\vec{s}$ is a column vector with entries $s_i = \langle\hat{\sigma}_i\rangle$, T a symmetric 3×3 matrix with entries $T_{ij} = \langle\hat{\sigma}_i \otimes \hat{\sigma}_j\rangle$ for $i,j = 1,2,3$, and $\bar{T} = O^T T O$. The condition (26) corresponds to $\mathrm{Tr}[T] = 1$. Since $-1 \leqslant \lambda_{ij} \leqslant 1$ holds, only one of the diagonal elements T_{ii} can be negative. Further, if one element is negative, then the other two diagonal elements have to be strictly positive.

B. CLU operations

Here, we consider pure symmetric entangled states under CLU operations. This is realized in a bosonic system where all particles can be in two external states, for instance. In this situation the states can be completely characterized with respect to their usefulness, and it turns out that any symmetric state is useful, apart from superpositions of $|0\rangle^{\otimes N}$ and $|1\rangle^{\otimes N}$ with significantly different weights. We directly state the result and present the proof afterward.

Proposition 1. For a pure, symmetric, and entangled state $|\psi_S\rangle$ there is a direction $\vec{n}$ such that $F_Q(|\psi_S\rangle, \hat{J}_{\vec{n}}) > N$ except for the following family of states of $N > 2$ qubits:

$$|\psi_S\rangle = \sqrt{q}|0\rangle^{\otimes N} + e^{i\phi}\sqrt{1-q}|1\rangle^{\otimes N} \tag{28}$$

up to a CLU operation and

$$q \leqslant \frac{1}{2}\left(1 - \sqrt{\frac{N-1}{N}}\right) \quad \text{or} \quad q \geqslant \frac{1}{2}\left(1 + \sqrt{\frac{N-1}{N}}\right). \tag{29}$$

Proof. The form of $F_Q(|\psi\rangle, \hat{J}_{\vec{n}})$ is

$$\begin{aligned} 4\langle(\Delta\hat{J}_{\vec{n}})^2\rangle &= \left\langle \sum_{k,l}\hat{\sigma}_{\vec{n}}^{(k)}\hat{\sigma}_{\vec{n}}^{(l)}\right\rangle - \left\langle\sum_k \hat{\sigma}_{\vec{n}}^{(k)}\right\rangle^2 \\ &= N - \sum_k \left\langle\hat{\sigma}_{\vec{n}}^{(k)}\right\rangle^2 + 2\sum_{k<l}\left\langle\hat{\sigma}_{\vec{n}}^{(k)}\hat{\sigma}_{\vec{n}}^{(l)}\right\rangle - \left\langle\hat{\sigma}_{\vec{n}}^{(k)}\right\rangle\left\langle\hat{\sigma}_{\vec{n}}^{(l)}\right\rangle. \end{aligned}$$

For symmetric states, the terms $\langle\hat{\sigma}_{\vec{n}}^{(k)}\rangle$ and $\langle\hat{\sigma}_{\vec{n}}^{(k)}\hat{\sigma}_{\vec{n}}^{(l)}\rangle$ do not depend on the sites k and l, and hence

$$\begin{aligned} 4\langle(\Delta\hat{J}_{\vec{n}})^2\rangle &= N(1 - \langle\hat{\sigma}_{\vec{n}}\rangle^2) + N(N-1)(\langle\hat{\sigma}_{\vec{n}}\hat{\sigma}_{\vec{n}}\rangle - \langle\hat{\sigma}_{\vec{n}}\rangle^2) \\ &= N + N(N-1)\langle\hat{\sigma}_{\vec{n}}\hat{\sigma}_{\vec{n}}\rangle - N^2\langle\hat{\sigma}_{\vec{n}}\rangle^2, \end{aligned}$$

where we left out the particle indices. It follows that for pure symmetric states

$$F_Q[\psi_S, \hat{J}_{\vec{n}}] > N \Leftrightarrow \langle\hat{\sigma}_{\vec{n}}\hat{\sigma}_{\vec{n}}\rangle > \frac{N}{N-1}\langle\hat{\sigma}_{\vec{n}}\rangle^2. \tag{30}$$

Hence, the task is to see whether it is possible for any pure symmetric state to find a CLU operation or a direction $\vec{n}$ such that condition (30) holds.

We can choose a CLU operation such that $\langle\hat{\sigma}_x\rangle = \langle\hat{\sigma}_y\rangle = 0$, $O^T\vec{s} = (0,0,\delta)^T$. Now we have to consider several cases: (i) Let us assume that the elements $\bar{T}_{ij}$, $i,j = 1,2$ are not all equal to zero. Since $\langle\hat{\sigma}_x\rangle = \langle\hat{\sigma}_y\rangle = 0$, if we can make $\langle\hat{\sigma}_x\hat{\sigma}_x\rangle$ or $\langle\hat{\sigma}_y\hat{\sigma}_y\rangle$ positive, then condition (30) is fulfilled for the respective direction. If there are nonzero elements $\bar{T}_{ij}$, $i,j = 1,2$, then the trace of this submatrix might be zero, but the eigenvalues will be different from zero. Then they have to be both different from zero, and only one of them can be negative. Hence, there is an orthogonal transformation $1 \oplus \tilde{O} \oplus 1$ which makes λ_{11} positive while keeping $\vec{s} = (0,0,\delta)^T$. Therefore, we can make $\langle\hat{\sigma}_x\hat{\sigma}_x\rangle$ positive and fulfill condition (30) for $\vec{n} = \hat{x}$.

(ii) If all elements $\bar{T}_{ij}$ are equal to zero for $i,j = 1,2$, then $\bar{T}_{33} = 1$ due to Eq. (26). What kind of states $|\psi_S\rangle$ are compatible with these values? Only those of the form of Eq. (28). This can be seen as follows: We can expand $|\psi_S\rangle = \sum_{m=-N/2}^{N/2} c_m|N/2,m\rangle_S$ in the basis of symmetric Dicke states. Then $1 = \langle\hat{\sigma}_z\hat{\sigma}_z\rangle = \sum_m |c_m|^2\langle N/2,m|\hat{\sigma}_z\hat{\sigma}_z|N/2,m\rangle = \sum_m |c_m|^2$, where the latter equality comes from the normalization of $|\psi_S\rangle$. It follows that c_m can only be different from zero if $\langle N/2,m|\hat{\sigma}_z\hat{\sigma}_z|N/2,m\rangle = 1$, which is the case for $m = \pm N/2$ only. These are the states of Eq. (28) with the notation $c_{N/2} = \sqrt{q}$ and $c_{-N/2} = e^{i\phi}\sqrt{1-q}$.

For $N > 2$, the coefficients s_i and T_{ij} $(i,j = 1,2)$ vanish for any value of q. In this case, $\langle\sigma_z\rangle = 2(q - \frac{1}{2})$, and condition (30) reads $(q-\frac{1}{2})^2 < \frac{(N-1)}{4N}$. This condition is violated if Eq. (29) holds. This suggests that if q is too close to 0 or to 1, then $F_Q \leqslant N$. What is left to show is that there is no other direction $\vec{n}$ where $F_Q > N$ for this state. This follows directly from Observation 1 since $\gamma_C = \mathrm{diag}(\frac{N}{4}, \frac{N}{4}, \frac{N^2}{4}[1-(2q-1)^2])$ is diagonal already. Hence, there is no better basis, and if Eq. (29) holds, then the entanglement of the state (28) is not useful for SSN interferometry in any direction $\vec{n}$.

In contrast, for $N = 2$, the coefficients s_i and T_{ij} for $i,j = 1,2$ vanish only if $q = 0$ or $q = 1$, that is, if $|\psi_S\rangle$ is a product state. It follows that any pure symmetric entangled two-qubit state is useful for SSN interferometry. ■

We point out three things concerning the states (28). (i) The region where the states are not useful shrinks with increasing N. (ii) When q is changed such that the states change from being useful to not being useful, then the optimal direction $\vec{n}$ changes from $\hat{z}$ to any direction in the x-y plane. This is not surprising, since for the product states $|0\rangle^{\otimes N}$ and $|1\rangle^{\otimes N}$, the variance of $\hat{J}_{\vec{n}}$ is maximized for $\vec{n}$ lying in the x-y plane, while the variance of the $N00N$ state $\frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$ is maximized for $\vec{n} = \hat{z}$ as seen in Sec. III B. However, one could have expected a smooth transition from $\vec{n} = \hat{z}$ to the x-y plane. (iii) The states of Eq. (28) are not separable with respect to any partition if $q \neq 0$ and $q \neq 1$, and hence genuinely multipartite entangled, but still of no use for SSN interferometry when condition (29) holds and only CLU can be applied to the input state.

C. LU operations

We have found that states of the form (28) are not useful for SSN interferometry if the condition (29) holds and if only CLU operations can be applied. It is natural to ask whether or not this can be changed by applying *arbitrary* LU operations on this state. It turns out, however, that this more general class of transformations does not help. This is the content of Proposition 2. Hence, not all pure entangled states are useful for SSN interferometry, even if arbitrary LU operations can be applied to the input state. The main results of this article regarding this question are summarized in Theorem 1.

Proposition 2. For a pure, symmetric, and entangled state $|\psi_S\rangle$ under LU operations the maximum quantum Fisher information is obtained by choosing a collective spin vector with $\vec{n}_{\max}$ determined as stated in Observation 1. For $N > 2$, any non-collective operation leads to a strictly smaller value of F_Q.

Proof. In order to apply Observation 2, we first have to construct γ_R as defined in Eq. (20). The terms $\langle\hat{\sigma}_i^{(k)}\rangle$ and $\langle\hat{\sigma}_i^{(k)}\hat{\sigma}_j^{(l)}\rangle$ do not depend on the sites k and l if $|\psi_S\rangle$ is symmetric. The resulting covariance matrix has the block form

$$\gamma_R = \begin{pmatrix} A & B & B & \cdots & B \\ B & A & B & \cdots & B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B & B & B & \cdots & A \end{pmatrix}, \tag{31}$$

where $A_{ij} = \delta_{ij} - \langle\hat{\sigma}_i\rangle\langle\hat{\sigma}_j\rangle$ and $B_{ij} = \langle\hat{\sigma}_i\hat{\sigma}_j\rangle - \langle\hat{\sigma}_i\rangle\langle\hat{\sigma}_j\rangle = B_{ji}$ are 3×3 matrices. With the notation introduced prior to Eq. (27), we can write $A = \mathbb{1} - \vec{s}\vec{s}^T$ and $B = T - \vec{s}\vec{s}^T$. The rank of γ_R is in general full, but there are at most six distinct eigenvalues. This can be seen as follows: If we find the three eigenvectors $\vec{a}_k$ of the matrix $[A + (N-1)B]$, then we can directly construct three eigenvectors of γ_R which are fully symmetric under interchange of the blocks, namely, $\vec{x}_k^T = (\vec{a}_k^T, \vec{a}_k^T, \ldots, \vec{a}_k^T)$, where $k = 1,2,3$. Furthermore, if we find three eigenvectors $\vec{b}_k$ of the matrix $[A - B]$, we obtain $3(N-1)$ linearly independent eigenvectors of γ_R of the form $[\vec{y}_k^{(j)}]^T = (\vec{b}_k^T, 0, \ldots, 0, -\vec{b}_k^T, 0, \ldots, 0)$, where the second vector $-\vec{b}_k$ is located at the positions of block j, $j = 2,3,\ldots,N$. These vectors are orthogonal to the vectors $\vec{x}_k$ by construction, so the spectrum of γ_R is given by the eigenvalues of the matrices $[A + (N-1)B]$ and $[A - B]$. Let us denote by λ_1 the largest eigenvalue of the first matrix and by λ_2 the largest eigenvalue of the second matrix. If $\lambda_1 \geqslant \lambda_2$, then the optimal F_Q can be reached by a collective spin operator with all spin operators pointing in the same direction, while if the inequality holds strictly, $\lambda_1 > \lambda_2$, then it is clear that the optimal F_Q reached with a collective operator is strictly larger than the largest F_Q that can be achieved with a non-collective spin operator.

Comparing λ_1 and λ_2 is equivalent to comparing the eigenvalues of $(N-1)T - N\vec{s}\vec{s}^T$ and $-T$. Let us denote the eigenvalues of the matrix T by t_i, $i = 1,2,3$ and order them increasingly. Due to Eq. (26), they fulfill $t_1 + t_2 + t_3 = 1$. The largest eigenvalue of $-T$ is hence given by $-t_1$. We have to consider several cases: (i) If $t_1 > 0$, then $-T$ has no positive eigenvalues, whereas $(N-1)T$ has only positive eigenvalues. Hence, there is always a vector $\vec{r}_\perp$ orthogonal to $\vec{s}$ such that $\vec{r}_\perp^T[(N-1)T - N\vec{s}\vec{s}^T]\vec{r}_\perp > 0 > -t_1$, which implies $\lambda_1 > \lambda_2$. (ii) If $t_1 \leqslant 0$, $-T$ has at most one positive eigenvalue $|t_1|$. There is a two-dimensional subspace $\tilde{S}$ such that for $\vec{r} \in \tilde{S}$, $\vec{r}^T T\vec{r} \geqslant |t_1|$ holds since $t_3 \geqslant t_2 \geqslant |t_1|$. The last inequality holds since $|t_i| \leqslant 1$ for all i. Hence there is a vector $\vec{r}_\perp \in \tilde{S}$ orthogonal to $\vec{s}$ with $\vec{r}_\perp^T[(N-1)T - N\vec{s}\vec{s}^T]\vec{r}_\perp \geqslant (N-1)t_2 \geqslant |t_1|$, implying $\lambda_1 \geqslant \lambda_2$.

So far we have shown that we can always choose a symmetric collective operator $\hat{J}_{\vec{n}}$. Let us focus now on the cases where $\lambda_1 = \lambda_2$ holds, where also nonsymmetric collective operators may reach the optimal F_Q. This may happen if $N = 2$ and $t_2 = |t_1|$ or if $N > 2$ and $t_1 = t_2 = 0$. In the first case, symmetric vectors $(\vec{n}^T, \vec{n}^T)^T$ and antisymmetric vectors $(\vec{n}^T, -\vec{n}^T)^T$ always reach the same optimum unless $t_1 = t_2 = 0$, when the state is separable. This can be seen by a direct calculation of λ with the general symmetric state $|\psi_S\rangle = c_1|1,1\rangle + c_{1,-1}|1,-1\rangle + c_0|1,0\rangle$ and by requiring that T be diagonal with $t_1 = -t_2$. In the second case, $T = \text{diag}[0,0,1]$. As mentioned in the proof of Proposition 1, this is only possible for states of the form (28), for which $\vec{s}^T = (0,0,\delta)$, where $\delta = 2(q - \frac{1}{2})$. Then the condition that $(N-1)T - N\vec{s}\vec{s}^T = \text{diag}[0,0,(N-1) - N\delta^2]$ has a strictly larger eigenvalue than $|t_1| = 0$ is fulfilled unless condition (29) holds. If it holds, then $F_Q = N$, and this can be reached by choosing $\vec{n}^{(k)} = (c_1^{(k)}, c_2^{(k)}, 0)^T$ for any $c_{1,2}^{(k)}$, as can be seen directly by writing down γ_R from Eq. (31) in this case. ■

Summarizing, this allows us to formulate a central result of this article.

Theorem 1. Allowing for general LU operations to be applied on the input state, then for $N = 2$, any pure entangled state is useful for SSN interferometry. For $N > 2$, there are pure entangled states which are not useful even if they can be transformed by arbitrary LU transformations. The pure entangled symmetric states which are not useful are completely characterized by Proposition 1.

Proof. Proposition 2 implies that even allowing for any LU operation does not make the states (28) useful for SSN interferometry if condition (29) holds. Therefore, for $N > 2$, there are pure entangled states which cannot be made useful. For $N = 2$ we have seen already that all states of the form $\sqrt{q}|00\rangle + e^{i\phi}\sqrt{1-q}|11\rangle$ are useful unless $q = 0$ or $q = 1$. In this case any state can be brought into this form by a local change of basis; therefore, any pure entangled state of two qubits is useful for SSN interferometry. ■

The result that all entangled states with $N = 2$ particles lead to $F_Q > N$ for some change of the local basis also follows directly from results obtained for the Wigner-Yanase skew information $I(\rho, \hat{H})$ depending on a state ρ and an observable $\hat{H}$ [69]. For any pure entangled state $|\psi_{\text{ent}}\rangle$ of $N = 2$ particles, it has been shown that $4I(|\psi_{\text{ent}}\rangle, \hat{H}) > 2$ can be achieved by local rotations [70]. This proof carries over to the Fisher information since for pure states, the quantities are related by $F(|\psi\rangle,\hat{H}) = 4I(|\psi\rangle,\hat{H})$.

D. More general local operations

So far we considered the scenario in which a single copy of a pure state is used to perform a phase estimation protocol. We allowed for local manipulations of this state prior to the experiment. Typically, investigations of quantum entanglement assume that the parties controlling the particles are very far apart and that they can only perform local operations on their system and classical communication (LOCC) [49]. In this scenario, the particles can be treated effectively as distinguishable [43] and general LU can be applied. More general local measurements can be performed when each party is allowed to add local particles, so-called ancillas, and to perform LU operations and measurements on the ancilla particles, discarding them after the operation [49]. They can be described with so-called Kraus operators $\hat{A}_i$, where i labels the results of the local measurements. They fulfill $\sum_i \hat{A}_i^\dagger \hat{A}_i = \mathbb{1}$ and transform the initial state as $|\psi_{\rm in}\rangle \to \sum_i \hat{A}_i|\psi_{\rm in}\rangle\langle\psi_{\rm in}|\hat{A}_i^\dagger$. With probability $\langle\psi_{\rm in}|\hat{A}_i^\dagger\hat{A}_i|\psi_{\rm in}\rangle$, the state is transformed as $|\psi_{\rm in}\rangle \to \hat{A}_i|\psi_{\rm in}\rangle$.

Let us assume that N parties share a state $|\psi\rangle$ of the form (28) with q such that the state is *not* useful, and let us choose $\phi = 0$ for convenience. Then a single party could perform the general measurement with the two-outcome measurement $\hat{A}_1 = \sqrt{1-q}|0\rangle\langle 0| + \sqrt{q}|1\rangle\langle 1|$ and $\hat{A}_2 = \sqrt{q}|0\rangle\langle 0| + \sqrt{1-q}|1\rangle\langle 1|$. With probability $P = 2q(1-q)$, the state is transformed into the $N00N$ state, while with probability $1-P$, the state $|\psi_2\rangle = (q|0\rangle^{\otimes N} + (1-q)|1\rangle^{\otimes N})/\sqrt{1-P}$ is obtained. Hence, in one case, the maximally useful $N00N$ state is obtained, while in the other case, the state is still as useful as the original state (namely, shot-noise limited). Therefore, the classification of usefulness changes in this situation. However, from an experimental point of view, CLU or LU operations are significantly easier to implement in general.

This result has an implication regarding a possible measure of entanglement which is useful for SSN interferometry. For LU operations $U_{\rm L}$, the quantity

$$e(\rho) = \max\left[0, \max_{U_{\rm L}} F_Q[\rho;\hat{J}_y] - N\right] \qquad (32)$$

defined for arbitrary mixed states ρ satisfies the following conditions which are typically required of an entanglement measure [71,72]: (i) $e(\rho) = 0$ for separable states and (ii) $e(\rho)$ is invariant under LU operations. However, the preceding example shows that it violates the postulate that the function should not increase on average under LOCC since

$$e(|\psi\rangle) < Pe(|N00N\rangle) + (1-P)e(|\psi_2\rangle) \qquad (33)$$

holds because we chose the initial state $|\psi\rangle$ such that $e(|\psi\rangle) = 0$.

V. EXAMPLES

A. Symmetric states

From Proposition 2 we know that that Observation 1 delivers the optimal F_Q for pure symmetric states. Hence, the results obtained for the three examples of symmetric states in Sec. III B are already optimal.

B. Singlet states

Singlet states of N qubits exist if N is even. By definition, these states fulfill (i) $U^{\otimes N}|\psi\rangle = e^{i\phi}|\psi\rangle$ for some phase ϕ and (ii) $\hat{\vec{J}}^2|\psi\rangle = 0$. It follows that $F_Q[|\psi\rangle;\hat{J}_{\vec{n}}] = 0$ holds for any direction $\vec{n}$. Hence, there is no CLU operation which makes these states useful for SSN interferometry with a Mach-Zehnder interferometer. Therefore, it is natural to ask whether they can be made useful with LU operations. This situation can only be achieved with bosons or fermions which can be individually addressed. In the case of Fermions occupying just two modes, the only entangled state which can occur in the particle picture is the two-particle singlet state $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, which is not useful for interferometry under CLU operations. An obvious example of a N qubit singlet state of individually addressable particles is the tensor product of $N/2$ two-qubit singlet states. For these states we already know that they can be made useful with LU operations from Theorem 1.

In the following, we consider the nontrivial family of N qubit singlet states defined in Ref. [73] as

$$\left|\mathcal{S}_N^{(2)}\right\rangle = \frac{1}{\frac{N}{2}!\sqrt{\frac{N}{2}+1}}\sum_{\mathcal{P}} z!\left(\frac{N}{2}-z\right)!(-1)^{N/2-z}\mathcal{P}[|01\rangle^{\otimes N/2}], \qquad (34)$$

where the sum runs over all permutations of the state $|01\rangle^{\otimes N/2}$ and z is the number of 0's in the first $\frac{N}{2}$ positions. Apart from (i) and (ii) they are (iii) multipartite entangled, (iv) invariant under the permutations $\mathcal{P}_{ij}|\mathcal{S}_N^{(2)}\rangle = |\mathcal{S}_N^{(2)}\rangle$ if $i,j \in [1,\ldots,\frac{N}{2}]$ or $i,j \in [\frac{N}{2}+1,\ldots,N]$, and (v) invariant up to the factor $(-1)^{N/2}$ under exchange of the first $\frac{N}{2}$ qubits and the second $\frac{N}{2}$ qubits. Due to the symmetries (iv) and (v), the covariance matrix of the states has the form

$$\gamma_R^{\rm singlet} = \begin{pmatrix} \tilde{A} & \tilde{C} \\ \tilde{C} & \tilde{A} \end{pmatrix}, \qquad (35)$$

where $\tilde{A}$ is a block matrix of the form of Eq. (31) and $\tilde{C}$ is a block matrix of 3×3 matrices C. Hence, we have to compute the matrices A, B, and C. Due to (i), the single-particle reduced states fulfill $U\rho^{(k)}U^\dagger = \rho^{(k)}$ for any unitary operation U. The only state of a single qubit with that property is $\mathbb{1}/2$, from which $\vec{s} = \vec{0}$ and $A = \mathbb{1}$ follows. Also due to (i), all reduced two-particle states fulfill $U\otimes U\rho^{(k,j)}U^\dagger\otimes U^\dagger = \rho^{(k,j)}$. The only states of two qubits with that property are the so-called Werner states [56],

$$\rho^{(k,l)} = f|\psi^-\rangle\langle\psi^-| + (1-f)\frac{\mathbb{1} - |\psi^-\rangle\langle\psi^-|}{3}, \qquad (36)$$

where $f \in [0,1]$. The two-qubit singlet state is the only pure state of two qubits fulfilling (i). It follows that

$$\lambda^{(k,l)} = \begin{pmatrix} 1 & \vec{0}^T \\ \vec{0} & \xi\mathbb{1} \end{pmatrix}, \qquad (37)$$

where $\xi = \frac{1}{3} - \frac{2}{3}f$ and the matrix is defined as in Eq. (27). Hence, the matrices A, B, and C are proportional to the identity. For reduced two-particle states within the sets considered in (iv), $f = 0$ has to hold since the reduced state is acting on the symmetric subspace only, and we obtain $B = \frac{1}{3}\mathbb{1}$. The missing parameter from C can be calculated by employing (ii) since it implies $\gamma_C = \hat{0}$, where γ_C is the collective covariance matrix introduced in Observation 1. Setting $C = \xi_C \mathbb{1}$, we find that this condition is fulfilled provided that

$$\xi_C = -\frac{2}{N^2}\left[\frac{2}{3}N + \frac{N^2}{6}\right]. \tag{38}$$

We can find the optimal directions $\{\vec{n}^{(k)}\}$ using Observation 2 by diagonalizing $\gamma_R^{\text{singlet}}$ and showing that the maximal eigenvector has the properties of $\vec{m}$ from Eq. (21). As expected, we find $\gamma_R^{\text{singlet}}\vec{m}^{(+)} = 0$ for symmetric eigenvectors $(\vec{m}^{(+)})^T = (\vec{n}^T,\vec{n}^T,\ldots,\vec{n}^T)$, while the vectors $(\vec{m}^{(-)})^T = (\vec{n}^T,\vec{n}^T,\ldots,-\vec{n}^T,\ldots,-\vec{n}^T)$ are eigenvectors with eigenvalue $\frac{4}{3} + \frac{1}{3}N$. In $\vec{m}^{(-)}$, the vectors for the particles $\frac{N}{2}+1,\ldots,N$ have the minus sign. Finally, the eigenvalue $\frac{2}{3}$ is shared by the vectors $\vec{x}_k^{(1)}$, which have vanishing elements except for a vector $\vec{n}$ at the positions of particle 1 and a vector $-\vec{n}$ at the entries of particle $k \in [2,\frac{N}{2}]$, and the vectors $\vec{x}_k^{(2)}$, which have vanishing elements except for a vector $\vec{n}$ at the positions of party $\frac{N}{2}+1$ and a vector $-\vec{n}$ at the entries of party $k \in [\frac{N}{2}+2,N]$. We did not further specify the vectors $\vec{n}$ because they are eigenvalues of the identity matrix in three dimensions since A, B, and C are proportional to $\mathbb{1}$.

Hence, the vectors $\vec{m}^{(-)}$ are the eigenvectors with the maximal eigenvalue, and we conclude that they lead to the maximal quantum Fisher information

$$F_Q^{\max}\big(\mathcal{S}_N^{(2)}\big) = \frac{N^2}{3} + \frac{4}{3}N, \tag{39}$$

surpassing the shot-noise limit for all N. This bound can be reached by keeping $|\mathcal{S}_N^{(2)}\rangle$ unchanged while choosing the collective operator $\hat{J}'$ such that $-\hat{\sigma}_y^{(k)}$ for the particles $k = 1,\ldots,\frac{N}{2}$, and $\hat{\sigma}_y^{(k)}$ for the remaining ones, for instance. If we consider instead a LU applied to the initial state and the Mach-Zehnder operator $\hat{J}_y$, then we can apply $\hat{\sigma}_z$ to the first $\frac{N}{2}$ parties only. Due to the definition of z in Eq. (34), $\frac{N}{2} - z$ is the number of 1's in the first $\frac{N}{2}$ positions. So the effect of this LU transformation is to remove the factor $(-1)^{N/2-z}$.

Any singlet state of N (N even) qubits can be obtained from superpositions of permutations of tensor-products of two-qubit singlet states. It is an interesting question whether *all* such states can be made useful for SSN interferometry with LU operations. The usefulness of singlet states in the mode picture has been considered recently in a different scenario in Ref. [12].

C. Graph states

Finally, we discuss the usefulness of the so-called graph states of N qubits, which recently have received large attention because of their importance for one-way quantum computation, quantum error correcting codes, studies of nonlocality,

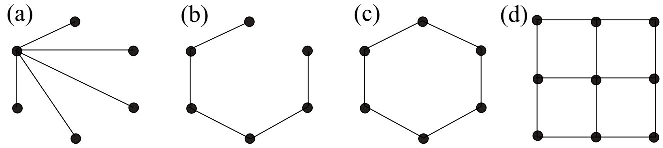

FIG. 4. Examples for graphs describing important graph states: (a) a GHZ or $N00N$ state (up to LU operations) [77,78], (b) a linear cluster graph, (c) a ring cluster graph, (d) a cluster graph in two dimensions.

and decoherence (see [74] and references within). After discussing general properties of graph states in relation to the usefulness for SSN interferometry, we consider the so-called cluster states and again the $N00N$ state (usually referred to as the GHZ state [75] in this context).

Let us first recall the definition of graph states. A graph G is a collection of N vertices and connections between them, which are called edges [74]. In a physical implementation, the vertices correspond to qubits and the edges record interactions (to be specified below) that have taken place between the qubits. For each vertex i we define the neighborhood $N(i)$, the set of vertices connected by an edge with i, and associate to it a stabilizing operator

$$\hat{K}_i = \hat{\sigma}_x^{(i)} \bigotimes_{j\in N(i)} \hat{\sigma}_z^{(j)}. \tag{40}$$

It is easy to see that all the stabilizing operators commute. The graph state $|G\rangle$ associated to the graph G is the unique N-qubit state fulfilling

$$\hat{K}_i|G\rangle = |G\rangle \quad \text{for } i = 1,2,\ldots,N. \tag{41}$$

From a physical point of view, one can also define a graph state as the state arising from $[(|0\rangle + |1\rangle)/\sqrt{2}]^{\otimes N}$, if between all connected qubits i,j the Ising-type interaction $\hat{H}_I = (\mathbb{1} - \sigma_z^{(i)}) \otimes (\mathbb{1} - \sigma_z^{(j)})$ acts for the time $t = \pi/4$. See Fig. 4 for examples of prominent graph states.

The group of products of the $\hat{K}_i$ is called stabilizer $\mathcal{S}$ [76]. The state $|G\rangle$ can be expressed with the elements of the stabilizer [77,78],

$$|G\rangle\langle G| = \frac{1}{2^N}\sum_{s\in\mathcal{S}} s. \tag{42}$$

This form is particularly useful for our purpose, because it makes it possible to read off directly the reduced one- and two-qubit density matrices in the form of Eq. (25).

Observation 3. (i) For the reduced state of p qubits, products of at most p stabilizers $\hat{K}_i$ contribute. (ii) For $p = 1$, the reduced state is $\rho_i^{(r)} = \frac{1}{2}\mathbb{1}$ unless qubit i is not connected to any other qubit, in which case $\rho_i^{(r)} = \frac{1}{2}(\mathbb{1} + \hat{\sigma}_x^{(i)})$. (iii) If ρ_{ij} is a reduced state of $p = 2$ qubits, then the stabilizers $\hat{K}_i$ (or $\hat{K}_j$) contribute if i is the only neighbor of j (or vice versa). Also, the products $\hat{K}_i\hat{K}_j$ contribute if the qubits i and j have the same neighbors, where it is irrelevant whether i and j are neighbors themselves.

Proof. All elements s of the stabilizer have the form $\pm \otimes_{i=1}^N \hat{\sigma}_{j^{(i)}}^{(i)}$. Hence $\text{Tr}_i[s] = 0$ unless $j^{(i)} = 0$ (since then $\hat{\sigma}_{j^{(i)}}^{(i)} = \mathbb{1}$) and, in analogy, if more than one qubit are traced out [74]. If we compute the reduced one- and two-qubit density matrices from Eq. (42), then only those s will contribute which act as the identity on the traced-out particles. More specifically, products of $\tilde{p}$ stabilizers are of the form

$$\prod_{k=1}^{\tilde{p}} \hat{K}_{i_k} \propto \left(\bigotimes_{k=1}^{\tilde{p}} \hat{\sigma}_x^{(i_k)}\right)\left(\bigotimes_{k=1}^{\tilde{p}} \left(\otimes_{j_k\in N(i_k)} \hat{\sigma}_z^{(j_k)}\right)\right). \quad (43)$$

Since $[\hat{K}_i,\hat{K}_j]=0$ and $\hat{K}_i^2=\mathbb{1}$ for all i and j, we only have to consider products of different stabilizers for a given $\tilde{p}$. Then (i) follows because if $\tilde{p}$ is larger than the number of qubits p in the reduced state, one or more $\hat{\sigma}_x$ operators remain acting on the rest, which remain traceless even when multiplied by the $\hat{\sigma}_z$ operators acting on the neighborhoods. While (ii) and the first part of (iii) follow directly, the second part of (iii) follows because otherwise $\hat{\sigma}_z$ operators would be left acting on qubits which are traced out. ■

Since we are not interested in the situation where a qubit is fully separable from the rest, we can assume that $\vec{s}^{(k)}=\vec{0}$ for any k in the following, since the reduced states are equal to $\frac{1}{2}\mathbb{1}$ in this case. From Observation 3 it follows directly that cluster states of all kinds are practically of no more use for SSN interferometry than product states.

Proposition 3. The maximal quantum Fisher information $F_Q^{\max}$ of linear cluster states with $N\geqslant 4$ particles is $N+4$. For $N\geqslant 5$ qubits, ring cluster states as well as cluster states in more than one dimension have $F_Q^{\max}=N$.

Proof. For $N=3$ the linear cluster state is LU equivalent to a GHZ state and for $N=4$ the ring cluster state is equivalent to a linear cluster state [74]. The claim for the ring cluster state and the cluster states in more than one dimension follows directly from Observation 3, as all reduced two-qubit density matrices are of the form $\rho^{(r)}=\frac{1}{4}\mathbb{1}$, and hence $\gamma_R=\mathbb{1}$. Then Observation 2 yields $F_Q^{\max}\leqslant N$. For linear cluster states, there are four off-diagonal elements of γ_R coming from the stabilizers $\hat{K}_1=\hat{\sigma}_x^{(1)}\otimes\hat{\sigma}_z^{(2)}$ and $\hat{K}_N=\hat{\sigma}_z^{(N-1)}\otimes\hat{\sigma}_x^{(N)}$ at the ends of the cluster. Writing down γ_R in the block order $1,2,N-1,N,3,4,\ldots,N-2$ yields

$$\gamma_R=\begin{pmatrix}\mathbb{1} & \hat{x}\hat{z}^T\\ \hat{z}\hat{x}^T & \mathbb{1}\end{pmatrix}\oplus\begin{pmatrix}\mathbb{1} & \hat{z}\hat{x}^T\\ \hat{x}\hat{z}^T & \mathbb{1}\end{pmatrix}\oplus\mathbb{1}, \quad (44)$$

where $\hat{x}^T=(1,0,0)$ and $\hat{z}^T=(0,0,1)$. This matrix can be reordered as $\gamma_R=(2|x+\rangle\langle x+|)\oplus(2|x+\rangle\langle x+|)\oplus\mathbb{1}_{N-4}$, where $|x+\rangle=\frac{1}{\sqrt{2}}\binom{1}{1}$. Hence, from Observation 2 it follows that $F_Q^{\max}\leqslant 2N$. However, this limit cannot be reached due to the restriction on $\vec{m}$. Due to the block-diagonal structure of γ_R, the largest expectation value is obtained by choosing $\vec{m}^T=(\hat{x}^T,\hat{z}^T,\vec{n},\ldots,\vec{n},\hat{z}^T,\hat{x}^T)$, where $\vec{n}$ may point in any direction, which leads to $F_Q=N+4$. ■

A phase estimation scheme for one-dimensional cluster states enabling SSN sensitivity was suggested in Ref. [19]. In contrast to the situation we considered, the authors suggested using superpositions of cluster states and a non-collective generator of the phase shift, showing that in this case SSN sensitivity is possible even in the presence of noise.

Let us finally illustrate why the GHZ states have the largest Fisher information possible from the point of view of graph states. From Fig. 4(a) we see that if the qubit 1 is connected to all the others then all $\hat{K}_i$, $i\neq 1$ contribute to the reduced two-qubit states since all those qubits have only one neighbor. Further, all the products $\hat{K}_i\hat{K}_j$ for $i,j\neq 1$ contribute since these qubits have all the same neighborhood. The covariance matrix γ_R then takes the form

$$\gamma_R=\begin{pmatrix}\mathbb{1} & \hat{z}\hat{x}^T & \hat{z}\hat{x}^T & \cdots & \hat{z}\hat{x}^T\\ \hat{x}\hat{z}^T & \mathbb{1} & \hat{x}\hat{x}^T & \cdots & \hat{x}\hat{x}^T\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \hat{x}\hat{z}^T & \hat{x}\hat{x}^T & \hat{x}\hat{x}^T & \cdots & \mathbb{1}\end{pmatrix}. \quad (45)$$

It can be directly checked that $\vec{m}^T=\frac{1}{\sqrt{N}}(\hat{z}^T,\hat{x}^T,\hat{x}^T,\ldots,\hat{x}^T,\hat{z}^T)$ is the eigenvector of γ_R corresponding to the maximal eigenvalue N, and hence $F_Q^{\max}=N^2$.

VI. CONCLUSION

We have studied linear two-mode interferometers from a quantum information theory perspective. In particular, we have addressed the question of whether all pure entangled states of N particles can achieve SSN sensitivity in such interferometers if they can be optimized by operations which are local in the particles. We used the Cramér-Rao theorem, which gives a lower bound on the optimal sensitivity *via* the quantum Fisher information F_Q. For $F_Q>N$, SSN sensitivity can be achieved in the central limit.

We have studied the maximal quantum Fisher information F_Q that can be achieved for a general two-state linear interferometer such as the Mach-Zehnder interferometer. We have found a simple way to determine the optimal CLU operation, and an upper bound for the optimal F_Q for LU operations, which is tight in many cases. The optimizations carry over directly to the mixed state case and are useful for the experimental optimization of the source if tomographic data of the state are available.

Using these results, we have fully characterized the pure symmetric entangled states which are of no more use than noncorrelated states under CLU operations. These states and operations are available in bosonic two-mode interferometers. Further, we have obtained that for symmetric states of particles which can be individually addressed, a CLU operation achieves the maximal F_Q even if arbitrary LU can be applied. From these results it follows that while for $N=2$ any entangled state can be made useful with LU operations, there are pure entangled states, and even fully N-partite entangled states, which are not useful for SSN interferometry. We briefly commented that this picture changes when more general local operations are available.

Finally, we discussed several interesting states from the literature, finding the optimal sensitivity that they can deliver. Among them, we find that the highly entangled cluster states, which comprise a resource for one-way quantum computation [74], are practically not more useful than separable states.

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[1] V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004).
[2] C. M. Caves, Phys. Rev. D **23**, 1693 (1981).
[3] B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A **33**, 4033 (1986).
[4] M. J. Holland and K. Burnett, Phys. Rev. Lett. **71**, 1355 (1993).
[5] J. P. Dowling, Phys. Rev. A **57**, 4736 (1998).
[6] A. Sørensen, L.-M. Duan, J. I. Cirac, and P. Zoller, Nature (London) **409**, 63 (2001).
[7] R. A. Campos, C. C. Gerry, and A. Benmoussa, Phys. Rev. A **68**, 023810 (2003).
[8] J. J. Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, Phys. Rev. A **54**, R4649 (1996).
[9] L. Pezzé and A. Smerzi, Phys. Rev. A **73**, 011801(R) (2006).
[10] L. Pezzé and A. Smerzi, Phys. Rev. Lett. **100**, 073601 (2008); e-print arXiv:1004.5486.
[11] H. Uys and P. Meystre, Phys. Rev. A **76**, 013804 (2007).
[12] H. Cable and G. A. Durkin, Phys. Rev. Lett. **105**, 013603 (2010).
[13] D. W. Berry and H. M. Wiseman, Phys. Rev. Lett. **85**, 5098 (2000).
[14] B. L. Higgins, D. W. Berry, S. D. Bartlett, H. M. Wiseman, and G. J. Pryde, Nature (London) **450**, 393 (2007).
[15] D. W. Berry, B. L. Higgins, S. D. Bartlett, M. W. Mitchell, G. J. Pryde, and H. M. Wiseman, Phys. Rev. A **80**, 052114 (2009).
[16] Y. P. Huang and M. G. Moore, Phys. Rev. Lett. **100**, 250406 (2008).
[17] U. Dorner, R. Demkowicz-Dobrzanski, B. J. Smith, J. S. Lundeen, W. Wasilewski, K. Banaszek, and I. A. Walmsley, Phys. Rev. Lett. **102**, 040403 (2009).
[18] R. Demkowicz-Dobrzanski, U. Dorner, B. J. Smith, J. S. Lundeen, W. Wasilewski, K. Banaszek, and I. A. Walmsley, Phys. Rev. A **80**, 013825 (2009).
[19] M. Rosenkranz and D. Jaksch, Phys. Rev. A **79**, 022103 (2009).
[20] M. W. Mitchell, J. S. Lundeen, and A. M. Steinberg, Nature (London) **429**, 161 (2004).
[21] P. Walther, J.-W. Pan, M. Aspelmeyer, R. Ursin, S. Gasparoni, and A. Zeilinger, Nature (London) **429**, 158 (2004).
[22] T. Nagata, R. Okamoto, J. L. O'Brien, K. Sasaki, and S. Takeuchi, Science **316**, 726 (2007).
[23] J. C. F. Matthews, A. Politi, A. Stefanov, and J. L. O'Brien, Nat. Photonics **3**, 346 (2009).
[24] W.-B. Gao, C.-Y. Lu, X.-C. Yao, P. Xu, O. Gühne, A. Goebel, Y.-A. Chen, C.-Z. Peng, Z.-B. Chen, and J.-W. Pan, Nat. Phys. **6**, 331 (2010).
[25] D. Leibfried *et al.*, Nature (London) **438**, 639 (2005).
[26] C. Orzel, A. K. Tuchman, M. L. Fenselau, M. Yasuda, and M. A. Kasevich, Science **291**, 2386 (2001).
[27] G.-B. Jo, Y. Shin, S. Will, T. A. Pasquini, M. Saba, W. Ketterle, D. E. Pritchard, M. Vengalattore, and M. Prentiss, Phys. Rev. Lett. **98**, 030407 (2007).
[28] J. Estève, C. Gross, A. Weller, S. Giovanazzi, and M. K. Oberthaler, Nature (London) **455**, 1216 (2008).
[29] C. Gross, T. Zibold, E. Nicklas, J. Estève, and M. K. Oberthaler, Nature (London) **464**, 1165 (2010).
[30] M. F. Riedel, P. Böhi, Y. Li, T. W. Hänsch, A. Sinatra, and P. Treutlein, Nature (London) **464**, 1170 (2010).
[31] T. Fernholz, H. Krauter, K. Jensen, J. F. Sherson, A. S. Sørensen, and E. S. Polzik, Phys. Rev. Lett. **101**, 073601 (2008).
[32] K. Goda, O. Miyakawa, E. E. Mikhailov, S. Saraf, R. Adhikari, K. McKenzie, R. Ward, S. Vass, A. J. Weinstein, and N. Mavalvala, Nat. Phys. **4**, 472 (2008).
[33] H. Vahlbruch, M. Mehmet, S. Chelkowski, B. Hage, A. Franzen, N. Lastzka, S. Goßler, K. Danzmann, and R. Schnabel, Phys. Rev. Lett. **100**, 033602 (2008).
[34] A. Luis, Phys. Lett. A **329**, 8 (2004).
[35] S. M. Roy and S. L. Braunstein, Phys. Rev. Lett. **100**, 220501 (2008).
[36] S. Boixo, S. T. Flammia, C. M. Caves, and J. M. Geremia, Phys. Rev. Lett. **98**, 090401 (2007).
[37] S. Boixo, A. Datta, M. J. Davis, S. T. Flammia, A. Shaji, and C. M. Caves, Phys. Rev. Lett. **101**, 040403 (2008).
[38] S. Choi and B. Sundaram, Phys. Rev. A **77**, 053613 (2008).
[39] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett. **96**, 010401 (2006).
[40] L. Pezzé and A. Smerzi, Phys. Rev. Lett. **102**, 100401 (2009).
[41] The connection between shot-noise limit and separability for a nonfixed number of particles has been discussed in P. Hyllus, L. Pezzé, and A. Smerzi, e-print arXiv:1003.0649.
[42] H. Häffner, C. F. Roos, and R. Blatt, Phys. Rep. **469**, 155 (2008).
[43] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1995).
[44] R. H. Dicke, Phys. Rev. **93**, 99 (1954).
[45] G. Tóth and O. Gühne, Phys. Rev. Lett. **102**, 170503 (2009).
[46] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
[47] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[48] M. G. A. Paris, Int. J. Quantum. Inform. **7**, 125 (2009).
[49] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, UK, 2000).
[50] H. Cramer, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, NJ, 1946), p. 500.
[51] W. K. Wootters, Phys. Rev. D **23**, 357 (1981).
[52] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).
[53] S. L. Braunstein, C. M. Caves, and G. J. Milburn, Ann. Phys. **247**, 135 (1996).
[54] M. L. Cohen, IEEE Trans. Inf. Theory **14**, 591 (1968).
[55] A. Fujiwara, Phys. Rev. A **63**, 042304 (2001).
[56] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[57] S. L. Braunstein, A. S. Lane, and C. M. Caves, Phys. Rev. Lett. **69**, 2153 (1992); A. S. Lane, S. L. Braunstein, and C. M. Caves, Phys. Rev. A **47**, 1667 (1993).
[58] It has been argued that for indistinguishable particles, the entanglement defined in this way is not physical since the individual particles cannot be addressed [59,60]. Nonetheless, in the context of interferometry, it is a useful resource for SSN sensitivity.
[59] P. Zanardi, Phys. Rev. A **65**, 042101 (2002).
[60] F. Benatti, R. Floreanini, and U. Marzolino, Ann. Phys. **325**, 924 (2010).
[61] J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, Reading, MA, 1994).
[62] The conversion from $\vec{n}_{\max}$ to $U_{\rm CL}^{\rm opt} = \otimes_{k=1}^{N} U^{\rm opt}$ can be done as follows. If $\vec{n}_{\max} = O^T \hat{y}$ has been found, then from $\hat{y} \cdot \vec{n}_{\max} = \cos(\phi)$ and $\hat{y} \times \vec{n}_{\max} = \sin(\phi)\hat{n}$. the angle ϕ and the axis of rotation $\hat{n}$ can be extracted and $U^{\rm opt} = \exp(i\phi\hat{\vec{\sigma}} \cdot \hat{n}/2)$ is (up

to a phase) the unitary matrix such that $(U^{\text{opt}})^\dagger \hat{\sigma}_{\hat{y}} U^{\text{opt}} = \hat{\sigma}_{\vec{n}_{\max}}$ [61]. The positive sign in the exponent is due to the fact that O^T appears in the definition of $\vec{n}_{\max}$.

[63] Á. Rivas and A. Luis, Phys. Rev. A **77**, 022105 (2008).

[64] G. Tóth, C. Knapp, O. Gühne, and H. J. Briegel, Phys. Rev. A **79**, 042334 (2009).

[65] To see this, note that Eq. (30d) in Ref. [64] implies that for separable states $\lambda_{\max}(N\gamma) \leqslant (N-1)\text{Tr}(\gamma) - N(N-2)/4$ since $\mathfrak{X} = N\gamma + \langle \tilde{\vec{J}} \rangle \langle \tilde{\vec{J}} \rangle^T \geqslant N\gamma$. Using Eq. (30b) implies then that $N\lambda_{\max}(\gamma) \leqslant N(N-1)/2 - N(N-2)/4$; hence, $\lambda_{\max}(\gamma) \leqslant N/4$. We used the notations of Ref. [64] here.

[66] H. Lee, P. Kok, and J. P. Dowling, J. Mod. Opt. **49**, 2325 (2002).

[67] A. Shimizu and T. Miyadera, Phys. Rev. Lett. **89**, 270403 (2002).

[68] T. Morimae, A. Sugita, and A. Shimizu, Phys. Rev. A **71**, 032317 (2005).

[69] E. P. Wigner and M. M. Yakanse, Proc. Natl. Acad. Sci. USA **49**, 910 (1963).

[70] Z. Chen, Phys. Rev. A **71**, 052302 (2005).

[71] M. B. Plenio and S. Virmani, Quantum Inf. Comput. **7**, 1 (2007).

[72] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[73] A. Cabello, Phys. Rev. A **68**, 012304 (2003).

[74] M. Hein, W. Dür, J. Eisert, R. Raussendorf, M. V. den Nest, and H.-J. Briegel, in *Proceedings of the International School of Physics "Enrico Fermi" on Quantum Computers, Algorithms and Chaos*, edited by P. Casati, D. Shepelyansky, and G. Beneti (IOS Press, Varenna, Italy, 2005); see also e-print arXiv:quant-ph/0602096.

[75] D. M. Greenberger, M. Horne, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).

[76] D. Gottesman, Phys. Rev. A **54**, 1862 (1996).

[77] M. Hein, J. Eisert, and H. J. Briegel, Phys. Rev. A **69**, 062311 (2004).

[78] W. Dür, H. Aschauer, and H.-J. Briegel, Phys. Rev. Lett. **91**, 107903 (2003).