Continuous-variable teleportation of a negative Wigner function

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Teleportation is a basic primitive for quantum communication and quantum computing. We address the problem of continuous-variable (unconditional and conditional) teleportation of a pure single-photon state and a mixed attenuated single-photon state generally in a nonunity-gain regime. Our figure of merit is the maximum negativity of the Wigner function, which demonstrates a highly nonclassical feature of the teleported state. We find that the negativity of the Wigner function of the single-photon state can be *unconditionally* teleported for an arbitrarily weak squeezed state used to create the entangled state shared in teleportation. In contrast, for the attenuated single-photon state there is a strict threshold squeezing one has to surpass to successfully teleport the negativity of its Wigner function. The *conditional* teleportation allows one to approach perfect transmission of the single photon for an arbitrarily low squeezing at a cost of decrease of the success rate. In contrast, for the attenuated single photon state, conditional teleportation cannot overcome the squeezing threshold of the unconditional teleportation and it approaches negativity of the input state only if the squeezing increases simultaneously. However, as soon as the threshold squeezing is surpassed, conditional teleportation still pronouncedly outperforms the unconditional one. The main consequences for quantum communication and quantum computing with continuous variables are discussed.

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I. INTRODUCTION

Quantum teleportation is a fundamental primitive in quantum information [1–3]. Principally, it allows remote decomposition of the quantum state to noise and the necessary classical information required to uncover that state from the noise. In the quantum key distribution, teleportation between distant parties combined with quantum repeaters can transmit fragile quantum resources over a large distance [4–6]. In quantum computation, teleportation allows for fault-tolerant deterministic implementation of a difficult quantum gate on an unknown quantum state [7,8].

For quantum states in an infinite-dimensional Hilbert space, quantum squeezing is an irreducible resource for universal quantum teleportation [2,3]. Quantum squeezed states are states with variance of a quadrature below the vacuum noise. Although the squeezed states are nonclassical, as they have no regular and positive Glauber-Sudarshan quasiprobability distribution, their nonclassicality can still be simulated by semiclassical methods. This results from the fact that squeezing is simply observable in a Gaussian approximation, since the squeezed states are represented there by positive and regular Wigner functions, which then play the role of probability distributions. Thus the Wigner function of a squeezed state can be obtained simply by deforming the stochastic phase-space evolution of an irreducible vacuum state [9–11]. It does not prevent, for example, quantum key distribution for a limited distance, but it does not allow universal quantum computing [12]. In contrast, nonclassicality substantially reflecting a discrete particle structure of quantum states cannot be efficiently simulated by these stochastic methods [11]. The corresponding Wigner function of the particlelike state can exhibit negative values, invalidating its interpretation as any kind of classical probability density. The negative values are considered as a clear experimental demonstration of quantum features beyond the semiclassical description [13].

Neither long-distance quantum key distribution nor quantum computing can be performed solely based on the squeezing resource. Highly nonclassical repeaters in the sequential teleportation protocol [4–6] or many highly nonclassical cubic phase gates [14] in a complex quantum computer change propagating states to non-Gaussian states with a negative Wigner function. In both cases, teleportation of the negative Wigner function with just squeezing as a resource is a basic element of the communication and computation tasks. In efficient quantum key distribution with quantum repeaters, Gaussian teleportation should, at least *probabilistically*, allow propagation of the negativity of the Wigner function produced by the repeater operation through the network toward the next quantum repeater. In quantum computation, it should even *deterministically* realize a basic highly nonlinear cubic phase gate [14], if such the offline gate is, at least probabilistically, feasible.

The quality of teleportation is mainly limited by the finite squeezing resource. How much squeezing resource is actually required to at least partially maintain the negativity of the Wigner function through the teleportation step? The answer to this question determines how much squeezing is necessary to decompose higher nonclassical states. Since teleportation is the basic primitive for quantum communication and computation, it also specifies the amount of squeezing needed to deterministically (or probabilistically) operate highly nonclassical states. In this paper, we give a clear and illustrative answer to this basic question. As the first testing state, the single-photon Fock state having the maximal possible negativity of the Wigner function is considered at the input of teleportation. The negativity is then lowered by a loss implemented on the single-photon state. Our attempt is to show directly the effect of Gaussian teleportation on the different

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values of negativity of the Wigner function, thus judging its possible application in long-distance quantum communication and quantum cryptography.

The mechanism of teleportation of a single-photon state (or superposition of the coherent states) was analyzed in Refs. [3] and [15–17], but always using the fidelity [18,19], entanglement fidelity [20], or photon-number distribution [21], as a figure of merit. For unity-gain teleportation, the fidelity benchmark 2/3 was found to be a necessary condition to achieve teleportation of a negative Wigner function [22]. This corresponds to −3 dB of squeezing required in entanglement preparation. However, fidelity does not tell directly if the teleported state still has some negativity of the Wigner function and how large it is.

In this paper we investigate the capability of the standard teleportation protocol [3] to successfully teleport negativity of the Wigner function in the origin of the single-photon Fock state and the convex mixture of the state with the vacuum state. Our goal is to find teleportation protocols minimizing the value of the output Wigner function in the origin. We find that for the single-photon Fock state an arbitrarily small nonzero squeezing suffices to successfully teleport a negative value of the Wigner function in the origin if the gain of teleportation is chosen suitably. In contrast, to teleport a negative value of the Wigner function in the origin of the attenuated single-photon Fock state, one has to surpass a strict threshold level of squeezing. For both cases of input states one can attain a substantially larger negative value of the output Wigner function in the origin by using conditional teleportation with a reasonably high success probability. The postselection cannot improve the squeezing threshold, however, if it is surpassed, a higher negativity of the Wigner function can be achieved. The sufficient tolerance of conditional teleportation to impurity of the squeezed states used to produce the shared entangled state demonstrates the feasibility of conditional teleportation.

The paper is organized as follows. Section II deals with unity gain, optimal nonunity gain, and conditional teleportation of a single-photon and squeezed single-photon Fock state. In Sec. III we study unity gain, optimal nonunity gain, and conditional teleportation of a convex mixture of a single-photon Fock state and the vacuum state. Section IV reports the conclusions.

II. TELEPORTATION OF A SINGLE-PHOTON FOCK STATE

At the outset we focus on understanding the basic effects of continuous-variable teleportation on the negativity of the Wigner function. For this purpose we start with the simple case of teleportation of a single-photon Fock state $|1\rangle$. The state is described by the following Wigner function [23]:

$$W_{\rm in}(r_{\rm in}) = \frac{1}{\pi}\left(2r_{\rm in}^T r_{\rm in} - 1\right)\exp\left(-r_{\rm in}^T r_{\rm in}\right), \qquad (1)$$

where $r_{\rm in} = (x_{\rm in}, p_{\rm in})^T$ is the radius vector in phase space. In the origin the function attains the minimum possible negative value allowed by quantum mechanics, equal to $W_{\rm in}(0) = -1/\pi \doteq -0.3181$, where here 0 stands for a zero 2×1 vector.

We consider a standard continuous-variable teleportation protocol [3,24] in the nonunity-gain regime [25]. An input mode characterized by the quadrature operators $x_{\rm in}, p_{\rm in}$ satisfying the canonical commutation rules $[x_{\rm in}, p_{\rm in}] = i$ prepared in the Fock state $|1\rangle$ is teleported by Alice (A) to Bob (B). Initially, Alice and Bob hold modes A and B, respectively, described by the quadratures x_i, p_i, $i = A,B$, in a pure two-mode squeezed vacuum state with squeezed Einstein-Podolsky-Rosen variances $\langle[\Delta(x_A - x_B)]^2\rangle = \langle[\Delta(p_A + p_B)]^2\rangle = e^{-2r}$, where r is the squeezing parameter. The state can be prepared by mixing of two pure squeezed states with squeezed variances $V_{\rm sq} = \langle(\Delta p_A)^2\rangle = \langle(\Delta x_B)^2\rangle = e^{-2r}/2$ on a balanced beam splitter. Next, Alice superimposes the input mode with mode A of the shared entangled state on an unbalanced beam splitter with reflectivity $\sqrt{R}$ and transmissivity $\sqrt{T}$ ($R + T = 1$) and measures the quadratures $x_u = \sqrt{R}x_{\rm in} - \sqrt{T}x_A$ and $p_v = \sqrt{T}p_{\rm in} + \sqrt{R}p_A$ at the outputs of the beam splitter. She then sends the measurement outcomes $\bar{x}_u, \bar{p}_v$ via the classical channel to Bob who displaces his mode B as $x_B \to x_B + g_x\bar{x}_u$, $p_B \to p_B + g_p\bar{p}_v$, where g_x, g_p are electronic gains, thereby partially re-creating the input state on mode B.

From the mathematical point of view nonunity-gain teleportation belongs to the class of single-mode trace-preserving completely positive Gaussian maps [26]. On the level of Wigner functions such a map transforms the Wigner function of the input state $W_{\rm in}(r_{\rm in})$ according to the integral formula [27]:

$$W_{\rm out}(r_{\rm out}) = 2\pi\int_{-\infty}^{+\infty} W_\chi(r_{\rm in}, r_{\rm out})W_{\rm in}(\Lambda r_{\rm in})dr_{\rm in}, \qquad (2)$$

where $r_{\rm out} = (x_{\rm out}, p_{\rm out})^T$, $\Lambda = {\rm diag}(1,-1)$, and W_χ is the following two-mode Gaussian kernel:

$$W_\chi(r_{\rm in}, r_{\rm out}) = \frac{1}{2\pi^2\sqrt{\det Q}}\exp(-\Delta r^T Q^{-1}\Delta r), \qquad (3)$$

where $\Delta r = r_{\rm out} - S\Lambda r_{\rm in}$, Q is a real symmetric positive semidefinite 2×2 matrix, S is a real 2×2 matrix, and the matrices satisfy the inequality $Q + iJ - iSJS^T \geqslant 0$, where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For nonunity-gain teleportation we have, in particular,

$$S = \begin{pmatrix} g_x\sqrt{R} & 0 \\ 0 & g_p\sqrt{T} \end{pmatrix} \qquad (4)$$

and $Q = {\rm diag}(Q_x, Q_p)$, where

$$\begin{aligned} Q_x &= \cosh(2r) + g_x^2 T\cosh(2r) - 2g_x\sqrt{T}\sinh(2r), \\ Q_p &= \cosh(2r) + g_p^2 R\cosh(2r) - 2g_p\sqrt{R}\sinh(2r). \end{aligned} \qquad (5)$$

Substituting Eqs. (4) and (5) into Eq. (3) and calculating the integral in Eq. (2) for the input Wigner function given by Eq. (1), we find the output Wigner function in the origin in the form

$$W_{\rm out}(0) = \frac{\det Q - (\det S)^2}{\pi[\det(SS^T + Q)]^{\frac{3}{2}}}. \qquad (6)$$

From a practical point of view it is important to know the largest negative value of the Wigner function that can be obtained at the output of teleportation for a given level of shared entanglement. This requires minimization of function (6) for a fixed r over three variables g_x, g_p, and T, which can barely be done analytically. Numerical minimization, however, indicates that as one would expect, optimal performance of teleportation is achieved when the beam splitter is balanced, that is, $\sqrt{R} = \sqrt{T} = 1/\sqrt{2}$, and when the teleportation adds noise symmetrically into position and momentum quadrature, that is, $g_x = g_p = g$. Under these assumptions and introducing the normalized gain $G = g/\sqrt{2}$, we can express the Wigner function in the origin as

$$W_{\text{out}}(0) = \frac{\alpha(G) - G^2}{\pi[\alpha(G) + G^2]^2}, \tag{7}$$

where $\alpha(G) = \cosh(2r)(1 + G^2) - 2G\sinh(2r)$. From the nominator in Eq. (7) it follows that for $r > 0$ there always exists a gain G for which the Wigner function (7) is negative. Solving the extremal equation $dW_{\text{out}}(0)/dG = 0$ with respect to G, one can find the optimal gain as a root of the following third-order polynomial:

$$G^3 + aG^2 + bG + c = 0, \tag{8}$$

where

$$a = -3\coth(r), \quad b = 2 + \coth^2(2r) + 3\frac{\cosh(2r)}{\sinh^2(2r)}, \\ c = -\coth(2r). \tag{9}$$

The polynomial has three real roots of the form $G_{1,2,3} = y_{1,2,3} + \coth(r)$, where

$$y_1 = 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\phi}{3}\right), \quad y_{2,3} = -2\sqrt{-\frac{p}{3}}\cos\left(\frac{\phi \pm \pi}{3}\right), \tag{10}$$

where $\cos\phi = -(q/2)\sqrt{-27/p^3}$ and

$$p = b - \frac{a^2}{3}, \quad q = c - \frac{ab}{3} + \frac{2a^3}{27}. \tag{11}$$

Substituting the roots $G_{1,2,3}$ back into the right-hand side of Eq. (7) and plotting the dependence of the obtained function on the squeezed variance V_{sq}, one finds the optimal gain G_{opt} minimizing the output Wigner function in the origin to be $G_{\text{opt}} = G_2$. The dependence of the output Wigner function in the origin on the squeezed variance V_{sq} for optimal nonunity-gain teleportation is depicted by the solid curve in Fig. 1. The figure reveals that the Wigner function in the origin is a monotonously decreasing function of the squeezing approaching the minimum value of $W_{\text{in}}(0) = -1/\pi \doteq -0.3181$ in the limit of infinitely large squeezing. Figure 1 further shows that optimal nonunity-gain teleportation transfers the negative values of the Wigner function successfully for arbitrarily small nonzero squeezing $r > 0$. The latter finding should be contrasted with the unity-gain regime that is recovered for $\sqrt{R} = \sqrt{T} = 1/\sqrt{2}$ and $g_x = g_p = \sqrt{2}$. Then equations (4) and (5) give $S = \mathbb{1}$, $Q = 2e^{-2r}\mathbb{1}$, which leads, using Eq. (6), to the output Wigner function in the origin, in the form

$$\tilde{W}_{\text{out}}(0) = \frac{2e^{-2r} - 1}{\pi(2e^{-2r} + 1)^2}. \tag{12}$$

Hence it immediately follows that in the unity-gain regime the output Wigner function in the origin is negative only if $e^{-2r} < 1/2$, that is, if the squeezing is larger than −3 dB (see also the dashed curve in Fig. 1), which corresponds to the

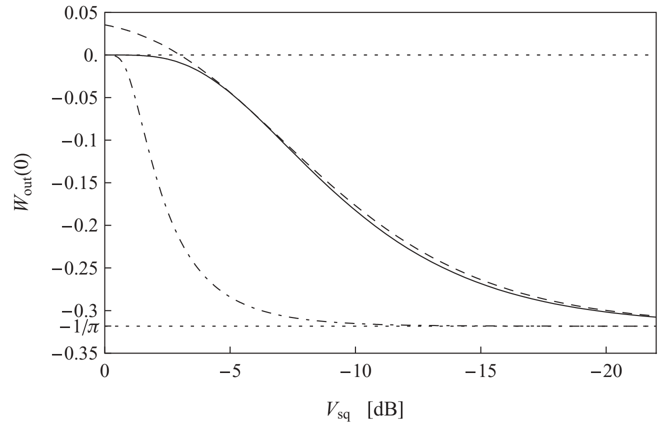

FIG. 1. Output Wigner function in the origin versus the squeezed variance V_{sq} for optimal nonunity-gain teleportation (solid curve), unity-gain teleportation (dashed curve), and optimal conditional teleportation with $K = 0.3$ (dash-dotted curve) of a single-photon Fock state. See text for details.

fidelity benchmark $F = 2/3$ [22]. Thus while nonunity-gain teleportation allows teleportation of a negative Wigner function of the Fock state $|1\rangle$ for an arbitrarily small squeezing, unity-gain teleportation requires more than −3 dB squeezing to accomplish this task. For comparison we mention explicitly the value of the output Wigner function in the origin for nonunity- and unity-gain regimes for several values of squeezing. For −3 dB squeezing the optimal nonunity-gain teleportation gives $W_{\text{out}}(0) \doteq -0.0091$, while unity-gain teleportation yields $\tilde{W}_{\text{out}}(0) \doteq 0.0002$; for −5 dB we get $W_{\text{out}}(0) \doteq -0.0442$ and $\tilde{W}_{\text{out}}(0) \doteq -0.0439$; for −7 dB we get $W_{\text{out}}(0) \doteq -0.0993$ and $\tilde{W}_{\text{out}}(0) \doteq -0.0977$; and for −10 dB we get $W_{\text{out}}(0) \doteq -0.1826$ and $\tilde{W}_{\text{out}}(0) \doteq -0.1768$.

Summarizing the obtained results we see that for the single-photon Fock state at the input of teleportation, we can get a state with a negative Wigner function in the origin at the output of teleportation for arbitrarily small squeezed variance V_{sq} provided that the gain of the teleportation is adjusted suitably. Achievement of a reasonably high negativity, not less than an order of magnitude lower than the negativity at the input, however, requires squeezed variances larger than −5 dB. Substantially larger negative values for lower squeezed variances are obtained by using *conditional* teleportation, where we accept the output state only when the outcome of Alice's measurement $\beta \equiv (\bar{x}_u + i\bar{p}_v)/\sqrt{2}$ falls inside a circle centered in the origin with radius K, that is, falls into the set $\Omega = \{\beta, |\beta| \leqslant K, K > 0\}$. If a measurement outcome β was detected, then the unnormalized output state is [18]

$$|\psi(\beta)\rangle = \sqrt{1-\lambda^2}e^{-(1-\lambda^2)\frac{|\beta|^2}{2}}D[(G-\lambda)\beta] \\ \times [(1-\lambda^2)\beta^*|0\rangle + \lambda|1\rangle], \tag{13}$$

where $\lambda = \tanh r$ and $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ is the displacement operator. The probability of finding the outcome in the set Ω then reads

$$P_\Omega = \frac{1}{\pi}\int_\Omega \langle\psi(\beta)|\psi(\beta)\rangle d^2\beta \\ = 1 - [1 + (1-\lambda^2)^2K^2]e^{-(1-\lambda^2)K^2}, \tag{14}$$

and the normalized density matrix of the output state is

$$\rho_\Omega = \frac{1}{\pi P_\Omega}\int_\Omega |\psi(\beta)\rangle\langle\psi(\beta)| d^2\beta. \qquad (15)$$

The Wigner function in the origin of the state is then easy to calculate as the expectation value $W_\Omega(0) = \mathrm{Tr}[\rho_\Omega(-1)^n]/\pi$ [28] of the parity operator $(-1)^n$. Substituting into the latter formula from Eqs. (13) and (15) and performing the integration over β, we arrive at the following output Wigner function in the origin:

$$W_\Omega(0) = \frac{(1-\lambda^2)}{\pi P_\Omega}\left\{-\frac{\lambda^2}{a}\left(1-e^{-aK^2}\right) + \frac{(\lambda^2-2G\lambda+1)^2}{a^2}\left[1-(1+aK^2)e^{-aK^2}\right]\right\}, \qquad (16)$$

where $a = (1-\lambda^2) + 2(G-\lambda)^2$. As the postselection interval K vanishes, the role of optimized displacement becomes negligible and the Wigner function in the origin approaches the original value $W_\Omega(0) = -1/\pi$ of the single-photon state, irrespective of the squeezing used to produce the shared entangled state. It corresponds to the result obtained previously for the fidelity of teleportation [16,18]. We performed numerical optimization of the gain G and depicted the Wigner function in the origin (16) by the dash-dotted curve in Fig. 1 for $K = 0.3$. The corresponding success probability P_Ω is depicted by the solid curve in Fig. 2.

The figure shows that conditional teleportation substantially outperforms the optimal unconditional teleportation, of course, at the expense of the probabilistic nature of the protocol. For example, conditional teleportation with $K = 0.3$ gives for -3 dB squeezing $P_\Omega = 0.0112$ and $W_\Omega(0) \doteq -0.2174$, for -5 dB we get $P_\Omega = 0.0187$ and $W_\Omega(0) \doteq -0.284$, for -7 dB we get $P_\Omega = 0.0223$ and $W_\Omega(0) \doteq -0.3056$ and for -10 dB we get $P_\Omega = 0.0198$ and $W_\Omega(0) \doteq -0.3152$. The obtained values indicate that conditional teleportation allows the achievement of high negative values of the Wigner function in the origin even for moderate levels of squeezing, equal to approximately -3 dB, at the cost of a roughly 1.1% probability of success.

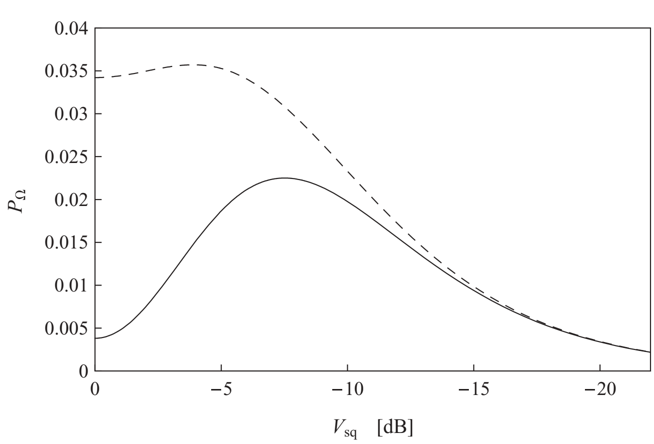

FIG. 2. Success probability P_Ω versus squeezed variance V_{sq} for conditional teleportation of the single-photon Fock state (solid curve) and the state ρ_η with $\eta = 0.6304$ (dashed curve) for $K = 0.3$. See text for details.

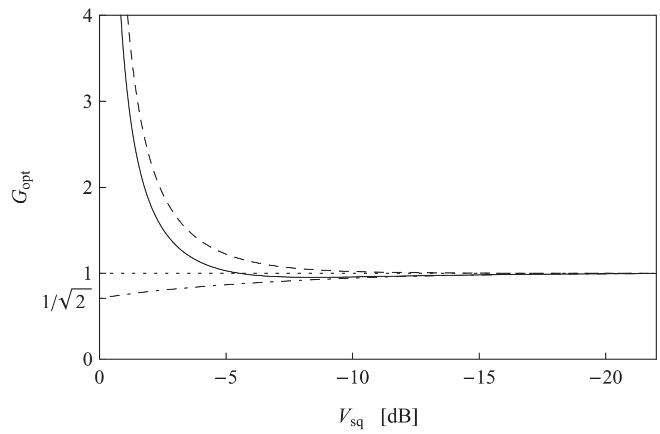

FIG. 3. Optimal normalized gain G_{opt} (solid curve) versus squeezed variance V_{sq} for teleportation of a single-photon Fock state. The dashed curve corresponds to the gain of teleportation upon minimizing the added output noise [29] and the dash-dotted curve corresponds to the gain of teleportation upon maximizing the average teleportation fidelity [21]. See text for details.

To gain deeper insight into the performance of teleportation that is optimal for negativity of the Wigner function in the origin for Fock state $|1\rangle$, we display by the solid curve in Fig. 3 the optimal gain G_{opt} as a function of the squeezed variance V_{sq}. We see from the figure that for squeezing of less than -5.52 dB, optimal teleportation works as a phase-insensitive amplifier, while for larger squeezing it is a weak attenuator, approaching the unity-gain regime in the limit of infinitely large squeezing. It is of interest to compare the optimal gain G_{opt} with the gain G'_{opt} of teleportation that is optimal in the sense that it adds for a given noise in Alice's measurement outcomes the least possible noise into the output state [29]. In this protocol the optimal gain depends on the squeezing of the shared two-mode squeezed vacuum state as $G'_{opt} = \coth(2r)$ and it is depicted by the dashed curve in Fig. 3. It is clearly visible from the figure that teleportation adding minimum noise is not optimal for teleportation of a Wigner function in the origin of the Fock state $|1\rangle$. While the first teleportation is a phase-insensitive amplifier for all levels of squeezing, the latter one acts like a phase-insensitive attenuator for squeezing larger than -5.52 dB. We should also stress here that our teleportation protocol that is optimal from the point of view of the output Wigner function in the origin differs from the optimal teleportation of the single-photon Fock state maximizing the average teleportation fidelity that was investigated in [21]. In the latter protocol the optimal normalized gain, depicted by the dash-dotted curve in Fig. 3, always lies between $1/\sqrt{2}$ and 1, and therefore the teleportation maximizing the average teleportation fidelity realizes a phase-insensitive attenuator for all levels of squeezing.

Up to now we have considered teleportation of Fock state $|1\rangle$. In practice, states with a negative Wigner function are prepared by a single-photon subtraction from a squeezed state [30]. The subtraction is implemented by mixing of a squeezed state squeezed in the position quadrature x_{in} with variance $\langle(\Delta x_{\rm in})^2\rangle = e^{-2s}/2$ on a beam splitter with amplitude transmissivity $\sqrt{\tau}$, followed by projection of one of its outputs on Fock state $|1\rangle$. As a result we obtain the squeezed single-photon Fock state $S(t)|1\rangle$, where $S(t) = \exp[(t/2)(a^2 - a^{\dagger 2})]$ is the squeezing operator and t is the squeezing parameter satisfying $\tanh t = \tau \tanh s$. The state has the Wigner function in the origin of the form

$$W_{\rm in}^{(\rm sq)}(r_{\rm in}) = \frac{1}{\pi}\left(2r_{\rm in}^T\gamma^{-1}r_{\rm in} - 1\right)\exp\left(-r_{\rm in}^T\gamma^{-1}r_{\rm in}\right), \quad (17)$$

where $\gamma = \mathrm{diag}(e^{-2t}, e^{2t})$. Substituting the Wigner function into formula (2) and carrying out the integration, we arrive at the following output Wigner function:

$$W_{\rm out}^{(\rm sq)}(r_{\rm out}) = \left[2r_{\rm out}^T Z r_{\rm out} + \frac{\det Q - (\det S)^2}{\det\tilde{\gamma}}\right] \times \frac{\exp\left(-r_{\rm out}^T\tilde{\gamma}^{-1}r_{\rm out}\right)}{\pi\sqrt{\det\tilde{\gamma}}}, \quad (18)$$

where $Z = \tilde{\gamma}^{-1}S\gamma S^{\rm T}\tilde{\gamma}^{-1}$, $\tilde{\gamma} = S\gamma S^{\rm T} + Q$, and the matrices S and Q are given in Eqs. (4) and (5). In the origin the output Wigner function reads

$$W_{\rm out}^{(\rm sq)}(0) = \frac{\det Q - (\det S)^2}{\pi[\det(S\gamma S^T + Q)]^{\frac{3}{2}}} \quad (19)$$

and it has the same nominator as in Eq. (6). Teleportation of the squeezed single-photon Fock state can be easily transformed into the optimal teleportation of the single-photon Fock state. Obviously, it is sufficient if the teleportation simply compensates the squeezing represented by the covariance matrix (CM) γ, and simultaneously its overall normalized gain is equal to $G_{\rm opt}$. Indeed, returning to the more general protocol with transmissivity $\sqrt{T}$ and gains $g_{x,p}$ and setting the gains as $g_x = e^t G_{\rm opt}/\sqrt{R}$, $g_p = e^{-t}G_{\rm opt}/\sqrt{T}$ and the transmissivity such that $T/R = e^{-2t}$, one finds that $S = G_{\rm opt}\mathrm{diag}(e^t, e^{-t})$, $S\gamma S^T = G_{\rm opt}^2\mathbb{1}$, and $Q = \alpha(G_{\rm opt})\mathbb{1}$, where $\alpha(G)$ is given following Eq. (7). Substitution of the latter expressions for $S\gamma S^T$ and Q into Eq. (19) leads, finally, to the minimal-output Wigner function in the origin for Fock state $|1\rangle$. To illustrate the marked difference between the value of the output Wigner function in the origin as well as its shape for the optimal nonunity-gain teleportation of the squeezed single-photon state and unity-gain teleportation of the state, we plot all of the output Wigner functions for the two scenarios in Figs. 4 and 5.

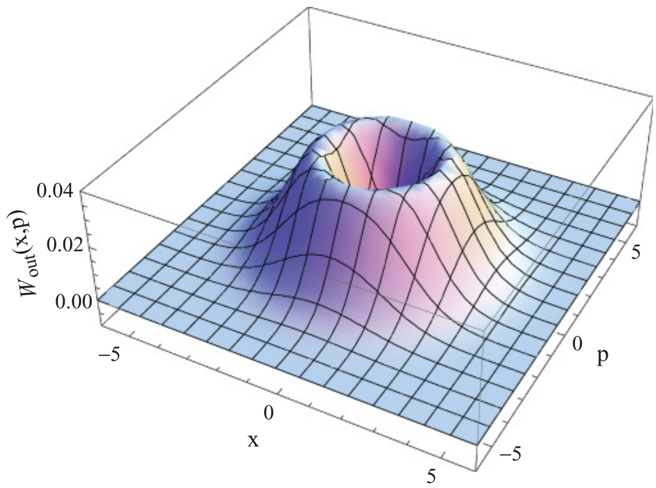

FIG. 4. (Color online) Wigner function of the output state for the optimal nonunity-gain teleportation for $e^{2t} = 2$, $V_{\rm sq} = -3$ dB, and transmissivity $\sqrt{T} = 1/\sqrt{3}$. The Wigner function in the origin attains the negative value of $W_{\rm out}^{(\rm sq)}(0) \doteq -9.10^{-3}$. See text for details.

III. TELEPORTATION OF AN ATTENUATED SINGLE-PHOTON FOCK STATE

Quantum states with a negative Wigner function prepared currently in a laboratory have a substantially reduced negativity in comparison with Fock state $|1\rangle$ and they are mixed. From an experimental point of view it is therefore imperative to know the bounds one has to surpass to successfully teleport mixed states with a negative Wigner function. In an experiment the main source of mixedness is losses, which, in the case of Fock state $|1\rangle$, can be most simply modeled by a purely lossy channel that transmits the state with probability η and replaces it with a vacuum state with probability $1-\eta$. At the output of the channel we get the mixed state

$$\rho_\eta = \eta|1\rangle\langle 1| + (1-\eta)|0\rangle\langle 0| \quad (20)$$

with the Wigner function in the origin equal to $W_{\rm in}^{(\eta)}(0) = (1-2\eta)/\pi$, which is negative if $\eta > 1/2$. Making use of formula (2), where we set $S = G\mathbb{1}$ and $Q = \alpha(G)\mathbb{1}$, we arrive at the output Wigner function in the origin in the form

$$W_{\rm out}^{(\eta)}(0) = \frac{1}{\pi}\left\{\eta\frac{\alpha(G) - G^2}{[\alpha(G) + G^2]^2} + \frac{1-\eta}{\alpha(G) + G^2}\right\}. \quad (21)$$

This formula allows us to calculate, for a given probability η, the threshold value of the squeezing above which the output state has a negative Wigner function in the origin. From the condition $W_{\rm out}^{(\eta)}(0) < 0$ we therefore obtain,

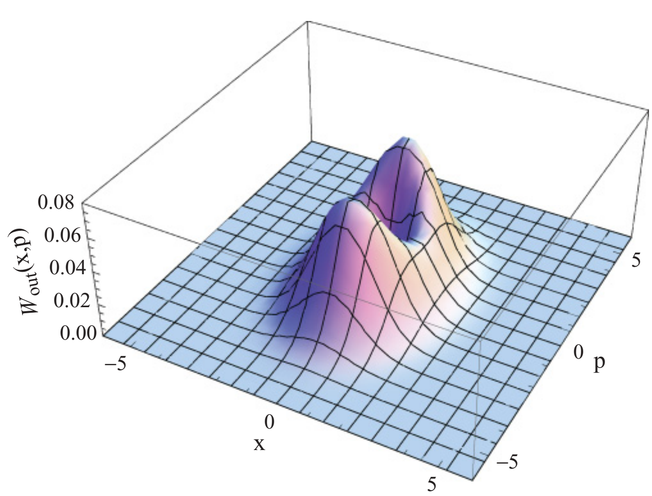

FIG. 5. (Color online) Wigner function of the output state for the unity-gain teleportation for $e^{2t} = 2$ and $V_{\rm sq} = -3$ dB. The Wigner function is equal to 0 in the origin.

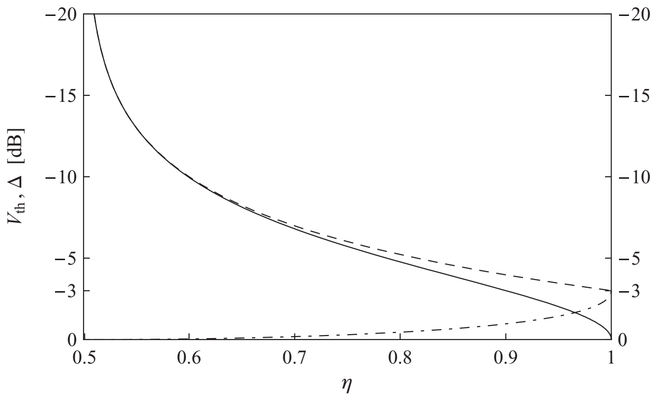

FIG. 6. Threshold squeezed variances $V_{\rm th}^{(G)}$ (solid curve) and $V_{\rm th}^{(1)}$ (dashed curve) and their difference, $\Delta = V_{\rm th}^{(1)} - V_{\rm th}^{(G)}$ (dash-dotted curve), versus the probability η. See text for details.

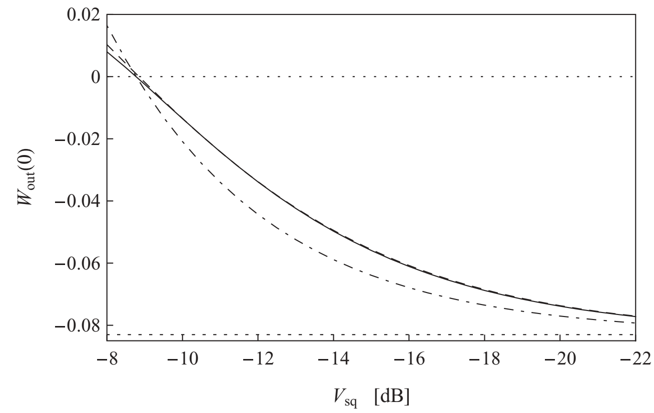

FIG. 7. Output Wigner function in the origin versus the squeezed variance $V_{\rm sq}$ for optimal nonunity-gain teleportation (solid curve), unity-gain teleportation (dashed curve), and optimal conditional teleportation with $K = 0.3$ (dash-dotted curve) for input state $\eta|1\rangle\langle 1| + (1-\eta)|0\rangle\langle 0|$ with $\eta = 0.6304$. The lower dotted curve corresponds to an input Wigner function in the origin $W_{\rm in}^{(\eta)}(0) \doteq -0.083$. See text for details.

after some algebra, that the Wigner function in the origin, (21), is negative if the squeezing parameter r satisfies $r > r_{\rm th}^{(G)} = \text{arctanh}\sqrt{(1-\eta)/\eta}$. By setting $G = 1$ in formula (21) and repeating the preceding calculation, one finds, in contrast, that the output Wigner function in the origin for unity-gain teleportation is negative if the squeezing parameter r satisfies $r > r_{\rm th}^{(1)} = \ln\sqrt{2/(2\eta-1)}$. In Fig. 6 we plot the dependence of the threshold squeezed variances $V_{\rm th}^{(G)} = e^{-2r_{\rm th}^{(G)}}/2$ and $V_{\rm th}^{(1)} = e^{-2r_{\rm th}^{(1)}}/2$ on the probability η. It is apparent from the figure that starting at $\eta = 1$, the squeezing costs increase slowly with decreasing probability η up to $\eta \approx 0.6$. For probabilities less than approximately 0.6, which correspond to the negative values of the Wigner function already demonstrated experimentally, the squeezing costs increase dramatically as η approaches $\eta = 0.5$. As an illustrative example, consider the state ρ_η with $\eta = 0.6304$, corresponding to $W_{\rm in}^{(\eta)}(0) \doteq -0.083$, which was recently achieved experimentally [31]. To have the output Wigner function in the origin negative for the state, we need the squeezing parameter $r > r_{\rm th}^{(G)} = 1.0098$, corresponding to a squeezed variance $V_{\rm sq}$ larger than −8.77 dB. Further, the threshold squeezing is apparently lower for the optimal nonunity-gain teleportation than for unity-gain teleportation, and the difference increases with increasing probability η up to −3 dB for $\eta = 1$. In Fig. 7 we plot the output Wigner function in the origin for the state with $\eta = 0.6304$ versus the squeezed variance $V_{\rm sq}$. Figure 7 reveals a relatively steep decrease in the Wigner function in the origin with increasing squeezing for squeezed variance up to $V_{\rm sq} \approx -14$ dB. For larger squeezing a saturation effect occurs when a small decrease in the value of the Wigner function in the origin requires a large increase in the squeezing. Figure 7 also illustrates that the observation of a reasonably large negativity of the Wigner function at the output of teleportation will require the highest squeezing levels ever achieved. For example, a squeezed variance $V_{\rm sq} = -10$ dB, which was recently observed experimentally [32], would yield an output Wigner function in the origin for optimal nonunity-gain teleportation of $W_{\rm out}^{(\eta)}(0) \doteq -0.0135$. Further improvement can be reached again by using the conditional teleportation with optimized gain. For the state ρ_η at the input we get the output Wigner function in the origin in the form

$$W_\Omega^{(\eta)}(0) = \eta W_\Omega^{(1)}(0) + (1-\eta) W_\Omega^{(0)}(0), \tag{22}$$

where $W_\Omega^{(1)}(0) = (P_\Omega/P_\Omega^{(\eta)}) W_\Omega(0)$ and

$$W_\Omega^{(0)}(0) = \frac{(1-\lambda^2)}{\pi P_\Omega^{(\eta)} a}\left(1 - e^{-aK^2}\right), \tag{23}$$

where

$$P_\Omega^{(\eta)} = 1 - [1 + \eta(1-\lambda^2)^2 K^2] e^{-(1-\lambda^2)K^2} \tag{24}$$

is the success probability, P_Ω is defined in Eq. (14), $W_\Omega(0)$ is defined in Eq. (16), and a is defined following Eq. (16). As the postselection interval K vanishes, the Wigner function in the origin approaches its lowest value,

$$W_{\Omega,K=0}^{(\eta)}(0) = \frac{1}{\pi}\frac{1-\eta-\eta\lambda^2}{1-\eta+\eta\lambda^2}, \tag{25}$$

which can be achieved by conditional Gaussian teleportation of the attenuated single-photon state, at the expense of the success rate. The threshold for preserving the negativity of the Wigner function is clearly the same as for non-unity gain unconditional teleportation, that is, $r_{\rm th}^{(\rm cond)} = r_{\rm th}^{(G)} = \text{arctanh}\sqrt{(1-\eta)/\eta}$. For large squeezing levels the Wigner function, (25), can be expanded in the parameter λ around the point 1 as

$$W_{\Omega,K=0}^{(\eta)}(0) \approx \frac{1}{\pi}[1 - 2\eta + 4\eta(1-\eta)(1-\lambda)]. \tag{26}$$

Hence it follows that, compared to the ideal single-photon Fock state, the value of the Wigner function in the origin of the state after teleportation approaches the initial value $W_{\rm in}^{(\eta)}(0) = (1-2\eta)/\pi$ only in the limit of infinitely large squeezing used to prepare the entangled state, that is, for $\lambda \to 1$. How much squeezing is required is clearly visible from Fig. 8. This is a substantial difference from the idealized single-photon state, for which conditional teleportation approaches unit fidelity for

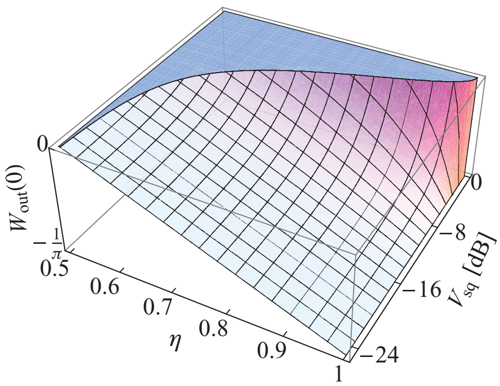

FIG. 8. (Color online) Asymptotic Wigner function in the origin, (25), for the attenuated single-photon state ρ_η after conditional teleportation dependent on the probability η and the squeezed variance $V_{\rm sq}$.

an arbitrary small squeezing used for production of the shared entangled state.

To compare the negativities of the output Wigner function for the attenuated single-photon state for conditional versus unconditional teleportation, we plotted the output Wigner function in the origin, (22), for $K = 0.3$ by the dash-dotted curve in Fig. 7, and the corresponding success probability, (24), by the dashed curve in Fig. 2. The gain in the conditional teleportation is optimized numerically and it differs from the optimal gain in the unconditional teleportation at, at most, the third decimal place. It is apparent from the figure that conditional teleportation markedly outperforms the optimal unconditional nonunity-gain teleportation. For instance, for the state ρ_η with $\eta = 0.6304$ and squeezed variance $V_{\rm sq} = -10$ dB, we obtain $W_\Omega^{(\eta)}(0) \doteq -0.0209$, and the corresponding success probability is $P_\Omega^{(\eta)} \doteq 0.0233$, in comparison with $W_{\rm out}^{(\eta)}(0) \doteq -0.0135$, which is obtained for the unity-gain regime. The figure also reveals that for both teleportations, the output Wigner function in the origin becomes negative at the same value of the threshold squeezed variance, -8.77 dB, which is in full accordance with our finding of the impossibility of reducing the threshold squeezing by resorting to a conditional protocol. It is also worth mentioning that the optimal teleportation is, again, an amplifier for squeezed variance of less than -10.84 dB and then changes to a weak attenuator for larger squeezing, which finally approaches the unity-gain regime in the limit of infinitely large squeezing. In comparison with the case for the pure Fock state $|1\rangle$ depicted in Fig. 1, the advantage of the nonunity-gain regime for the mixed state ρ_η is wiped out.

IV. CONDITIONAL TELEPORTATION WITH EXCESS NOISE

As stated in Sec. I quantum teleportation is a basic building block for quantum computation and long-distance quantum communication. In quantum computation unconditional quantum teleportation can be used for implementation of a deterministic gate on an arbitrary quantum state. Therefore, deterministic transmission of a negative Wigner function by unconditional teleportation is a necessary prerequisite for successful gate operation. Previous analysis indicates that even for an ideal input Fock state $|1\rangle$, the state at the output of unconditional teleportation will have, for realistic squeezing levels, a substantially reduced negative value of the Wigner function in the origin. In addition, it will be mixed so that successful transmission of the negativity of the Wigner function of this state through the next gate will require even larger squeezing, owing to the existence of a strict bound on the minimum squeezing needed to teleport a negative Wigner function of a mixed state.

In contrast, for quantum communication purposes it suffices to implement just conditional teleportation, which will only reduce the success rate of in any case probabilistic communication protocol. Since conditional teleportation gives, for the currently achievable levels of squeezing, better results than unconditional teleportation and is already applied in quantum communication, in this section we restrict ourselves to the analysis of conditional teleportation. In previous sections we assumed an ideal case of pure shared entanglement produced by mixing of two pure squeezed states. Here we take another step toward a more realistic scenario by considering impure squeezed states with excess noise in the antisqueezed quadrature. We show that conditional teleportation of the negative Wigner function of Fock state $|1\rangle$ is tolerant of realistic values of the noise excess.

Let us therefore consider the squeezed states of modes A and B to be momentum and position squeezed states with squeezed quadratures $V_{\rm sq} = \langle(\Delta p_A)^2\rangle = \langle(\Delta x_B)^2\rangle$ and the antisqueezed quadratures $V_{\rm an} = \langle(\Delta x_A)^2\rangle = \langle(\Delta p_B)^2\rangle$. Let us further denote the input state $\rho_{\rm in}$, the shared entangled state ρ_{AB}, and, by $\Pi_{{\rm in}A}(\beta)$, the projector onto the Bell state $|\beta\rangle_{\rm inA} = \sum_{n=0}^{\infty} D_{\rm in}(\beta)|n\rangle_{\rm in}|n\rangle_A$ [33], where $D(\beta)$ is the displacement operator defined following Eq. (13). The state at the output of teleportation, conditioned on the measurement outcome $\beta = (\bar{x}_u + i\bar{p}_v)/\sqrt{2}$ and displaced according to the measurement outcome with normalized gain G by Bob, then reads

$$\tilde{\rho}_{\rm B}(\beta) = D_B(G\beta){\rm Tr}_{{\rm in}A}[\rho_{\rm in} \otimes \rho_{AB}\Pi_{{\rm in}A}(\beta)]D_B^\dagger(G\beta). \quad (27)$$

The state is not normalized and its norm, $P(\beta) = {\rm Tr}_B[\tilde{\rho}_{\rm B}(\beta)]$, gives the probability density of finding the outcome β. The probability that the measurement outcome β falls into the set Σ then reads $P_\Sigma = (1/\pi)\int_\Sigma P(\beta)d^2\beta$, and the corresponding conditionally prepared normalized density matrix is given by $\rho_\Sigma = \frac{1}{\pi P_\Sigma}\int_\Sigma \tilde{\rho}_B(\beta)d^2\beta$. Making use of the formula for the Wigner function of state ρ_Σ of the form $W_\Sigma(0) = (1/\pi){\rm Tr}_B[\rho_\Sigma(-1)^n]$ [28], we find that the output Wigner function in the origin can be expressed as the integral,

$$W_\Sigma(0) = \frac{1}{\pi P_\Sigma}\int_\Sigma W_{\tilde{\rho}_B(\beta)}(0)d^2\beta, \quad (28)$$

of the Wigner function in the origin $W_{\tilde{\rho}_B(\beta)}(0)$ of state (27).

First, we analyze the most simple case, when $\rho_{\rm in} = |1\rangle_{\rm in}\langle 1|$. Then the probability density $P(\beta)$ reads

$$P(\beta) =_A \langle 1|D_A(\beta^*)\rho_A D_A^\dagger(\beta^*)|1\rangle_A, \quad (29)$$

where $\rho_A = {\rm Tr}_B\rho_{AB}$ is the reduced state of mode A. The reduced state is a thermal state with a mean number of thermal photons $\langle n\rangle = (V-1)/2$, where $V = V_{\rm an} + V_{\rm sq}$, and the probability density is therefore an overlap of the displaced thermal state with Fock state $|1\rangle$. Expressing the overlap in terms of the Wigner functions and performing the needed integration, we arrive at a probability density of the form

$$P(\beta) = \frac{\langle n\rangle}{(1+\langle n\rangle)^2}\left[1 + \frac{|\beta|^2}{\langle n\rangle(1+\langle n\rangle)}\right] e^{-\frac{|\beta|^2}{1+\langle n\rangle}}. \quad (30)$$

For the sake of computational simplicity we assume the set Σ to be a square in the plane $[\bar{x}_u, \bar{p}_v]$ of measurement outcomes centered in the origin with sides of length $2a$ parallel to the coordinate axes. Integration of the probability density, (30), over the square then yields the success probability P_Σ, the explicit form of which, in terms of the error function, is given by Eq. (A1) in the Appendix. For measurement outcome β the output state, (27), attains the form

$$\tilde{\rho}_B(\beta) = D_B(G\beta)_A\langle 1|D_A(\beta^*)\rho_{AB}D_A^\dagger(\beta^*)|1\rangle_A D_B^\dagger(G\beta), \quad (31)$$

where we used the relation ${}_{\rm in}\langle 1|\Pi_{{\rm in}A}(\beta)|1\rangle_{\rm in} = D_A^\dagger(\beta^*)|1\rangle_A\langle 1|D_A(\beta^*)$. The Wigner function in the origin of the state, (31), needed to calculate the Wigner function in the origin, (28), can then be calculated from the overlap formula,

$$W_{\tilde{\rho}_B(\beta)}(0) = 2\pi\int_{-\infty}^{+\infty} W_{AB}(\xi_A - \bar{\xi}_A, -\bar{\xi}_B)W_A(\xi_A)d\xi_A, \quad (32)$$

where $\xi_A = (x_A, p_A)^T$, $\bar{\xi}_A = (\bar{x}_u, -\bar{p}_v)^T$, $\bar{\xi}_B = G(\bar{x}_u, \bar{p}_v)^T$, and $W_A(\xi_A)$ is the Wigner function of Fock state $|1\rangle$ given in Eq. (1), and

$$W_{AB}(\xi) = \frac{1}{4\pi^2 V_{\rm sq}V_{\rm an}} e^{-\xi^T\gamma_{AB}^{-1}\xi}, \quad (33)$$

where $\xi = (x_A, p_A, x_B, p_B)^T$ and γ_{AB} is the CM of the shared state ρ_{AB} of the form

$$\gamma_{AB} = \begin{pmatrix} V\mathbb{1} & C\sigma_z \\ C\sigma_z & V\mathbb{1}\end{pmatrix}, \quad (34)$$

where $C = V_{\rm an} - V_{\rm sq}$. Note that we use notation in which the CM of a vacuum state is equal to $\gamma_{\rm vac} = \mathbb{1}$ and the matrix, (34), is a legitimate CM of a quantum state. Namely, denoting its submatrices $A = B = V\mathbb{1}$ and $D = C\sigma_z$, one can easily show, using the Heisenberg uncertainty relations $V_{\rm sq}V_{\rm an} \geqslant 1/4$, that matrix (34) satisfies the necessary and sufficient conditions for a matrix to be a CM of a quantum state given by the inequalities $A, B > 0$, $\det A + \det B + 2\det D \leqslant 1 + \det\gamma_{AB}$, and $2\sqrt{\det A \det B} + (\det D)^2 \leqslant \det\gamma_{AB} + \det A\det B$ [34].

Now performing the integration in Eq. (32) using Eqs. (33) and (34) and substituting the obtained formula into Eq. (28), we finally get, after integration over β, the Wigner function in the origin $W_\Sigma(0)$, which is explicitly given by formula (A2) in the Appendix.

A. Conditioning on the outcome $\beta = 0$

Let us first analyze the simplest case, where we accept only the measurement outcome $\beta = 0$. Then, using Eq. (31), we obtain the normalized output state in the form ${}_A\langle 1|\rho_{AB}|1\rangle_A/P(0)$, where $P(0) = \langle n\rangle/(1+\langle n\rangle)^2$ is obtained from Eq. (30). The

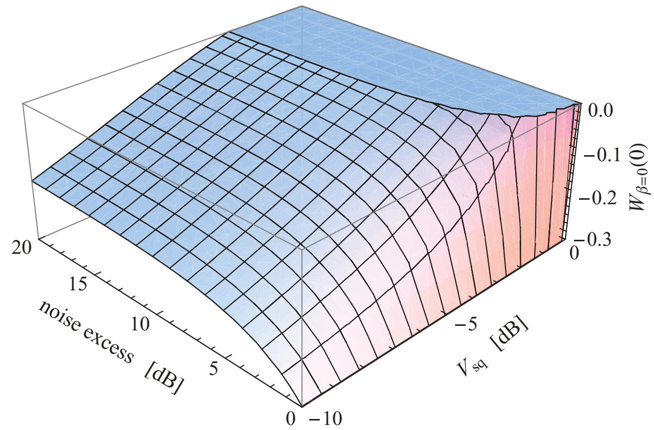

FIG. 9. (Color online) Output Wigner function in the origin versus the squeezed variance $V_{\rm sq} = \langle(\Delta p_A)^2\rangle = \langle(\Delta x_B)^2\rangle$ and the noise excess in the antisqueezed quadratures $V_{\rm an} = \langle(\Delta x_A)^2\rangle = \langle(\Delta p_B)^2\rangle$ for conditional teleportation of Fock state $|1\rangle$ where we accept only the result $\beta = (\bar{x}_u + i\bar{p}_v)/\sqrt{2} = 0$.

Wigner function in the origin of the state can be derived with the help of formula (32) in the form

$$W_{\beta=0}(0) = -\frac{(V - 4V_{\rm sq}V_{\rm an})}{\pi(V-1)}\left(\frac{V+1}{V+4V_{\rm sq}V_{\rm an}}\right)^2, \quad (35)$$

and it is depicted in Fig. 9 as a function of the squeezed variance $V_{\rm sq}$ and the excess noise defined as the sum of the antisqueezed quadrature in decibels and the squeezed quadrature in decibels. Figure 9 shows, again, that for no noise excess, Fock state $|1\rangle$ is perfectly teleported for arbitrarily small nonzero squeezed variance $V_{\rm sq}$, as is also apparent from Eq. (13). For a nonzero noise excess there is a threshold squeezed variance $V_{\rm th}$ one has to overcome to have a negative output Wigner function in the origin. The threshold squeezed variance can be determined from the condition $V - 4V_{\rm sq}V_{\rm an} = 0$. We calculated the threshold squeezed variance for several values of the noise excess in Table I. The calculated values indicate that the threshold squeezed variance increases very slowly with increasing noise excess and approaches the limit value $V_{{\rm th},\infty} = 1/4$, corresponding to -3 dB in the limit of infinitely large noise excess. Thus in the case of postselection of the measurement outcome $\beta = 0$, the teleportation of a negative Wigner function is strongly tolerant of the noise excess.

B. Finite postselection interval

Next we focus on the case of a finite postselection interval. The success probability and the Wigner function in the origin are given by Eqs. (A1) and (A2) in the Appendix, and they are displayed in Figs. 10 and 11 for $a = 0.3$ and a unity-gain regime ($G = 1$). Inspection of the graph in Fig. 11 reveals that it is just the graph in Fig. 9 displaced along the z axis. In

TABLE I. Threshold squeezed variance $V_{\rm th}$ for a given noise excess if we postselect the measurement outcome $\beta = 0$.

Noise excess (dB)	1	2	3	4	5
$V_{\rm th}$ (dB)	−1.62	−2.06	−2.32	−2.49	−2.62

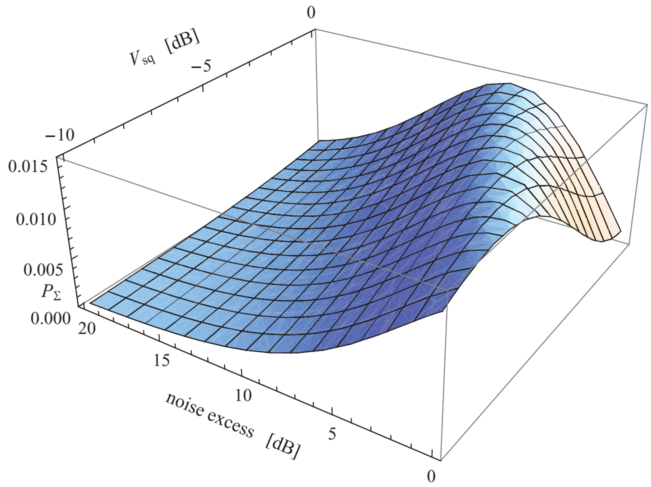

FIG. 10. (Color online) Success probability P_Σ versus the squeezed variance $V_{sq} = \langle(\Delta p_A)^2\rangle = \langle(\Delta x_B)^2\rangle$ and the noise excess in the antisqueezed quadratures $V_{an} = \langle(\Delta x_A)^2\rangle = \langle(\Delta p_B)^2\rangle$ for conditional unity-gain teleportation of Fock state $|1\rangle$, where we accept only the outcomes of the Bell measurement $[\bar{x}_u, \bar{p}_v]$ that fall into the square centered in the origin with side $2a$, where $a = 0.3$.

other words, conditioning on measurement outcomes from a finite postselection interval leads to a uniform reduction of the value of the output Wigner function in the origin in comparison with the case where we accept only the outcome $\beta = 0$. This naturally entails the emergence of a nonzero threshold on squeezing that has to be overcome to successfully teleport a negative Wigner function. The threshold still increases slowly with increasing noise excess as is apparent from Table II.

C. Attenuated single-photon state

Finally, we discuss the situation where we have the attenuated single-photon state at the input, that is, $\rho_{in} = \rho_\eta$, where ρ_η is defined in Eq. (20), and we condition on the measurement outcome $\beta = 0$.

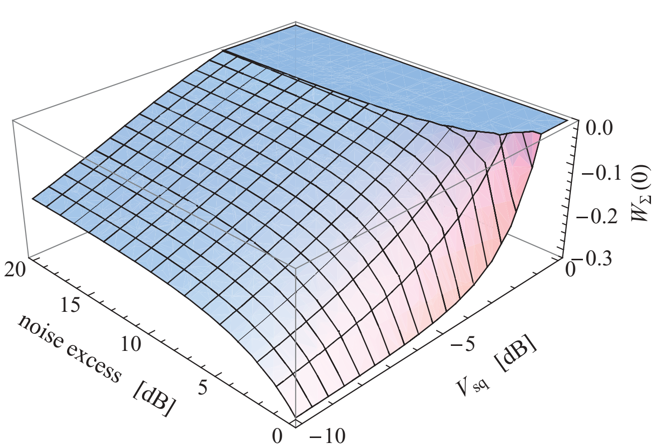

FIG. 11. (Color online) Output Wigner function in the origin versus the squeezed variance $V_{sq} = \langle(\Delta p_A)^2\rangle = \langle(\Delta x_B)^2\rangle$ and the noise excess in the antisqueezed quadratures $V_{an} = \langle(\Delta x_A)^2\rangle = \langle(\Delta p_B)^2\rangle$ for conditional unity-gain teleportation of Fock state $|1\rangle$, where we accept only the outcomes of the Bell measurement $[\bar{x}_u, \bar{p}_v]$ that fall into the square centered in the origin with side $2a$, where $a = 0.3$.

TABLE II. Threshold squeezed variance V_{th} for a given noise excess for a finite postselection interval.

Noise excess (dB)	0	1	2	3	4	5
V_{th} (dB)	−1.13	−1.77	−2.12	−2.35	−2.51	−2.63

In this case the probability density reads

$$P^{(\eta)}(\beta) = \eta P(\beta) + (1 - \eta) P^{(0)}(\beta), \tag{36}$$

where $P(\beta)$ is given in Eq. (30) and

$$P^{(0)}(\beta) = {}_A\langle 0|D_A(\beta^*)\rho_A D_A^\dagger(\beta^*)|0\rangle_A = \frac{e^{-\frac{|\beta|^2}{1+\langle n\rangle}}}{1 + \langle n\rangle}. \tag{37}$$

For $\beta = 0$ we get, in particular, $P^{(\eta)}(0) = 2(V + 1 - 2\eta)/(V + 1)^2$. Further, conditioned on the measurement outcome $\beta = 0$, we get the unnormalized output state in the form

$$\tilde{\rho}_B^{(\eta)}(0) = \eta\, {}_A\langle 1|\rho_{AB}|1\rangle_A + (1 - \eta)\, {}_A\langle 0|\rho_{AB}|0\rangle_A. \tag{38}$$

Calculating, finally, the Wigner function in the origin of the normalized state $\tilde{\rho}_B^{(\eta)}(0)/P^{(\eta)}(0)$, we obtain

$$W_{\beta=0}^{(\eta)}(0) = \frac{4V_{sq}V_{an} + V(1 - 2\eta)}{\pi(V + 1 - 2\eta)} \left(\frac{V + 1}{V + 4V_{sq}V_{an}}\right)^2. \tag{39}$$

Hence, one can calculate the threshold squeezed variance from the condition $4V_{sq}V_{an} + V(1 - 2\eta) = 0$. Expressing the antisqueezed quadrature as $V_{an} = (1/2)(N/V_{sq})$, where N

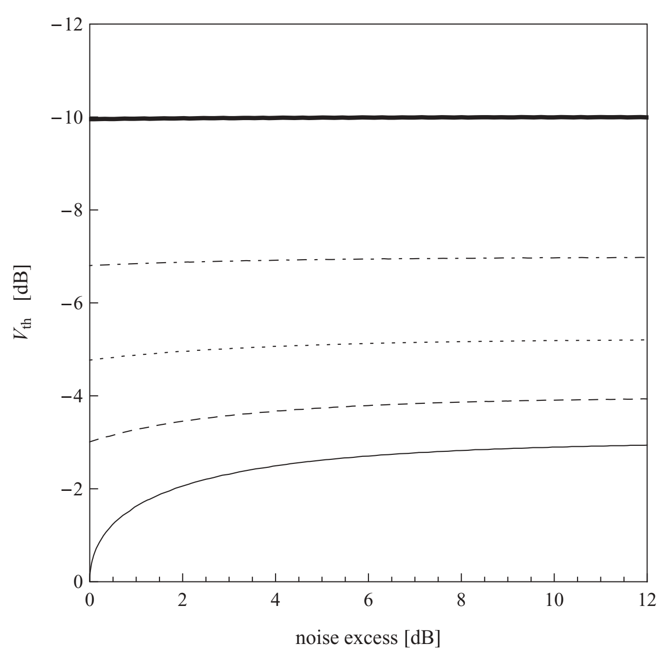

FIG. 12. Threshold squeezed variance V_{th} versus the noise excess for conditional teleportation of input state $\eta|1\rangle\langle 1| + (1 - \eta)|0\rangle\langle 0|$ for $\eta = 1$ (solid curve), $\eta = 0.9$ (dashed curve), $\eta = 0.8$ (dotted curve), $\eta = 0.7$ (dash-dotted curve), and $\eta = 0.6$ (thick solid curve), where we accept only the result $\beta = (\bar{x}_u + i\bar{p}_v)/\sqrt{2} = 0$.

stands for the noise excess ($N = 1/2$ for no noise excess), we get the threshold squeezed variance in the form

$$V_{\mathrm{th}} = \frac{2N - \sqrt{4N^2 - 2N(2\eta - 1)^2}}{2(2\eta - 1)}. \quad (40)$$

The dependence of the threshold variance on the noise excess is plotted in Fig. 12 for several values of the probability η. The threshold squeezing dramatically increases with decreasing probability of Fock state $|1\rangle$, but for realistic values of the probability η in the interval $0.6 < \eta < 0.7$, it still remains in the region of achievable squeezing levels. As in the previous cases a saturation effect is observed and the threshold squeezed variance approaches $V_{\mathrm{th},\infty} = (2\eta - 1)/4$ in the limit of infinitely large noise excess. To illustrate the tolerance of the teleportation of a negative Wigner function of noise excess, even for mixed input states, we consider, again, the example of the input state ρ_η with $\eta = 0.6304$ discussed previously and a noise excess of 4 dB. Using Eq. (40) we then get a threshold squeezed variance V_{th} equal to -8.82 dB, and it approaches $V_{\mathrm{th},\infty} = -8.85$ dB with an increasing noise excess.

V. DISCUSSION AND CONCLUSIONS

We have studied teleportation of a pure single-photon Fock state and a mixed attenuated single-photon Fock state by the standard continuous-variable teleportation protocol [3]. We optimized analytically the gain of the teleportation so as to minimize the output Wigner function in the origin. For the single-photon Fock state we found that an arbitrarily weak squeezing used to create the shared entangled state is sufficient for successful teleportation of the negative value of its Wigner function in the origin. For an attenuated single-photon state we have shown that there is a strict bound on the squeezing that has to be overcome to have a negative output Wigner function in the origin. In both cases the negative value of the output Wigner function in the origin can be increased by using a conditional teleportation with a reasonably high success rate. However, in the case of the attenuated single-photon state, the bound on squeezing one has to surpass to observe a negative output Wigner function is the same for both unconditional and conditional teleportation, and its initial value can be reached only asymptotically in the limit of a high squeezing. Finally, we have taken into account noise excess in the antisqueezing and we have shown that conditional teleportation of a negative Wigner function exhibits a strong tolerance of the noise excess.

Now let us discuss the consequences of our observations for quantum computation and quantum communication. For quantum computation, quantum teleportation was considered as a possible scenario of how to carry out a *deterministic* operation on an unknown arbitrary quantum state. A basic requirement is to be able to preserve the negativity of the Wigner function through the operation: in our simplest case, through teleportation. Therefore, we can take the preservation of the negativity as a necessary condition for quantum gate performance. We observed that, to preserve the negativity of the Winger function, either the input state has to be very close to a single-photon state or, for the imperfect single-photon state (the attenuated version), an extremely high squeezing is required. If the negativity of the Wigner function must be preserved (e.g., at 95% of its original value), the requirement is even more demanding. Clearly, it is very important to protect the input state against even the loss, since it substantially increases the squeezing required for teleportation. In summary, the cost (the squeezing required to prepare the entangled state in the teleportation) of implementing the operation in this measurement-induced way is quite high. It can stimulate a further increase in the squeezing in the experiments, but this resource seems also to be practically limited.

In contrast, for quantum communication with repeaters, the preservation of negativity of the Wigner function through Gaussian teleportation seems to be just a reasonable condition to efficiently extend the quantum key distribution between two distant repeaters. Further, it is enough to implement *conditional* teleportation, since the key distribution is a probabilistic protocol anyway. The minimal squeezing required to keep the negativity of the Wigner function is practically unchanged by postselections. However, for an almost-perfect single-photon state, the threshold squeezing is low and the input negativity of the Wigner function can be archived for the experimentally feasible values of squeezing (up to -10 dB of squeezing). Advantageously, for a strongly attenuated single-photon state, the postselection improves the value of the negativity up to a maximum for the given squeezing used to produce entanglement. However, to reach the original value, postselection must be combined with enhancement of the squeezing. In summary, the cost (the squeezing required to prepare the entangled state in the teleportation) is lower for the communication application of the teleportation of highly nonclassical states if the negativity of the Wigner function is already preserved by teleportation.

Finally, we want to point out one important issue. Since teleportation with finite squeezing always lowers the negativity of the Wigner function, the next teleportation will transfer an already imperfect version of the highly nonclassical state and the squeezing threshold (or squeezing required to almost maintain the negativity) will become more demanding for the implementation. This means that small imperfections or errors are actually amplified through multiple Gaussian teleportations and any correction mechanism (quantum error correction or quantum repeater) has to be applied very frequently and very efficiently to maintain the threshold squeezing in a feasible range to be able to transmit the negativity of the Wigner function.

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APPENDIX

The success probability P_Σ is given by

$$P_\Sigma = \mathrm{erf}(b)\left[\mathrm{erf}(b) - \frac{2be^{-b^2}}{\sqrt{\pi}(1 + \langle n\rangle)}\right], \quad (A1)$$

where $b = a/\sqrt{2(1+\langle n\rangle)}$ and $\mathrm{erf}(z) = (2/\sqrt{\pi})\int_0^z e^{-t^2}dt$ is the error function.

The conditional output Wigner function in the origin $W_\Sigma(0)$ reads

$$W_\Sigma(0) = \frac{\mathrm{erf}(a\sqrt{q})}{4\pi P_\Sigma V_{\mathrm{sq}} V_{\mathrm{an}} \alpha q}\left[\left(\frac{2}{\alpha} + \frac{2\delta^2}{\alpha^2 q} - 1\right) \times \mathrm{erf}(a\sqrt{q}) - \frac{4\delta^2 a}{\alpha^2\sqrt{\pi q}} e^{-qa^2}\right], \quad \text{(A2)}$$

where

$$q = \frac{V(1+G^2) - 2CG + G^2}{V + 4V_{\mathrm{sq}}V_{\mathrm{an}}}, \quad \alpha = \frac{V + 4V_{\mathrm{sq}}V_{\mathrm{an}}}{4V_{\mathrm{sq}}V_{\mathrm{an}}},$$

$$\delta = \frac{CG - V}{4V_{\mathrm{sq}}V_{\mathrm{an}}}.$$

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