Quantum counting algorithm and its application in mesoscopic physics

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We discuss a quantum counting algorithm which transforms a physical particle-number state (and superpositions thereof) into a binary number. The algorithm involves two quantum Fourier transformations. One transformation is in physical space, where a stream of $n < N = 2^K$ (charged) particles is coupled to K qubits, rotating their states by prescribed angles. The second transformation is within the Hilbert space of qubits and serves to read out the particle number in a binary form. Applications include a divisibility check characterizing the size of a finite train of particles in a quantum wire and a scheme allowing one to entangle multiparticle wave functions in a Mach-Zehnder interferometer, generating Bell, Greenberger-Horne-Zeilinger, or Dicke states.

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I. INTRODUCTION

Quantum mechanics offers novel algorithms that make it possible to speed up the solution of specific computational tasks, some modestly, such as sorting a list [1,2], whereas others, such as prime factorization [1,3], are accelerated exponentially. While applications in quantum cryptography [1,4] are close to commerical realization [5], the endeavor of building a universal quantum computer with thousands of quantum bits lies in the distant future, if ever realized. In this situation, it is interesting to consider special tasks which are less demanding in their requirement with regard to the number of qubits and the complexity of their network. An example of such an application is the use of a qubit as a measuring device in the realization of full counting statistics [6] in mesoscopic transport. In the present article, we expand the concept of counting statistics in mesoscopic systems and analyze the problem of precise counting of particles in a mesoscopic setting. This task combines both algorithmic aspects as well as physical aspects of its implementation; while the algorithmic task might appear trivial at first glance, closer inspection unveils a number of interesting features relating the task of (quantum, nondemolition) counting with the quantum Fourier transformation.

Consider electrons flowing through a (quantum) wire—the task of counting particles consists of attributing to a finite set (or "train") of electrons a number (the cardinality of the set). In an information-theoretic sense, this task corresponds to transforming a unary number (represented by the physical set of electrons) into a binary (or ternary, quaternary, quinary, etc.) number. Within a mesoscopic setting, we wish to achieve this task without perturbing the electron train, that is, to transform the input into a number with a nondemolition measurement. Furthermore, given a superposition of number states, an ideal algorithm provides a final state for readout which encodes a superposition of numbers as well. Such quantum superpositions of number states quite naturally appear in mesoscopic devices involving a splitter, such as a Mach-Zehnder interferometer; sending a stream of particles into such a device, the quantum state within one arm will constitute a superposition of number states.

It turns out that such a quantum counting algorithm can be realized by coupling the stream of electrons to a register of qubits, for example, spin qubits or charge qubits made from double-dot systems. The cardinality of the particle train then is encoded into the qubit register via proper rotations of all the qubit states, whereby the first qubit in the register measures the number's parity, the second qubit measures the number of pairs of electrons, the third qubit measures the number of electron quartets, etc., passing by. Ideally, a transverse coupling is used between the electrons flowing in the wire and the qubits, leading to a mere accumulation of phases (or rotation of qubits, which is a nonclassical operation); choosing such a mode of operation immediately connects the task of counting with the quantum Fourier transformation (counting in phase space rather than in real space). Hence, the task of quantum counting in mesoscopics unveils interesting elements that go beyond simple counting in a classical setting.

Using more technical terms, in our quantum counting algorithm the information in a physical number state $|n\rangle_\Phi$ is entangled with the set of K counter-qubits which we describe with the help of the usual computational basis $|0\rangle_Q,|1\rangle_Q,\dots,|2^K-1\rangle_Q$ (cf. Ref. [1]); the indices Φ and Q refer to the physical Hilbert space $\mathcal{H}_\Phi$ of particle states and the Hilbert space $\mathcal{H}_Q$ of K-qubit states (cf. Fig. 1). The passage of the particles in the wire transforms the initial state $\mathcal{F}(|0\rangle_Q)$, that is, the lowest harmonic $\mathcal{F}(|0\rangle_Q) \propto \sum_{j=0}^{2^K-1}|j\rangle_Q$ of the K-qubit register, into the state $\mathcal{F}(|n\rangle_Q) \propto \sum_{j=0}^{2^K-1}\exp(2\pi i nj/2^K)|j\rangle_Q$, the quantum Fourier transform of the state $|n\rangle_Q = |n_1,n_2,\dots,n_K\rangle$, where n is written in binary form, $n = n_1n_2\dots n_K = n_1 2^{K-1} + n_2 2^{K-2} + \dots + n_K 2^0$ (cf. Ref. [1]). The full information encoded into the K-qubit register is made available to a single-shot readout through a second (inverse) quantum Fourier transformation of the qubit state $\mathcal{F}(|n\rangle_Q)$, taking it back into the state $|n\rangle_Q$. A simultaneous measurement of the K qubits then provides the particle number n in binary form. In case of a superposition $\sum_j c_j\,|j\rangle_\Phi$ of particle-number states, the backward Fourier transformation $\mathcal{F}^{-1}$ acts on the state $\sum_j c_j\,|j\rangle_\Phi\mathcal{F}[|j\rangle_Q]$ and generates the final result $\sum_j c_j\,|j\rangle_\Phi|j\rangle_Q$, with the number state entangled with the qubit register; the measurement of the qubit register then will execute a projection onto one component. The usefulness of sequential forward and backward Fourier transformations is well appreciated in the fields of imaging and particularly in crystallography.

Performing an inverse quantum Fourier transformation on the state $\mathcal{F}(|n\rangle_Q)$ requires a quantum computer. Here, instead, we use a procedure which is basically identical to the semiclassical Fourier transform suggested by Griffiths and Niu [7], a conditional measurement algorithm involving a sequential readout, where the jth reading depends on the results of the previous $j-1$ measurements (the availability of such a semiclassical algorithm relies on the fact that the quantum Fourier transform associated with a number n presents itself as a simple product state of qubits). Alternatively, a simultaneous (rather than conditional) readout of the state $\mathcal{F}(|n\rangle_Q)$ counts the power of 2 in the cardinality n of the particle train (divisibility check by 2^k, $k < K$). The basic concepts entering our quantum counting algorithm are schematically illustrated in Fig. 1. The algorithm provides an exponential savings in resources as compared to a straightforward scheme where the number $n < N$ is trivially encoded into a set of $M = N^2/\pi^2$ qubits, $|n\rangle_\Phi \to [|0\rangle_Q + e^{in\varphi}|1\rangle_Q]^{\otimes M}/\sqrt{2^M}$, with $\varphi < 2\pi/N$ being a small rotation angle [8]; note, that the no-cloning theorem [9,10] prevents us from using one spin and then cloning it after the passage of the n particles. Below, we will drop the indices Φ and Q that distinguish states in the (isometric) Hilbert spaces describing physical particle number states and qubit states.

The use of two-level systems as clocks or counters has a long history: using the Larmor precession of a spin as a clock attached to the particle, Baz' [11] and Rybachenko [12] proposed to measure the tunneling time through a barrier. In the context of full counting statistics in mesoscopic physics, Levitov and Lesovik [13] introduced the idea of using a stationary spin as a measurement device to count the electrons flowing in a nearby quantum wire. The use of a qubit as a measuring device (counter) in the realization of full counting statistics has been proposed in Ref. [6]. In quantum optics, Brune *et al.* [14] proposed to make use of atoms excited to Rydberg states as atomic clocks to count photons in a cavity using a straightforward measurement scheme (cf. Ref. [8]), a proposal that has been experimentally realized recently [15]. Our algorithm can be used to count photons as well; in our dual setup the counters are fixed and (microwave) photon pulses propagate in a transmission line.

Below, we first introduce our setup and algorithm including the sequential and single-shot measurement schemes (cf. Sec. II). We then proceed with some remarks regarding the implementation of the algorithm with charge qubits in Sec. III and then discuss an application of our algorithm (Sec. IV) where we combine our counter with a Mach-Zehnder interferometer in a "which path" setup [16] in order to fabricate entangled many-particle wave functions.

II. SETUP AND ALGORITHM

We consider the setup in Fig. 2 where the $n < N = 2^K$ particles to be counted flow in a quantum wire along x. The K spins or qubits (we will use these terms synonymously) are initially polarized along the positive y axis [i.e., along $|+y\rangle = (|\uparrow\rangle + i\,|\downarrow\rangle)/\sqrt{2}$, with the states $|\uparrow\rangle$ and $|\downarrow\rangle$ polarized along the z axis]. The simplest case makes use of transverse coupling between the charged particle and the spin: upon passage of a charge, the induced B field, locally directed along the z axis, rotates the

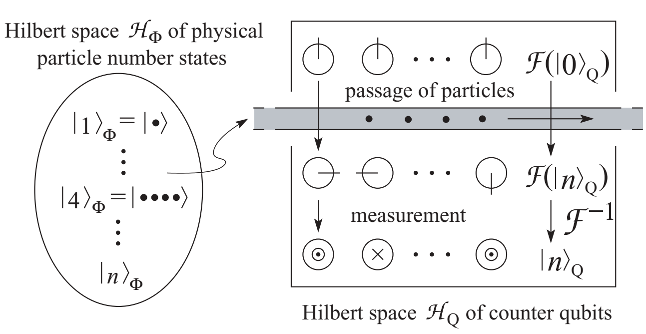

FIG. 1. Schematic representation of the quantum counting algorithm. A particle number state or a superposition of particle number states is fed into a quantum wire to undergo quantum counting. The interaction between the charged particles and the qubits rotates the qubit states, thereby generating the first Fourier transformation $\mathcal{F}$ taking a number state $|n\rangle_\Phi$ from the Hilbert space $\mathcal{H}_\Phi$ of particles to the Hilbert space $\mathcal{H}_Q$ of qubits. The second Fourier transform $\mathcal{F}^{-1}$ operates within the space of qubit states and generates the number state $|n\rangle_Q$ in binary form, which may be used in a further quantum computation or undergo a single-shot measurement for readout. Rather then relying on a quantum computer, we make use of a semiclassical Fourier transformation with a sequential readout.

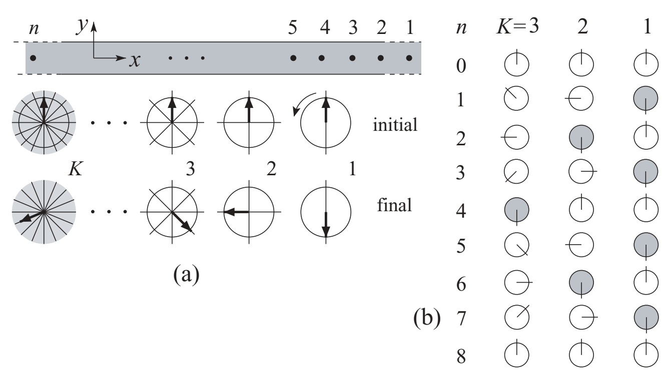

FIG. 2. Illustration of the quantum algorithm to transform the cardinality n of a set of particles into a binary number. (a) Initially, all spins point in the $+y$ direction. Upon passage of one particle, the jth spin is rotated (anticlockwise) by $\phi_j = 2\pi/2^j$. After the passage of all particles, the first spin is measured along the y axis and provides the number's parity. Depending on the parity, the second spin is measured along the y axis (even parity) or the $-x$ axis (odd parity); a measurement along (opposite to) the axis is encoded with a 0 (1). The further iteration is straightforward; see text. The figure shows the reading after passage of 5 particles with $K = 4$. (b) Divisibility check: qubit states after passage of $n = 0, \ldots, 8$ electrons for $K = 3$; for $n = 1, \ldots, 7$ there is exactly one qubit ending up in the $|\downarrow\rangle$ state (shaded), signaling that the cardinality n of the sequence is not divisible by $2^3 = 8$. The divisibility by 2^K can be tested in a single-shot measurement.

spins in the x-y plane. The couplings of the spin counters to the wire are chosen such that the jth spin is rotated (anticlockwise) by the amount $\phi_j = 2\pi/2^j$ [a rotation by $U_z(\phi_j) = \exp(-i\phi_j\sigma_z/2)$, with σ_z being a Pauli matrix]. The passage of n particles then rotates the jth spin by the amount $n\phi_j$. In particular, the first spin is rotated by the angle $n\pi$ and points either upward if the number's parity is even (we store a "0" in the first position of the binary number) or downward (we store a "1") if the parity is odd. Hence the measurement of the first spin along the y axis provides already the parity of the number. In addition, the state of the first spin determines the axis in the measurement of the second spin: for an even n, we measure ϕ_2 along the y axis [and store a 0 (1) if the spin is pointing up (down)], while for an odd-parity n, we measure ϕ_2 along the $-x$ axis.

The iteration of the algoritm is straightforward (see also Ref. [7]): the jth spin is measured along the direction $m_{j-1}\phi_j$ with the integer $m_{j-1} = n_{K-j+2},\ldots,n_{K-1}n_K$ corresponding to the binary number encoded in the $j-1$ previous measurements. The jth position in the binary register then assumes the value $n_{K-j+1} = 0$ or $n_{K-j+1} = 1$, depending on the measurement result, 0 for a spin pointing along the axis and 1 for a spin pointing opposite. Rather than rotating the axis of measurement, the spins are rotated by the corresponding angles. These rotations by $-m_{j-1}\phi_j$ are conveniently done incrementally: after measurement of the jth spin with outcome 0 or 1, all spins $J > j$ are rotated by $-n_{K-j+1}2^{j-1}\phi_J$. After the passage of the particles (the first "analog" quantum Fourier transform which lasts exponentially long in K), the second semiclassical quantum Fourier transform then requires $\sim K^2$ steps (measurements and rotations) and provides the desired result, the cardinality of the set of electrons in binary form, $n = n_1 2^{K-1} + n_2 2^{K-2} + \cdots + n_K 2^0$.

A. Divisibility by 2^k

A variant of the aforementioned counting algorithm provides a test for divisibility by powers of 2: given a finite train of electrons and K qubits, we check whether the number n of electrons in the train is divisible by 2^K. As a corollary, we find the factor 2^k, $k \leqslant K$, in n. Using the aforementioned setup, the divisibility check involves only a single-shot measurement along the y axis at the end of the train's passage (rather than the conditional measurement presented previoulsy): the train's cardinality is divisible by 2^K, if all spins are pointing up, that is, along the positive y axis [cf. Fig. 2(b)]. The nondivisibility is signalled by the "opposite" outcome; that is, there is at least one spin pointing down.

The previous statement relies on the fact that after the passage of $n = 2^K$ particles all counters end up in the spin-up state, while for $n \neq 2^K$ there is exactly one spin residing in a spin-down state [cf. Fig. 2(b)] (note the difference in having counters in up and down states with defined measurement outcomes and statistical results of up and down measurements for counters pointing away from the y direction). Here is the formal proof. Starting in the initial state $|+y\rangle = (|\uparrow\rangle + i\,|\downarrow\rangle)/\sqrt{2}$, after passage of n particles, the jth spin ends up (up to an overall phase) in the state $|\mathrm{f}\rangle = [|\uparrow\rangle + ie^{2\pi i n/2^j}|\downarrow\rangle]/\sqrt{2}$. The probability to measure this spin along the $+y$ direction is $|\langle +y|\mathrm{f}\rangle|^2 = \cos^2(\pi n/2^j)$, $j = 1,\ldots,K$. There is exactly one spin $1 \leqslant j^* \leqslant K$, for which this probability vanishes: this follows from the statement, that any number $0 < n < 2^K$ can be represented in the form $2^m I$ with $0 \leqslant m < K$ and I being an odd integer. Then, for the spin $j^* = m+1$ (and only for this spin) the phase $\pi n/2^{j^*} = \pi I/2$ is an odd multiple of $\pi/2$ and hence the probability $|\cos(\pi I/2)|^2$ to find it pointing along $+y$ vanishes; that is, the spin is pointing down. For all other spins $j \neq m+1$, the phase is a multiple of π (for $j < m+1$, the spin is pointing up) or a fraction $I/2^{j-m-1}$ of $\pi/2$ (for $j > m+1$, the spin is not pointing down). Furthermore, a reduced power $k < K$ of 2 in n is easily read off the register's state after the passage of the particles: if the first k qubits (from the left) are pointing up, the cardinality n contains the factor 2^k.

B. Equivalence with quantum Fourier transformation

Next, we establish the equivalence of our algorithm with the quantum Fourier transformation and discuss the problem of error correction. Given the computational basis $|0\rangle,\ldots,|2^K-1\rangle$ defined by the K qubits, we rewrite the state $|n\rangle$ in binary form $|n\rangle = |n_1,n_2,\ldots,n_K\rangle$. Second, we define the binary fraction $0.n_j n_{j+1}\ldots n_K = n_j/2 + n_{j+1}/4 + \cdots + n_K/2^{K-j+1}$. The quantum Fourier transform of $|n\rangle$ then reads (cf. Ref. [1])

$$\mathcal{F}(|n\rangle) = \prod_{j=1}^{K} \frac{|0\rangle + \exp(2\pi i\, 0.n_{K-j+1}\ldots n_K)|1\rangle}{\sqrt{2}}. \quad (1)$$

In our counting algorithm, the passage of n particles rotates the jth qubit (initially prepared in the state $|+y\rangle = [|\uparrow\rangle + i\,|\downarrow\rangle]/\sqrt{2})$ by $U_z(\phi_j)$ to produce the state $[|\uparrow\rangle + i\,\exp(2\pi i\, n/2^j)\,|\downarrow\rangle]/\sqrt{2}$ (up to an overall phase). Identifying $|\uparrow\rangle \leftrightarrow |0\rangle$ and $i\,|\downarrow\rangle \leftrightarrow |1\rangle$ and using the binary representation $n = n_1 2^{K-1} + \cdots n_K 2^0$, this qubit state coincides with the factor $[|0\rangle + \exp(2\pi i\, 0.n_{K-j+1}\ldots n_K)|1\rangle]/\sqrt{2}$ in the Fourier transform (1). Hence, after the passage of the n particles, the K-qubit register resides in the state $\mathcal{F}(|n\rangle)$. Second, our sequential readout scheme provides the number $n = n_1 n_2 \ldots n_K$ in binary form. The corresponding final state $|n\rangle = |n_K,n_{K-1},\ldots,n_1\rangle$ (obtained after rotation of the qubits back to the z axis, that is, a rotation around x by $\pi/2$, and subsequent measurement along the z axis defining the computational basis) is characterized by the reverse sequence. The semiclassical readout sequence then corresponds to an inverse Fourier transformation combined with a permutation (cf. Ref. [7]). The simpler divisibility check relies upon the fact that the quantum Fourier transform $\mathcal{F}(|0\rangle) = \mathcal{F}(|2^K\rangle)$ is distinguishable from $\mathcal{F}(|n\rangle)$, $0 < n < 2^K$, by a single-shot measurement in the computational basis (after rotation of the y axis to the z axis).

Not different from others, our algorithm suffers from errors, particularly those accumulated in the phase ϕ during rotation of the qubit. Given a phase error $\delta\phi$ (during rotation in the x-y plane), the probability for a false readout of the qubit is given by $p_e = 1 - \cos^2(\delta\phi/2)$ (the phase error $\delta\phi$ remains the same after rotation around the x axis). Such erroneous phase drifts may be systematic in origin (due to a wrong coupling of the qubit to the wire or due to the usual phase decoherence of the qubit); the resulting bit errors destroy the result of the computation. Given the one-bit error probability p_e, the probability to measure a wrong result is given by $P_e = 1 - (1 - p_e)^K$; for small phase errors $\delta\phi$, the total error adds up to $P_e \approx K\delta\phi^2/8$, quadratic in $\delta\phi$. Further improvement can be gained with a classical multiqubit error correction scheme combined with a simple majority rule; for example, for a three-qubit code, p_e is to be replaced by p_e^2. Quantum error correction is required only for the case where the inverse Fourier transform is performed by a quantum computer.

III. IMPLEMENTATION WITH CHARGE QUBITS

The spins required in the aforementioned counting and divisibility check algorithms can be implemented using various types of qubits; note that, while the special nature of our algorithm avoids the large number of qubits and the huge network complexity of a quantum computer, we do require individual qubits with high performance. Most qubits naturally couple to the electrons in the quantum wire, either via the gauge field (current) or via the scalar potential (charge). The coupling to charge is strong, with typical rotation angles ϕ allowing for an $\sim\pi$-phase rotation upon passage of one unit of charge. Transverse coupling via the current is weak, usually requiring enhancement with a flux transformer (cf. the discussion in Ref. [6]).

To fix ideas, we consider an implementation with charge qubits in the form of double quantum dots (DQD) as one attractive possibility. DQDs have been implemented in GaAs/Al$_x$Ga$_{1-x}$As heterostructures [17,18] or as an isolated (leadless) version in Si technology [19], the former with typical oscillation frequencies in the few GHz regime and nanosecond phase decoherence times, resulting in quality factors of orders 1 to 10. Alternatively, one may consider superconducting charge qubits, for example, the "Quantronium" [20], with a decoherence time reaching nearly a μs; this value, measured at the "sweet spot," will be reduced when choosing a working point which is sensitive to charge. At present, the resolution in the competition between a suitable charge sensitivity to achieve rotation angles of order π and the decoherence due to fluctuating charges in the environment remains a technological challenge. On the other hand, today's best solid-state qubit (with a decoherence time above 2 μs), the transmon [21,22], could be used as a photon counter in the microwave regime [23].

The aforementioned qubit characteristics have to be compared with the typical time scale of electronic transport in the wire. While under dc bias conditions subsequent electrons are separated by the voltage time $\tau = h/\mathrm{eV}$, single-electron wave packets can be generated by unit-flux voltage pulses of Lorentzian shape [24,25]. Recently, an alternative scheme has been used by Fève *et al.* [26], who have been injecting individual electrons from a quantum dot into an edge channel formed in the quantum Hall regime. Typical time scales $\tau = h/T\delta$ of single-electron pulses in their experiment range between 0.1 and 10 ns [26], where T and δ denote the tunneling probability and the level separation between states in the dot feeding the quantum wire. We conclude that today's charge qubits are at the border of becoming useful for the proposed electron counting experiments.

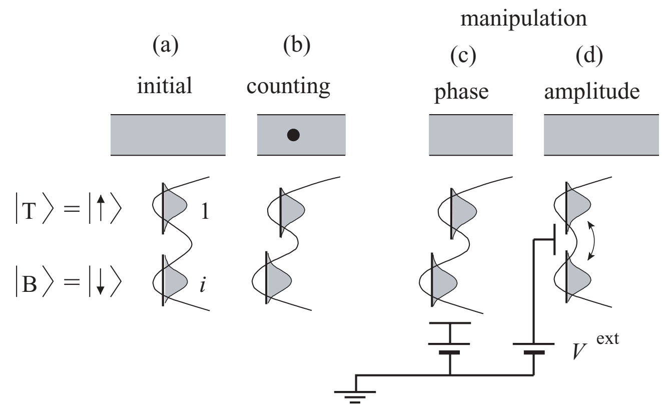

FIG. 3. Elements of the counting algorithm with double-dot charge qubits modeled as double-well systems. (a) The initial state $|+y\rangle = [|\uparrow\rangle + i\,|\downarrow\rangle]/\sqrt{2}$ involves balanced weights and phases 1 and i. (b) Particles are counted via their associated voltage pulses generating an unbalanced energy state of the qubit and thereby a phase shift between the states $|\uparrow\rangle$ and $|\downarrow\rangle$ (rotation around z). The initialization and readout involve manipulations of phase [see panel (c) where the levels $|\uparrow\rangle$ and $|\downarrow\rangle$ are disbalanced producing a rotation around z] and of amplitude [see panel (d) where the barrier between $|\uparrow\rangle$ and $|\downarrow\rangle$ is lowered, producing a rotation around x].

For the implementation of the charge qubit counter, we assume the two dots aligned perpendicular to the wire, such that they couple differently to the electron charge in the wire. We model the double dot as a two-well potential with quasiclassical states $|\mathrm{T}\rangle \equiv |\uparrow\rangle$ (top well, see Fig. 3; we use spin language in our analysis below) and $|\mathrm{B}\rangle \equiv |\downarrow\rangle$ (bottom well) and ground and excited states $|\pm\rangle = [|\uparrow\rangle \pm |\downarrow\rangle]/\sqrt{2}$ separated by the gap Δ. We consider a "phase mode operation" of the counter with a large barrier separating the quasiclassical states, resulting in an exponentially small tunneling amplitude $\propto \Delta$. To prepare the qubits in the state $|+y\rangle = [|\uparrow\rangle + i\,|\downarrow\rangle]/\sqrt{2}$, we measure and subsequently rotate the states around x by an angle of $-\pi/2$ ($\pi/2$) if the state $|\uparrow\rangle$ ($|\downarrow\rangle$) is measured (this rotation involves a lowering of the barrier, allowing for an amplitude shift, cf. Fig. 3). The passage of electrons in the wire generates a state $[|\uparrow\rangle + i e^{i\phi_j n}|\downarrow\rangle]/\sqrt{2}$ (up to an overall phase). The divisibility check involves a rotation around the x axis by an angle of $\pi/2$ and tests for the presence of all dot electrons in the state $|\uparrow\rangle$; if the answer is positive, the number n of particles passing the K double-dots is divisible by 2^K. In order to find the exact value of the cardinality n, another rotation around the z axis by an angle of $-m_{j-1}\phi_j$ has to be performed before rotating around x; for example, for the third qubit $j = 3$, after passage of seven electrons, the measurement of the first two qubits provides the binary number $(1,1)$, hence $m_2 = 3$, and a rotation by $-3\pi/4$ around z makes the third spin point along the $-y$ direction; storing a 1 as the third digit of the binary number we obtain $m_3 = 7$ [cf. Fig. 2(b)]. Using the aforementioned phase mode operation, the double dot does not act back on the electrons in the wire since the charge distribution of the qubits remains unchanged during the detection phase. When using an "amplitude mode operation" of the charge qubit, the moving charge of the qubit acts back on the wire, what can be exploited in the detection scheme (cf. Ref. [8]).

IV. APPLICATION: ENTANGLING PARTICLES

As an application, we discuss how to make use of our quantum counting algorithm to generate multiqubit entangled states. The basic idea is to generate a superposition of number states in the arms of a Mach-Zehnder interferometer and to use our counting device in order to project the system to a desired entangled state—depending on the reading of the counting register, we end up with different entangled states such as Greenberger-Horne-Zeilinger (GHZ) or Dicke states.

The standard way to entangle quantum degrees of freedom makes use of the interaction between the constituents. An alternative is provided by a projection technique, where a measurement selects the desired entangled state. In some cases, the projection makes use of the entangled state but simultaneously implies its destruction—more useful for quantum information processing are those schemes which entangle qubits for further use after the projection. Examples for the latter have been proposed using various arrangements of double-dot charge qubits combined with a quantum point contact serving as a quadratic detector [27,28] or (free) flying spin qubits tracked via a charge detector, where the charge provides an additional nonentangled degree of freedom associated with the entangled spins [29,30]. Here, we generate multiqubit orbital entanglement of flying qubits via their entanglement with our spin counters serving as ancillas; after reading of the counter states, the entangled multiqubit state can be further used.

Consider a particle entering the Mach-Zehnder interferometer from the lower-left lead and propagating along one of the two leads U or D (see Fig. 4). The wave function can propagate along two trajectories: the upper arm U where the particle picks up a phase φ_U and the spin counter is flipped, or the lower arm D accumulating a phase φ_D and leaving the spin unchanged. The total wave function evaluated at the position A then assumes the form

$$\Psi_{1A} = t\, e^{i\varphi_U}|\Uparrow\rangle \otimes |\downarrow\rangle + r\, e^{i\varphi_D}|\Downarrow\rangle \otimes |\uparrow\rangle, \tag{2}$$

where t and r denote transmission and reflection coefficients of the beam splitter and we have introduced a pseudospin notation to describe the propagation of the particles along the two arms:

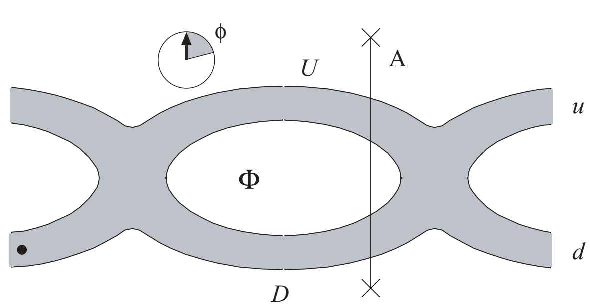

FIG. 4. Mach-Zehnder interferometer with spin counter. Particles enter the interferometer through the left leads (here the bottom lead) and are measured on the right. The spin counter in the upper arm U detects the passage of particles via a rotation by the angle ϕ. The magnetic flux Φ through the loop allows one to tune the phase difference when propagating along different arms.

a pseudospin $\Uparrow$ ($\Downarrow$) refers to propagation in the upper (lower) arm.

Next, we inject two particles (from the bottom left) into the Mach-Zehnder (MZ) loop with the spin-counter flipping by $\phi = \pi$ upon passage of one particle in the upper arm. We assume the two wave functions describing the initial state to be well separated in space, allowing us to ignore exchange effects in our (MZ) geometry. Assuming scattering coefficients for a symmetric beam splitter, for example, $t^2 = 1/2$ and $r^2 = (-1)/2$ and $r\,t = (\pm i)/2$, the wave function at the position A reads

$$\Psi_{2A} = \{[[-1]e^{2i\varphi_U}|\Uparrow,\Uparrow\rangle + (-1)e^{2i\varphi_D}|\Downarrow,\Downarrow\rangle] \otimes |\uparrow\rangle + (\pm i)e^{i(\varphi_U+\varphi_D)}[|\Downarrow,\Uparrow\rangle + |\Uparrow,\Downarrow\rangle] \otimes |\downarrow\rangle\}/2. \tag{3}$$

The factor $[-1]$ accounts for the phase π picked up in the rotation of the spin state by 2π (for a qubit, this phase can be tuned and assumes the value $[-1]$ if the coupling shifts the qubit levels symmetrically up and down), while the factor (-1) accounts for the additional scattering phases $(\pm i)$ in the reflection process. Choosing $\varphi_U = \varphi_D$, the measurement of the spin counter in the $|\uparrow\rangle$ state projects the particle wave function to the Bell state $|\Uparrow,\Uparrow\rangle + |\Downarrow,\Downarrow\rangle$, while the measurement of the $|\downarrow\rangle$ state generates the state $|\Downarrow,\Uparrow\rangle + |\Uparrow,\Downarrow\rangle$. The remaining Bell states can be obtained by injecting the two particles through the different leads on the left or applying a flux Φ (cf. Ref. [8]). Note that the indistinguishability of particles exploited in the aforementioned entanglement process is an "artificial" one defined by the qubit detector, rather than the "fundamental" one of identical particles.

The aforementioned scheme for entangling two particles with one spin-counter is easily extended to 2^K particles and an array of K spin-counters. As an illustration, we consider the case $K = 2$, four particles and two spin-counters. We use the shorthand $|j\rangle$, $j = 0,1,2,3,4$, with the identification $|0\rangle = |4\rangle = |\uparrow,\uparrow\rangle$ for the four different counter states, assume again a symmetric splitter, $\varphi_U = \varphi_D$, and injection from the bottom left; then

$$\begin{aligned}\Psi_{4A} = \{&[[-1]|\Uparrow,\Uparrow,\Uparrow,\Uparrow\rangle + |\Downarrow,\Downarrow,\Downarrow,\Downarrow\rangle] \otimes |0\rangle \\ &+ (\pm i)[|\Uparrow,\Uparrow,\Uparrow,\Downarrow\rangle + \cdots] \otimes |3\rangle \\ &+ [-1](-1)[|\Uparrow,\Uparrow,\Downarrow,\Downarrow\rangle + \cdots] \otimes |2\rangle \\ &+ (\mp i)[|\Uparrow,\Downarrow,\Downarrow,\Downarrow\rangle + \cdots] \otimes |1\rangle\}/4\end{aligned} \tag{4}$$

and proper projection provides us with specific entangled states with all pseudospins aligned (Greenberger-Horne-Zeilinger states [31]) or superpositions with exactly one-, two-, and three pseudospins pointing downward, among them the Dicke states [32] with an equal number of pseudospins pointing upward and downward.

Letting the particles propagate beyond the line A, the many-particle wave function undergoes mixing in the second beamsplitter (cf. Fig. 4). By properly choosing the transmission ($t = \cos\theta$) and reflection ($r = i\sin\theta$) coefficients of the second splitter, the pseudospins can be rotated into any direction, though all of the pseudospins are rotated equally. Different rotations of the pseudospins can be implemented by changing the characteristics of the splitter in time—the time separation of the particle wave packets can be enlarged, while compromising between leaving sufficient time for the manipulation of the splitter and keeping the system coherent.

We end with the discussion of a simple, though not rigorous, test for the presence of entanglement due to the action of the spin counter. We inject two (time delayed, to avoid exchange effects) particles through the two different leads on the left and measure the cross-correlator $\langle N_u N_d \rangle$ on the right ($N_{u,d} \in \{0,1,2\}$ denote the number of particles observed in the leads u and d). Without the counter, the product state generates the result $\langle N_u N_d \rangle = P_{1u,1} P_{1d,2} + P_{1d,1} P_{1u,2}$, where $P_{1x,i}$ denotes the probability to find the particle i in the lead x. With $P_{1u,1} = |t^2 + r^2 \exp(2\pi i \Phi/\Phi_0)|^2$ and $P_{1u,2} = |r\,t[1 + \exp(2\pi i \Phi/\Phi_0)]|^2$ and assuming symmetric splitters with $t^2 = 1/2$, $r^2 = -1/2$, we find that $\langle N_u N_d \rangle = [1 + \cos^2(2\pi\Phi/\Phi_0)]/2$. On the other hand, with the spin-counter selecting the singlet state $|\Uparrow,\Downarrow\rangle - |\Downarrow,\Uparrow\rangle$, only paths where the combined trajectories encircle the loop survive and the correlator is independent of Φ, $\langle N_u N_d \rangle = P_{2ud} = |t^4 + r^4|^2 + |2t^2r^2|^2 = 1/2$ for symmetric splitters. Hence post-selecting the spin-flipped events entangles the particles and quenches the Aharonov-Bohm oscillations in the cross correlator $\langle N_u N_d \rangle$. Such an analysis, although not as rigorous as the standard Bell-inequality test but much simpler to implement, may nevertheless serve as a preliminary indicator for the presence of entanglement.

V. SUMMARY AND CONCLUSION

We have introduced a base-2 quantum counting algorithm with qubits, providing the binary number representation of the cardinality of a finite stream of particles (electrons) passing in a nearby quantum wire. Our scheme exerts a minimal perturbation where both the counter and the particle merely pick up a phase in the counting process. Such a minimal coupling establishes the counting procedure as a sequence of forward and backward quantum Fourier transformations, a physical method often used in other contexts, for example, in imaging or crystallography; in the latter case, the x-ray beam generates a Bragg image (the forward Fourier transformation), which then has to be converted back into a real space structure (via a back transformation). A simpler single-shot measurement (rather than an inverse Fourier transformation) provides us with a divisibility test (by 2^k) for the train's cardinality. We have discussed aspects of a possible implementation with charge qubits and have shown how to make use of the algorithm in entangling particles in a Mach-Zehnder interferometer. Finally, we remark that our algorithm can be generalized to base-d counting with qudits, disclosing further interesting features of quantum counting in mesoscopics [33].

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