Quantum arrival time for open systems

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We extend previous work on the arrival time problem in quantum mechanics, in the framework of decoherent histories, to the case of a particle coupled to an environment. The usual arrival time probabilities are related to the probability current, so we explore the properties of the current for general open systems that can be written in terms of a master equation of the Lindblad form. We specialize to the case of quantum Brownian motion, and show that after a time of order the localization time of the current becomes positive. We show that the arrival time probabilities can then be written in terms of a positive operator-valued measure (POVM), which we compute. We perform a decoherent histories analysis including the effects of the environment and show that time-of-arrival probabilities are decoherent for a generic state after a time much greater than the localization time, but that there is a fundamental limitation on the accuracy δt, with which they can be specified which obeys $E\delta t \gg \hbar$. We confirm that the arrival time probabilities computed in this way agree with those computed via the current, provided there is decoherence. We thus find that the decoherent histories formulation of quantum mechanics provides a consistent explanation for the emergence of the probability current as the classical arrival time distribution, and a systematic rule for deciding when probabilities may be assigned.

I. INTRODUCTION

Questions involving time in quantum theory have a rich history, and there is still much debate about their status [1]. Quantities such as arrival and dwell times, despite being measurable [2], still lack concrete grounding within "standard" quantum theory, and there has been considerable interest in understanding these quantities in the framework of various interpretations of quantum theory. Arrival times, in particular, have attracted much interest, as the natural procedure of quantizing the appropriate classical quantity gives rise to an operator which is not self-adjoint and thus, in standard quantum theory at least, cannot easily be considered as an observable.

Despite these difficulties, if one considers a free particle in an initial state $\rho = |\psi\rangle \langle\psi|$ localized in $x > 0$ consisting entirely of negative momenta, and asks for the probability $p(t_1, t_2)$ that the particle crosses the origin during the time interval $[t_1, t_2]$ there *is* a semiclassical answer, given by the difference between the probability of being in $x > 0$ at the initial and final times [1]. Defining $P(t) = \theta(\hat{x}(t))$,

$$p(t_1, t_2) = \text{Tr}[P(t_1)\rho] - \text{Tr}[P(t_2)\rho]$$
$$= -\int_{t_1}^{t_2} dt \, \text{Tr}(P\dot{\rho}_t) = \int_{t_1}^{t_2} dt \, J(t), \quad (1.1)$$

where

$$J(t) = \frac{i\hbar}{2m}\left[\psi^*(0,t)\frac{\partial\psi(0,t)}{\partial x} - \frac{\partial\psi^*(0,t)}{\partial x}\psi(0,t)\right],$$

is the standard quantum-mechanical probability current. (We denote the state at time t by ρ_t.) This can also be written in the following two forms,

$$p(t_1, t_2) = \int_{t_1}^{t_2} dt \int dp\,dq \frac{(-p)\delta(q)}{m} W_t(p,q), \quad (1.2)$$

where $W_t(p,q)$ is the Wigner function, defined later, and

$$p(t_1, t_2) = \text{Tr}(C\rho)$$
$$C = \int_{t_1}^{t_2} dt \frac{(-1)}{2m}[\hat{p}\delta(\hat{x}(t)) + \delta(\hat{x}(t))\hat{p}]$$
$$= P(t_1) - P(t_2). \quad (1.3)$$

These expressions agree with the classical result, with the Wigner function W replaced by the classical distribution function w, provided that the classical trajectories are straight lines. They are also correctly normalized to 1 as $t_2 \to \infty$ with $t_1 = 0$, but they are not generally positive, even for states consisting entirely of negative momenta. This genuinely quantum phenomenon is called backflow and arises because the operator C, positive classically, has negative eigenvalues [3–5]. This means we cannot generally regard Eq. (1.1) as providing an acceptable arrival time distribution. There is, in addition, a more fundamental problem with Eq. (1.1), which is that probabilities in quantum theory should be expressible in the form [6]

$$p(\alpha) = \text{Tr}(P_\alpha\rho). \quad (1.4)$$

Here P_α is a projection operator, or more generally a positive operator-valued measure (POVM), associated with the outcome α. Equation (1.1) cannot be expressed in this form, and we therefore conclude that it is not a fundamental expression in quantum theory, so must be the result of some approximation.

An interesting clue as to how to improve on Eq. (1.1) is provided by the expression for the current in terms of the Wigner function, Eq. (1.2). Negativity of the Wigner function is a necessary condition for the negativity of the current (in the sense of backflow, as discussed in [7]), but it is known that evolution in the presence of an environment typically renders the Wigner function positive after a short time [8]. This suggests that something like Eq. (1.1) may be an acceptable, if still heuristic, arrival time distribution for a system coupled to an environment, at least after some time. The first aim of this paper is to derive the correct analog of Eq. (1.1) for a system coupled to an environment, and to prove that it can

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indeed be regarded as an arrival time distribution after a short time. In particular, we show that the arrival time probabilities computed in this way may be written as the trace of the density matrix times a POVM.

However, although this may allow us to interpret the current as the arrival time distribution, we are still left with the task of deriving Eq. (1.1) from some more fundamental quantity. This has previously been achieved, in some approximation, in Refs. [7,9] in the context of the decoherent histories approach to quantum theory [10–14], for the case of a free particle, and the derivation was shown to hold for states and intervals exhibiting sufficient decoherence. In decoherent histories the probability of arriving in an interval $[t_1, t_2]$ is approximately given by

$$p(t_1, t_2) = \text{Tr}(C\rho C^\dagger), \qquad (1.5)$$

with C given by Eq. (1.3), and this reduces to Eq. (1.1) under the somewhat special condition of decoherence. However, it was also shown that there exist states for which decoherence of histories cannot be obtained for reasonable levels of coarse graining, and for which therefore we cannot assign a time-of-arrival probability distribution. States exhibiting backflow are good examples of this.

Lack of decoherence for a general state is a situation frequently encountered in the literature on decoherent histories. The solution lies in the observation that realistic systems are always coupled to an environment, and that as such they are most fundamentally described by open system dynamics. Because by definition an environment consists of degrees of freedom about which we have no knowledge, and over which we have no control, it is natural to coarse-grain over these degrees of freedom. Such coarse-graining generally results in the recovery of approximate classical behavior and thus decoherence. We therefore anticipate that decoherence of histories corresponding to arrival times may be achieved for a generic state, provided that particle is coupled to a suitable environment. The second aim of this paper is to examine this scheme. We show that the probabilities calculated in this way are approximately decoherent, and approximately equal to those computed via the analog of Eq. (1.1), valid in the case of an environment.

Once we introduce an environment, however, the correspondence is with a classical stochastic theory, rather than with a deterministic one. The classical trajectories may now cross the origin many times due to fluctuations. The arrival time distribution for such a classical theory was computed in Ref. [15], and is given by

$$p(t_1, t_2) = \int_{t_1}^{t_2} dt \int_{-\infty}^{0} dp \int_{-\infty}^{\infty} dq \frac{(-p)}{m} \delta(q) w_t^r(p, q), \quad (1.6)$$

where $w_t^r(p, q)$ is the initial phase-space distribution, evolved with a type of restricted propagator valid in the case of an environment, and with boundary conditions, [15]

$$w_t^r(p, 0) = 0, \quad \text{for } p > 0. \qquad (1.7)$$

In Ref. [15] Halliwell and Zafiris presented a decoherent histories analysis of the corresponding quantum case. Although the conclusion they reached is sensible, their analysis actually contains a small error. This was the result of a lack of appreciation of the role of the quantum Zeno effect in these

calculations. We will show in this paper how the analysis can be corrected to give a correct expression for the arrival time probability in decoherent histories. We shall be interested in a particular limit of this general case where the stochastic trajectories are sharply peaked about the deterministic trajectories we would have in the absence of any environment. In this case the restricted propagator may be replaced with an unrestricted one, and we will show that quantization in this limit yields an expression of the form Eq. (1.2), but with the Wigner function evolved in the presence of an environment.

It will eventually be necessary to specialize to a particular model of system-environment coupling to obtain quantitative results, but we will begin by considering general models which can be written in terms of a master equation of the Lindblad form. This kind of system has been extensively studied in the decoherence literature, and the properties of such evolution, such as suppression of interference effects, loss of entanglement, and an approximate recovery of classical behavior have been discussed in [8,16–18].

This paper is arranged as follows, in Sec. II we explore some properties of the arrival time distribution for general open systems of the Lindblad form, our aim being to derive the corrections to the current resulting from the environmental dynamics. In Sec. III we discuss quantum Brownian motion and specialize the results of Sec. II to this case. In Sec. IV we derive some properties of the arrival time distribution for quantum Brownian motion, and in particular, we prove that the current becomes positive after a finite time. In Secs. V and VI we discuss the decoherent histories approach to quantum theory, and the expressions for the arrival time distribution that arise in this context. In Sec. VII we then examine whether histories corresponding to arriving in different intervals of time are decoherent. We summarize our results in Sec. VIII.

II. ARRIVAL TIME FOR OPEN QUANTUM SYSTEMS

We consider an open quantum system consisting of a free particle coupled to an environment with a master equation of the following Lindblad form [19],

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H, \rho] - \frac{1}{2}\sum_j (\{L_j^\dagger L_j, \rho\} - 2L_j \rho L_j^\dagger), \qquad (2.1)$$

where L_j are the Lindblad operators and $H = H_0 + H_1$ is the free Hamiltonian plus a possible extra term that depends on the L_j. (This term gives rise to frequency renormalization in oscillating systems.) Specific forms for the L_j may be computed for particular models, but for the moment we leave this general. The Lindblad equation represents the most general master equation for a Markovian system which preserves the properties of the density matrix, in particular positivity.

We now extend the derivation of Eq. (1.1) to this system. The probability of crossing during the interval $[t_1, t_2]$ is

$$p(t_1, t_2) = \text{Tr}(P\rho_{t_1}) - \text{Tr}(P\rho_{t_2}) = \int_{t_1}^{t_2} dt\, \text{Tr}(P\dot{\rho}_t)$$

$$= \frac{-1}{2m} \int_{t_1}^{t_2} dt\, \text{Tr}\{[\hat{p}\delta(\hat{x}) + \delta(\hat{x})\hat{p}]\rho_t\}$$

$$-\frac{i}{\hbar}\int_{t_1}^{t_2}dt\,\mathrm{Tr}([H_1,P]\rho_t)$$

$$+\frac{1}{2}\sum_j\int_{t_1}^{t_2}dt\,\mathrm{Tr}([L_j^\dagger,P]L_j\rho_t+L_j^\dagger[P,L_j]\rho_t),$$

$$(2.2)$$

where $P=\theta(\hat{x})$. The first term is the standard current expression, although with the state evolved according to Eq. (2.1), and we therefore recover Eq. (1.1) when all the L_j are 0. The second and third terms depend on the Lindblad operators L_j and thus on the form of the system-environment coupling.

To proceed further we specialize to the case where L is a linear combination of $\hat{x}$ and $\hat{p}$, $L=a\hat{x}+ib\hat{p}$, where a and b are real constants. The master equation Eq. (2.1) is then

$$\frac{\partial\rho_t}{\partial t}=-\frac{i}{\hbar}[H_0,\rho_t]-iab[\hat{x},\{\rho_t,\hat{p}\}]-\frac{a^2}{2}[\hat{x},[\hat{x},\rho_t]]$$

$$-\frac{b^2}{2}[\hat{p},[\hat{p},\rho_t]].$$

$$(2.3)$$

Note that this equation is also identical in form to the exact master equation for a particle in a gas environment, given in [20].

To derive the arrival time distribution we could simply substitute $L=a\hat{x}+ib\hat{p}$ into Eq. (2.2), but the algebra is somewhat clumsy, and there are a number of terms which vanish for reasons not immediately apparent from this expression. An equivalent approach is to start from Eq. (2.3), multiplying this expression by P and taking the trace. Because the second and third terms on the right of this expression have the form $[\hat{x},\hat{A}]$ their contribution is proportional to

$$\mathrm{Tr}([\hat{x},\hat{A}]P)=\mathrm{Tr}(\hat{A}[P,\hat{x}])=0,$$

so that only the first and last terms on the right-hand side of Eq. (2.3) contribute. The final term on the right of Eq. (2.3) gives a contribution

$$-\frac{b^2}{2}\mathrm{Tr}([\hat{p},[\hat{p},\rho_t]]P)=-\frac{b^2}{2}\mathrm{Tr}([\hat{p},[P,\hat{p}]]\rho_t)$$

$$=-i\hbar b^2\mathrm{Tr}([\hat{p}\delta(\hat{x})-\delta(\hat{x})\hat{p}]\rho_t),\quad(2.4)$$

so we arrive at the expression,

$$p(t_1,t_2)=\frac{-1}{2m}\int_{t_1}^{t_2}dt\,\mathrm{Tr}([\hat{p}\delta(\hat{x})+\delta(\hat{x})\hat{p}]\rho_t)$$

$$-i\hbar b^2\int_{t_1}^{t_2}dt\,\mathrm{Tr}([\hat{p}\delta(\hat{x})-\delta(\hat{x})\hat{p}]\rho_t).\quad(2.5)$$

The second term in this expression has a somewhat unusual form and its significance is not immediately clear. We will see in the following that this term is related to diffusion in position. There is a connection here to a recent paper by Genkin, Ferro, and Lindroth [21], in which the authors sought to examine the effects of an environment on the arrival time distribution. In that paper the authors used the standard expression for the current valid in the unitary case, Eq. (1.1), ignoring the extra terms that arise because of the environment. Although this may be a good approximation when we can neglect the effects of dissipation, it is clear that there may be significant corrections

to the current for strong dissipation. They also pointed out that these corrections are equivalent to the presence of extra terms in the continuity equation, although they did not compute these explicitly. For the sake of completeness, and also because it helps to understand the nature of the extra terms in Eq. (2.5), we will derive them here.

To derive the continuity equation we multiply Eq. (2.3) by $\delta(\hat{x}-x)$ and perform the trace. If we neglect the final three terms on the right we arrive at the standard result

$$\frac{\partial\rho_t}{\partial t}(x,x)+\frac{\partial J}{\partial x}(x,t)=0,\quad(2.6)$$

with $J(x,t)$ defined by an obvious extension of Eq. (1.3). Turning to the extra terms that result from the inclusion of the environment, the second and third terms vanish in exactly the same way as for the current above, and the correction term is therefore given by

$$-\frac{b^2}{2}\mathrm{Tr}([\hat{p},[\hat{p},\rho_t]]\delta(\hat{x}-x))=2\hbar^2b^2\frac{\partial^2\rho_t}{\partial x^2}(x,x),\quad(2.7)$$

so the continuity equation now reads

$$\frac{\partial\rho_t}{\partial t}(x,x)+\frac{\partial}{\partial x}[J(x,t)+J_D(x,t)]=0,\quad(2.8)$$

where we have identified the diffusive current,

$$J_D(x,t)=-2\hbar^2b^2\frac{\partial\rho_t}{\partial x}(x,x).\quad(2.9)$$

This is a specific example of a modification to a conservation equation resulting from open system dynamics. For a more general discussion of these issues see [22].

The analysis presented previously can also be phrased in terms of phase-space distributions. The Wigner function corresponding to ρ_t is defined as [23]

$$W_t(p,q)=\frac{1}{2\pi\hbar}\int d\xi\,e^{-\frac{i}{\hbar}p\xi}\rho_t(q+\xi/2,q-\xi/2).\quad(2.10)$$

The Wigner transform of the master equation Eq. (2.3) is

$$\frac{\partial W_t}{\partial t}=-\frac{p}{m}\frac{\partial W_t}{\partial q}+2\hbar^2ab\frac{\partial(pW_t)}{\partial p}+\frac{\hbar^2a^2}{2}\frac{\partial^2W_t}{\partial p^2}$$

$$+\frac{\hbar^2b^2}{2}\frac{\partial^2W_t}{\partial q^2}.\quad(2.11)$$

The first term is the standard unitary term, while the second term represents dissipation, and the third and fourth terms represent diffusion. The arrival time distribution can be written in terms of the Wigner function by taking the Wigner transform of Eq. (2.5)

$$p(t_1,t_2)=\int_{t_1}^{t_2}dt\int dp\,dq\left[\frac{(-p)}{m}\delta(q)W_t(p,q)\right.$$

$$\left.+\frac{\hbar^2b^2}{2}\delta(q)\frac{\partial W_t}{\partial q}(p,q)\right].\quad(2.12)$$

Equations (2.5) and (2.12) are the sought for generalization of Eq. (1.2) for the case of a particle coupled to an environment.

III. QUANTUM BROWNIAN MOTION

Quantum Brownian motion [16,17,24] is a commonly used form for the environment of an open system, partly because

it is exactly solvable and partly because, in many cases, it is a good approximation to a realistic environment. For the quantum Brownian motion model [17] we have Eq. (2.3) with

$$a = \sqrt{2D/\hbar^2}, \quad b = \frac{\gamma}{\sqrt{2D}}. \tag{3.1}$$

Here $D = 2m\gamma kT$, where T is the temperature of the environment, and γ is a phenomenological damping constant, see [24]. Equation (2.3) may be written as

$$\frac{\partial \rho_t}{\partial t}(x,y) = \frac{i\hbar}{2m}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\rho_t(x,y) - \frac{D}{\hbar^2}(x-y)^2\rho_t(x,y)$$

$$- \gamma(x-y)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\rho_t(x,y)$$

$$- \frac{\hbar^2\gamma^2}{D}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2\rho_t(x,y). \tag{3.2}$$

Although we could, in principle, work with this general case, it is useful to specialize to the case of negligible dissipation. There are two reasons for this, the first is that the analysis is considerably simplified, and this helps us to see the important effects more clearly. The second is that we have a particular aim in mind here, and that is to understand how a sensible classical result emerges from the quantum case. The classical case we have in mind is that of a heavy particle following an essentially classical, deterministic trajectory, but subject to small quantum fluctuations. We therefore restrict our analysis to time scales much shorter than the relaxation time γ^{-1}, and so we can drop the final two terms in the previous master equation.

This master equation may then be solved in terms of the propagator [16]

$$\rho_t(x,y) = \int dx_0\,dy_0\,J(x,y,t|x_0,y_0,0)\rho_0(x_0,y_0) \tag{3.3}$$

$$J(x,y,t|x_0,y_0,0) = \left(\frac{m}{2\pi\hbar t}\right)\exp\left(\frac{im}{2\hbar t}[(x-x_0)^2 - (y-y_0)^2]\right.$$

$$- \frac{Dt}{3}[(x-y)^2 + (x-y)(x_0-y_0)$$

$$\left. + (x_0-y_0)^2]\right). \tag{3.4}$$

Taking the same limit in the equation for the Wigner function, Eq. (2.11), gives

$$\frac{\partial W_t}{\partial t} = -\frac{p}{m}\frac{\partial W_t}{\partial q} + D\frac{\partial^2 W_t}{\partial p^2}. \tag{3.5}$$

Evolution of the Wigner function may also expressed in terms of a propagator [16],

$$W_t(p,q) = \int dp_0\,dq_0\,K(q,p,t|q_0,p_0,0)W_0(p_0,q_0) \tag{3.6}$$

$$K(q,p,t|q_0,p_0,0)$$

$$= N\exp\left[-\alpha(p-p_0)^2 - \beta\left(q - q_0 - \frac{p_0 t}{m}\right)^2\right.$$

$$\left. + \epsilon(p-p_0)\left(q - q_0 - \frac{p_0 t}{m}\right)\right], \tag{3.7}$$

where N, α, β, and ϵ are given by

$$\alpha = \frac{1}{Dt}, \quad \beta = \frac{3m^2}{Dt^3}, \quad \epsilon = \frac{3m}{Dt^2}, \quad N = \left(\frac{3m^2}{4\pi^2 D^2 t^4}\right)^{1/2}. \tag{3.8}$$

For later convenience we note that we can make the simple change of variables here, $q_0 \to q_0 - p_0 t/m$, so that

$$W_t(p,q) = \int dp_0\,dq_0\,\tilde{K}(q,p,t|q_0,p_0,0)\tilde{W}_0(p_0,q_0) \tag{3.9}$$

$$\tilde{K}(q,p,t|q_0,p_0,0) = N\exp[-\alpha(p-p_0)^2 - \beta(q-q_0)^2$$

$$+ \epsilon(p-p_0)(q-q_0)]$$

$$\tilde{W}(p,q) = W(p,q - pt/m). \tag{3.10}$$

This form makes it clear that the evolution consists of two effects. The first is a shifting along the classical trajectories, while the second is a spreading in phase space.

It is useful to consider this process in more detail. In the presence of an environment the width of the momentum distribution becomes time dependent, and we have,

$$(\Delta p)_t^2 = (\Delta p)_0^2 + Dt,$$

where $(\Delta p)_0$ is the momentum width of the initial state. We recognize an immediate difficulty here. Even for initial distributions consisting entirely of left-moving momenta, W_t will develop support on $p > 0$ under evolution. This means we cannot strictly regard the current as an arrival time distribution since a typical trajectory will now cross the origin many times. Differently put, the arrival time distribution is strictly a measurement of the first passage time, and this is no longer equal to the current. (Note that although this result is similar to the problems created by backflow, the reasons behind it are very different. The spreading of momentum induced by evolution in an environment is a purely classical effect.)

However, all is not lost. Although the current is no longer strictly the arrival time distribution, it may be a very good approximation to it. This is because the deviation of the current from the "true" arrival time distribution will be related to the probability that the state has the "wrong" sign momenta. This means that, provided we are in the "near-deterministic" limit,

$$(\Delta p)_t^2 \ll p_0^2, \tag{3.11}$$

where p_0 is the momentum around which the initial state is tightly peaked, the current will still be a very good approximation to the true arrival time distribution. The time scale on which this analysis breaks down is given by the "stochastic" time

$$\tau_s = p_0^2/D.$$

After this time we must therefore revert to using the exact expression for the arrival time given by Eq. (1.6). We see from the definition of D that,

$$\tau_s \gamma = \frac{p_0^2}{2mkT} \gg 1, \tag{3.12}$$

for the states we are interested in. This means the stochastic time τ_s is much longer than the relaxation time γ^{-1}, so that

working in the near-deterministic limit does not impose any further constraints compared with neglecting dissipation.

Returning to our arrival probability, in the limit of negligible dissipation Eq. (2.5), becomes

$$p(t_2, t_1) = \int_{t_1}^{t_2} dt \, J(t) \tag{3.13}$$

$$J(t) = \int dq \, dp \frac{(-p)}{m} \delta(q) W_t(p, q). \tag{3.14}$$

This expression is now identical to the unitary case, Eq. (1.1), but with the Wigner function evolved under quantum Brownian motion.

We now turn to the question of whether inclusion of an environment helps ensure the positivity of Eq. (3.13).

IV. PROPERTIES OF THE ARRIVAL TIME DISTRIBUTION IN QUANTUM BROWNIAN MOTION

In the introduction we noted that evolution in an environment typically renders W_t positive after a short time. We now wish to examine the effect of this on our candidate arrival time distribution Eq. (3.13). To this end we introduce the notation [8]

$$\mathbf{z} = \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}, \tag{4.1}$$

and also the class of Gaussian phase-space functions

$$g(\mathbf{z}; A) = \frac{1}{2\pi |A|^{1/2}} \exp\left(-\frac{1}{2} \mathbf{z}^T A^{-1} \mathbf{z}\right), \tag{4.2}$$

where A is a 2×2 positive-definite matrix, with determinant $|A|$. $g(\mathbf{z}; A)$ will be a Wigner function if and only if

$$|A| \geqslant \frac{\hbar^2}{4}. \tag{4.3}$$

A useful result is that

$$\int d^2 \mathbf{z} g(\mathbf{z}_1 - \mathbf{z}; A) g(\mathbf{z} - \mathbf{z}_2; B) = g(\mathbf{z}_1 - \mathbf{z}_2; A + B). \tag{4.4}$$

In this notation we can write the propagation of the Wigner function, Eq. (3.6), as

$$W_t(\mathbf{z}) = \int d\mathbf{z}' g(\mathbf{z} - \mathbf{z}'; A) \tilde{W}_0(\mathbf{z}'), \tag{4.5}$$

where

$$A = Dt \begin{pmatrix} 2 & t/m \\ t/m & 2t^2/3m^2 \end{pmatrix}. \tag{4.6}$$

Since

$$|A| = \frac{D^2 t^4}{3m^2} \tag{4.7}$$

after a time

$$t = \left(\frac{3}{16}\right)^{1/4} \left(\frac{2m\hbar}{D}\right)^{1/2} = \left(\frac{3}{16}\right)^{1/4} \tau_l, \tag{4.8}$$

$g(\mathbf{z} - \mathbf{z}'; A)$ will be a Wigner function, and thus $W_t(\mathbf{z})$ will be positive because it is equal to the convolution of two Wigner functions. Here $\tau_l = \sqrt{2m\hbar/D}$ is the localization time.

This is a useful result. Expressing the current Eq. (3.14) in this new notation

$$J(t) = \int d\mathbf{z} \frac{(-z_0)}{m} \delta(z_1) W_t(\mathbf{z}), \tag{4.9}$$

we see that after the time given in Eq. (4.8), because $W_t > 0$ the current Eq. (4.9) will be positive if the state consists purely of negative momenta. This means that after this time Eq. (3.13) will be a positive arrival time distribution. This holds provided the times involved are much smaller than τ_s, as per the discussion below Eq. (3.10).

Now we turn to examining the properties of the current. We wish to find an expression for the current in the form $\text{Tr}(P\rho)$, where P is a projector or POVM. This would allow us to express the heuristic arrival time distribution, Eq. (1.2), in the same form as other probabilities in quantum theory. Starting from Eq. (4.9) it is useful to write

$$J(t) = \int d\mathbf{z} \int d\mathbf{z}' \frac{(-z_0)}{m} \delta(z_1) g(\mathbf{z} - \mathbf{z}'; A) \tilde{W}_0(\mathbf{z}'), \tag{4.10}$$

using Eq. (4.5). We can deconvolve the propagator into two Gaussians using Eq. (4.4), in particular we will let $A = A_0 + B$, where A_0 is a minimum uncertainty Gaussian, and B is the remainder

$$A_0 = \hbar \begin{pmatrix} s^2 & 0 \\ 0 & 1/4s^2 \end{pmatrix}, \tag{4.11}$$

$$B = Dt \begin{pmatrix} 2 & t/m \\ t/m & 2t^2/3m^2 \end{pmatrix} - A_0. \tag{4.12}$$

Here s is some real number. Using the convolution property Eq. (4.4) we can write the current Eq. (4.9) as

$$\begin{aligned} J(t) &= \int d\mathbf{z}'' \left[\int d\mathbf{z} \frac{(-z^0)}{m} \delta(z^1) g(\mathbf{z} - \mathbf{z}''; B) \right] \\ &\quad \times \left[\int d\mathbf{z}' \tilde{W}_0(\mathbf{z}') g(\mathbf{z}'' - \mathbf{z}'; A_0) \right] \\ &= \int d\mathbf{z}'' \left[\int d\mathbf{z} \frac{(-z^0)}{m} \delta(z^1 + z_0 t/m) g(\mathbf{z} - \mathbf{z}''; B) \right] Q(\mathbf{z}''), \end{aligned} \tag{4.13}$$

where $Q(\mathbf{z})$ is the Q function [25], and we have undone the change of variables implied in Eq. (3.10). The Q function can be written as

$$Q(\mathbf{z}) = \frac{1}{\pi} \langle \mathbf{z} | \rho | \mathbf{z} \rangle.$$

We can therefore express the current as

$$J(t) = \text{Tr}(F\rho), \tag{4.14}$$

where

$$\begin{aligned} F &= \frac{1}{\pi} \int d\mathbf{z}'' |\mathbf{z}''\rangle \langle \mathbf{z}''| \left[\int d\mathbf{z} \frac{(-z_0)}{m} \delta(z_1 + z_0 t/m) g(\mathbf{z} - \mathbf{z}''; B) \right] \\ &= \int d\mathbf{z} P_{\mathbf{z}} \frac{(-z_0)}{m} \delta(z_1 + z_0 t/m), \end{aligned} \tag{4.15}$$

and we have defined the POVM

$$P_{\mathbf{z}} = \frac{1}{\pi} \int d\mathbf{z}'' |\mathbf{z}''\rangle \langle \mathbf{z}''| g(\mathbf{z} - \mathbf{z}''; B), \tag{4.16}$$

which is clearly a phase-space operator localized around $\mathbf{z}$. F is therefore a smeared version of the object used to compute the current classically, $-p\delta(x_t)/m$. This holds for times

$$\left(\frac{3}{16}\right)^{1/4}\tau_l \leqslant t \ll \tau_s. \tag{4.17}$$

Assume for a moment that $B = 0$, and so $P_{\mathbf{z}} = |\mathbf{z}\rangle\langle\mathbf{z}|$. The current would then be given by

$$J(t) = \int d\mathbf{z}\frac{(-z_0)}{m}\delta(z_1 - z_0 t/m)Q(\mathbf{z}). \tag{4.18}$$

Since the Q function is positive by construction the current computed in this way will be positive to the extent that the Q function has vanishing support on $p > 0$ [$Q(\mathbf{z})$ cannot strictly vanish for $p > 0$ but it can be exponentially small, and this suffices here. See the comments below Eq. (3.10)]. For $B > 0$ the Q function is simply smeared further, and this will preserve the property of positivity. In fact one could imagine *postulating* Eq. (4.18) as the definition of the arrival time even in the unitary case since it clearly satisfies all the conditions required.

The probability of arriving in the interval $[t_1, t_2]$ is given by the integral of Eq. (4.14), and because $P_{\mathbf{z}}$ is time dependent (via B) this will not, in general, have a simple form. However, there is a separation of time scales here, the time scale of evolution of $P_{\mathbf{z}}$ is seen from Eq. (4.7) to be τ_l. So, if $t_2 - t_1 \ll \tau_l$ we have approximately

$$\begin{aligned}
p(t_1, t_2) &= \int_{t_1}^{t_2} dt\,\mathrm{Tr}(F\rho)\\
&\approx \int d\mathbf{z}\,\mathrm{Tr}\left[\rho P_{\mathbf{z}}\int_{t_1}^{t_2} dt\frac{(-z_0)}{m}\delta(z_1 + z_0 t/m)\right]\\
&= \mathrm{Tr}(E\rho),
\end{aligned} \tag{4.19}$$

where

$$E = \int d\mathbf{z}\,P_{\mathbf{z}}[\theta(z_1 + z_0 t_1/m) - \theta(z_1 + z_0 t_2/m)], \tag{4.20}$$

is a POVM representing arrival at $x = 0$ between t_1 and t_2. Note the close similarity between this object and the operator Eq. (1.3), but that crucially this phase-space operator is positive for $z_0 < 0$ (i.e., $p < 0$). In this case, therefore, the effect of the environment is simply to smear the unitary result over a region of phase space, given by $P_{\mathbf{z}}$ computed at t_1.

Equations (4.19) and (4.20) form the first significant result of this paper. Equation (4.19) expresses the heuristic arrival time distribution, Eq. (1.2), as the trace of an operator times a POVM, and thus has the same form as standard expressions for probability in quantum theory. The POVM, Eq. (4.20), arises because the environment effectively measures the system.

We have therefore discovered a range of times for which the current, Eq. (4.9), gives a positive arrival time distribution. After a time of order τ_l, interference effects vanish and we can regard Eq. (4.9) as the arrival time distribution, which can also be written in the form Eq. (4.14). Eventually, however, on a time scale of order τ_s the environment causes diffusion in momentum to such an extent that the trajectories of the particle are no longer sharply peaked around the classical trajectory computed in the absence of an environment. Since the particle is still behaving classically after this time there will exist an arrival time distribution of the form Eq. (1.6), but our simpler

expression Eq. (4.9) will no longer be a good approximation to this. It is easy to show that $\tau_s/\tau_l = E\tau_l/\hbar$, where E is the energy of the initial state, and thus these expressions are valid for a large interval if $E\tau_l \gg \hbar$.

V. THE DECOHERENT HISTORIES APPROACH TO THE ARRIVAL TIME PROBLEM

A. The decoherent histories approach to quantum theory

We begin by briefly reviewing the decoherent histories approach to quantum theory. More extensive discussions can be found in Refs. [10–14].

Alternatives at fixed moments of time in quantum theory are represented by a set of projection operators $\{P_a\}$, satisfying the conditions

$$\sum_a P_a = 1, \tag{5.1}$$

$$P_a P_b = \delta_{ab}P_a, \tag{5.2}$$

where we take a to run over some finite range. In the decoherent histories approach to quantum theory, histories are represented by class operators C_α which are time-ordered strings of projections

$$C_\alpha = P_{a_n}(t_n), \dots, P_{a_1}(t_1), \tag{5.3}$$

or sums of such strings [26]. Here the projections are in the Heisenberg picture and α denotes the string $(a_1, \dots, a_n)$. All class operators satisfy the condition

$$\sum_\alpha C_\alpha = 1. \tag{5.4}$$

Probabilities are assigned to histories via the formula

$$p(\alpha) = \mathrm{Tr}(C_\alpha \rho C_\alpha^\dagger). \tag{5.5}$$

However, probabilities assigned in this way do not necessarily obey the probability sum rules because of quantum interference. We therefore introduce the decoherence functional

$$D(\alpha, \beta) = \mathrm{Tr}(C_\alpha \rho C_\beta^\dagger), \tag{5.6}$$

which may be thought of as a measure of interference between pairs of histories. We require that sets of histories satisfy the condition of decoherence, which is

$$D(\alpha, \beta) = 0, \quad \alpha \neq \beta. \tag{5.7}$$

This ensures that all probability sum rules are satisfied.

We note briefly that when there is decoherence Eqs. (5.4) and (5.7) together imply that the probabilities $p(\alpha)$ are given by the simpler expressions

$$q(\alpha) = \mathrm{Tr}(C_\alpha \rho). \tag{5.8}$$

Decoherence ensures that $q(\alpha)$ is real and positive, even though it is not in general. In this way decoherent histories may reproduce probabilities of the form Eq. (1.3).

B. The decoherent histories approach to the arrival time problem

We turn now to the definition of the arrival time problem in decoherent histories. In Refs. [7,9] the class operators

corresponding to a first crossing of the origin in the time interval $[t_{k-1}, t_k]$ were computed to be

$$C_k = \bar{P}(t_k)P(t_{k-1}), \ldots, P(t_2)P(t_1), \qquad (5.9)$$

for $k \geqslant 2$, with $C_1 = \bar{P}(t_1)$ and where $\bar{P}(t) = \theta(-\hat{x}_t)$. These clearly describe histories which are in $x > 0$ at times $t_1, t_2, \ldots, t_{k-1}$ and in $x < 0$ at time t_k, so, approximately, describe a first crossing between t_{k-1} and t_k. We are prevented from taking the time between projections to zero by the Zeno effect [27]. This point was extensively discussed in [7,9].

In Ref. [9] these class operators were then simplified with the help of the following semiclassical approximation

$$P(t_n), \ldots, P(t_2)P(t_1)|\psi\rangle \approx P(t_n)|\psi\rangle. \qquad (5.10)$$

This is the statement that, given that the state is in $x > 0$ at t_n, it must also have been $x > 0$ at all previous times (we restrict attention here to states with $p < 0$). It is clear from the path integral representation of the propagator that this is only true semiclassically. From this we obtain the class operators for the first crossing between t_{k-1} and t_k as

$$C_k \approx \bar{P}(t_k)P(t_{k-1}) \qquad (5.11)$$
$$= P(t_{k-1}) - P(t_k)P(t_{k-1})$$
$$\approx P(t_{k-1}) - P(t_k), \qquad (5.12)$$

where we use the semiclassical approximation again to arrive at the last line. In the case of a free particle without an environment, this class operator reproduces Eq. (1.1) under the conditions of decoherence. This is because, assuming decoherence, the arrival time probability computed from decoherent histories is

$$p(t_2, t_1) = \text{Tr}(C\rho C^\dagger) = \text{Tr}(C\rho), \qquad (5.13)$$

using Eq. (5.8).

VI. DECOHERENCE AND THE SEMICLASSICAL APPROXIMATION FOR ARRIVAL TIME

In the previous section we introduced the arrival time problem in decoherent histories. The key step in making contact with the simple heuristic result of Eq. (1.1) was the approximation of the class operators Eq. (5.9) by Eq. (5.12). We now want to explore this approximation, and prove that it is indeed valid in some interesting limit.

For a free particle, the error in this approximation comes from the finite width of the wave packet. This means that

$$P(t_k)|\psi\rangle \neq |\psi\rangle, \qquad (6.1)$$

even for times t_k, significantly before the classical crossing time. However, it is clear that the largest error will come from the last projection that precedes the classical crossing time. Since the spacing between projections has to be greater than the Zeno time, which is also the time scale on which a wave packet may be said to "cross" the origin, only one or two projections stand any chance of significantly disturbing the wave packet.

Writing this in terms of the density matrix,

$$\text{Tr}[P(t_{k+1})P(t_k)\rho P(t_k)] = \text{Tr}[P(t_{k+1})\rho] - \text{Tr}[P(t_{k+1})\bar{P}(t_k)\rho] - \text{Tr}[\rho \bar{P}(t_k)P(t_{k+1})]$$

$$+ \text{Tr}[P(t_{k+1})\bar{P}(t_k)\rho \bar{P}(t_k)]$$
$$\approx \text{Tr}[P(t_{k+1})\rho],$$

where the last line defines the semiclassical approximation. Noting that with $t_m \geqslant t_k$

$$|\text{Tr}[P(t_m)\bar{P}(t_k)\rho]|^2 \leqslant \text{Tr}[P(t_m)\bar{P}(t_k)\rho \bar{P}(t_k)]. \qquad (6.2)$$

A sufficient condition for the validity of the semiclassical approximation is therefore that the object

$$\Delta_{k,m} := \text{Tr}[P(t_m)\bar{P}(t_k)\rho \bar{P}(t_k)], \qquad (6.3)$$

is much smaller than 1 for all $t_m > t_k$.

Now assume for a moment that the semiclassical approximation holds, so that the class operators are given by Eq. (5.12). The probability of first crossing between t_k and t_{k+1} is given by

$$\text{Tr}[C_k\rho C_k^\dagger] = \text{Tr}[C_k\rho] + 2\,\text{Tr}[\rho P(t_{k+1})] - \text{Tr}[\rho P(t_k)P(t_{k+1})]$$
$$- \text{Tr}[P(t_{k+1})P(t_k)\rho].$$

However, since we are assuming that the semiclassical approximation holds, the last three terms cancel and we obtain

$$\text{Tr}(C_k\rho C_k^\dagger) = \text{Tr}(C_k\rho) = \text{Tr}(\rho C_k^\dagger).$$

Furthermore, consider a general off-diagonal term in the decoherence functional, where without loss of generality we take $t_{k+1} < t_m$

$$\text{Tr}[C_k\rho C_m^\dagger] = \text{Tr}[P(t_m)P(t_k)\rho] - \text{Tr}[P(t_m)P(t_{k+1})\rho]$$
$$- \text{Tr}[P(t_{m+1})P(t_k)\rho] + \text{Tr}[P(t_{m+1})P(t_{k+1})\rho].$$
$$(6.4)$$

We see that this vanishes if the semiclassical approximation holds.

What we have shown therefore is that the semiclassical condition and the decoherence conditions are closely related. If the object given in Eq. (6.3) is much smaller than 1 then both conditions are satisfied. This means first that the class operators are approximately given by Eq. (5.12), and second that the histories described by these class operators are approximately decoherent.

In general, the probabilities for time of arrival as computed by decoherent histories differ from the heuristic ones obtained by the standard quantum-mechanical analysis. We do not expect agreement since, as discussed in the Introduction, the heuristic formula, Eq. (1.1), is not, in general, of the canonical form required for genuine quantum-mechanical probabilities. It is also the case that decoherent histories only ascribe probabilities to certain sets of histories. However, if the object $\Delta_{k,m}$ is small, then we have shown that the decoherent histories analysis reproduces the probabilities computed in the standard, heuristic way.

As well as clarifying the analysis of [9], note that all of the statements above apply to the case of a particle coupled to an environment. In this case the trace is to be taken over the system and environment, and $P(t_k) = P(t_k)_s \otimes 1_\epsilon$ is a projection onto the degrees of freedom of the system s tensored with the identity operator on the environmental degrees of freedom ϵ. This is the case of interest in this paper.

Since the object $\Delta_{k,m}$ plays a central role in our analysis it is interesting to ask if it has any physical interpretation. The answer is that it does. Recall that our class operator for crossing between t_{k-1} and t_k is given by Eq. (5.12). We see, however, that

$$P(t_{k-1}) - P(t_k) = \bar{P}(t_k)P(t_{k-1}) - P(t_k)\bar{P}(t_{k-1}). \quad (6.5)$$

The approximation leading from Eq. (5.11) to Eq. (5.12) is equivalent to dropping the second term on the right-hand side of Eq. (6.5). The error in this approximation can be estimated by computing the probability associated with the class operator $P(t_k)\bar{P}(t_{k-1})$. We see

$$\mathrm{Tr}([P(t_k)\bar{P}(t_{k-1})]\rho[P(t_k)\bar{P}(t_{k-1})]^{\dagger})$$
$$= \mathrm{Tr}[P(t_k)\bar{P}(t_{k-1})\rho\,\bar{P}(t_{k-1})] = \Delta_{k-1,k}.$$

However, Eq. (6.5) has a simple physical interpretation: It is the decomposition of the total current into right-moving and left-moving parts. The object $\Delta_{k,m}$ is therefore just the probability associated with the right-moving current. Classically this is small by construction since we will choose to work with wave packets tightly peaked in negative momentum. Quantum mechanically, however, this term need not be small, indeed the existence of the backflow effect [4] shows that this term can sometimes be larger than the left-moving current, even for wave packets composed entirely of negative momenta.

The semiclassical condition we are imposing, and by extension the decoherence condition, is therefore stronger than the condition for the current to be positive. Decoherence requires the absence of interference between crossings at different times, while the standard heuristic analysis includes some interference effects, provided they are not so large as to render the arrival time probabilities negative. The significance of this will be explored elsewhere.

VII. DECOHERENCE OF ARRIVAL TIMES IN QUANTUM BROWNIAN MOTION

A. General case

In the previous section we have shown that an arrival time distribution may be derived from decoherent histories and that it agrees with the current provided there is decoherence. We therefore turn now to the question of determining for which states and intervals decoherence is achieved. Recall that our aim is to show that the inclusion of an environment gives rise to decoherence of arrival time probabilities for generic initial states of the system. We will start our decoherent histories analysis with a discussion of the general case considered in Ref. [15], but most of our detailed results will concern the near-deterministic limit discussed in the Introduction and in the previous discussion of the Wigner function.

The decoherence functional may be written in path integral form as

$$\mathcal{D}(\alpha,\alpha') = \int_{\alpha}\mathcal{D}x\int_{\alpha'}\mathcal{D}y\exp\left(\frac{i}{\hbar}S[x] - \frac{i}{\hbar}S[y] + iW[x,y]\right)$$
$$\times \rho_0(x,y), \quad (7.1)$$

where α,α' represents the restriction to paths that are, for example, in $x > 0$ at t_1 and in $x < 0$ at t_2. $W[x,y]$ is the Feynman-Veron influence functional phase which summarizes

the effect of the environment, and is given in the case of negligible dissipation by

$$W[x,y] = \frac{iD}{\hbar^2}\int dt(x-y)^2. \quad (7.2)$$

It is this phase which is responsible for the suppression of interference between paths $x(t)$ and $y(t)$ that differ greatly and produces decoherence.

The decoherent histories analysis we are about to perform was first attempted by Halliwell and Zafiris in Ref. [15]. Their conclusions are reasonable, but the analysis leading to them, in fact, contains a small error. This error arose due to a lack of appreciation of the role of the Zeno effect, as discussed in the Introduction. We aim to show here how the analysis may be modified in line with the treatment of the arrival time problem presented here and in Ref. [9].

We start from the expression for the density matrix for states that do not cross the origin in a time interval $[0,t]$ (Ref. [15], Eq. (4.34))

$$\rho_{rr}(x_f,y_f) = \int_r\mathcal{D}x\int_r\mathcal{D}y\exp\left[\frac{im}{2\hbar}\int dt(\dot{x}^2 - \dot{y}^2)\right.$$
$$\left. - \frac{D}{\hbar^2}\int dt(x-y)^2\right]\rho_0(x_0,y_0). \quad (7.3)$$

Here the restriction is interpreted as $x,y > 0$ at times $\epsilon, 2\epsilon, \ldots$. The probability that the state does not cross the origin in this interval is then given by the trace of this expression. There are two interesting limits to be taken here. The first is letting $\epsilon \to 0$, and this recovers the notion of the restricted propagator. The second limit is that of strong decoherence effects, so that we can assume that the path integral is tightly peaked in $(x - y)$, this recovers the classical limit. The claim in Ref. [15] is that we can take these limits simultaneously. To see why this is problematic we write the restricted path integral as a product of propagators,

$$\rho_{rr}(x_f,y_f) = \prod_{k=0}^{n}\int_0^{\infty}dx_k\,dy_k\,J(x_{k+1},y_{k+1},t_{k+1}|x_k,y_k,t_k)$$
$$\times \rho_0(x_0,y_0), \quad (7.4)$$

where $t_k = k\epsilon$, $t_0 = 0$, $t_{n+1} = t$. Changing variables to $X = x + y$, $\xi = \frac{1}{2}(x-y)$ gives

$$\rho_{rr}(x_f,y_f) = \prod_{k=0}^{n}\int_0^{\infty}dX_k\int_{-X_k}^{X_k}d\xi_k\left(\frac{m}{\pi\hbar\epsilon}\right)$$
$$\times \exp\left\{\frac{im}{\hbar\epsilon}[(\xi_{k+1} - \xi_k)(X_{k+1} - X_k)]\right.$$
$$\left. - \frac{D\epsilon}{3\hbar^2}[\xi_{k+1}^2 + \xi_{k+1}\xi_k + \xi_k^2]\right\}\rho_0(X_0,\xi_0). \quad (7.5)$$

Now in order for the product of propagators to be equal to a restricted propagator, we need to take $\epsilon \to 0$. However, to recover the classical result we need to replace the limits of integration for ξ_k with $\pm\infty$, which requires that $D\epsilon \gg \hbar^2/l^2$ where l is some length scale. Clearly these two limits are incompatible for finite D.

The standard discussion of the Zeno effect emphasizes the role of the so-called "Zeno time" $\tau_z = \hbar(\Delta H)^{-1}$. It is generally

held that projections separated by times greater than this will not give rise to the Zeno effect. The issue is not without subtlety in the present case, however, first because the Zeno effect is usually discussed in the context of unitary evolution, and this is not the case here, and second because characteristic times associated with the Zeno effect usually refer to selective measurement, and a projection onto positive x is certainly nonselective. This casts doubt on the role of the Zeno time as a time scale in this problem. Studies of the arrival time problem defined using a complex potential [7] suggest that the relevant time scale might be given by $\hbar/E$. This issue will be taken up elsewhere [28].

In any case it is clear that there exists a short time scale on which we can still approximate the limits in the ξ integrals above in the manner indicted, but that is short enough to give nontrivial crossing probabilities. Following the steps in Ref. [15] then gives

$$p(t_1,t_2) = 1 - \mathrm{Tr}(\rho_{rr})$$
$$= \int_{t_1}^{t_2} dt \int_{-\infty}^{0} dp \int_{-\infty}^{\infty} dq \frac{(-p)}{m} \delta(q) W_t^r(p,q), \quad (7.6)$$

where the Wigner function W_t^r obeys

$$W_t^r(p,q) = \int dp_0 \, dq_0 K_r(p,q,t|p_0,q_0,0) W_0(p_0,q_0), \quad (7.7)$$

where K_r is the "restricted" propagator, defined by

$$K_r(p,q,t|p_0,q_0,0)$$
$$= \prod_{k=1}^{n} \int dp_k \int_0^{\infty} dq_k K(p_k,q_k,t_k|p_{k-1},q_{k-1},t_{k-1}), \quad (7.8)$$

with $t_k = k\epsilon$. However, because we are now dealing with an essentially classical system, this propagator is well approximated by its continuum limit. Understood in this sense, we see that the conclusions of Ref. [15] are, in fact, correct, even if the argument leading to them is not. The interesting subtleties that arise in this argument relate to the classical limit of the Zeno effect, and this will be explored elsewhere [28].

As important as this general case is, it is of interest to examine the simpler case of near-deterministic evolution, where the classical limit of the arrival time probability is simply expected to be the time integral of the current density. The analysis in this case simplifies considerably, and we will explicitly exhibit decoherence of histories for suitable intervals.

B. The near-deterministic limit

We have previously seen that, in the near-deterministic limit, if there is decoherence then the arrival time probabilities derived from decoherent histories agree with those computed from the current. We therefore turn now to discussing the conditions under which we have decoherence of crossing probabilities. The general picture we have in mind is illustrated in Fig. 1, we have an initial wave packet defined at $t = 0$, evolved in the presence of an environment until time t_1, and then we wish to compute the probability of crossing between t_1 and t_2.
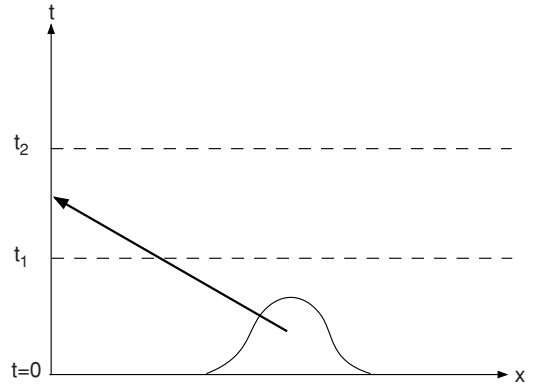

FIG. 1. The probability that a generic wave packet arrives between t_1 and t_2 can be expressed in terms of the state at t_1, or via the propagator, in terms of an initial time $t = 0$.

Our task is to compute the off-diagonal elements of Eq. (7.1), and this will be rather involved in general. However, we are saved from having to do this by the observation in Sec. VI about the relation between the semiclassical condition and the decoherence condition. It suffices therefore to compute the quantity $\Delta_{k,m}$ defined in Eq. (6.3), in the presence of an environment. We anticipate that this will be small simply from the form of Eq. (7.1). This is because the effect of the environment is to cause the density matrix to become tightly peaked around the classical path and classically, for $p < 0$ the probability given by $\Delta_{k,m}$ is zero. We will see how this works in a specific example in the following.

We take $k = 1$, $m = 2$ without loss of generality, and we drop the subscript from now on

$$\Delta = \mathrm{Tr}[P(t_2)\bar{P}(t_1)\rho\bar{P}(t_1)]$$
$$= \int_{\alpha} \mathcal{D}x \int_{\alpha} \mathcal{D}y \exp\left[\frac{i}{\hbar}S[x] - \frac{i}{\hbar}S[y] - \frac{D}{\hbar^2}\int dt(x-y)^2\right]$$
$$\times \rho_{t_1}(x_1,y_1), \quad (7.9)$$

where the histories α are those that start at $x_1, y_1 < 0$ and finish at $x_2 > 0$. In terms of the density matrix propagator, Eq. (3.4)

$$\Delta = \int_0^{\infty} dx_2 \int_{-\infty}^{0} dx_1 \int_{-\infty}^{0} dy_1 J(x_2,x_2,t_2|,x_1,y_1,t_1)\rho_{t_1}(x_1,y_1)$$
$$= \int_0^{\infty} dx_2 \int_{-\infty}^{0} dx_1 \int_{-\infty}^{0} dy_1 \left(\frac{m}{\pi\hbar t}\right)$$
$$\times \exp\left\{\frac{im}{2\hbar(t_2-t_1)}[(x_2-x_1)^2 - (x_2-y_1)^2]\right.$$
$$\left. - \frac{D(t_2-t_1)}{3\hbar^2}(x_1-y_1)^2\right\}\rho_{t_1}(x_1,y_1).$$

Transforming to new variables

$$X = \frac{x_1 + y_1}{2}, \quad \xi = x_1 - y_1, \quad (7.10)$$

and writing the density matrix in terms of the Wigner function via

$$\rho(x,y) = \int_{-\infty}^{\infty} dp \, e^{\frac{i}{\hbar}p(x-y)} W\left(p, \frac{x+y}{2}\right), \quad (7.11)$$

we obtain

$$\Delta = -\int_0^\infty dx_2 \int_{-\infty}^0 dX \int_{-X}^X d\xi \int_{-\infty}^\infty dp \left[\frac{m}{2\pi\hbar(t_2 - t_1)} \right]$$

$$\times \exp\left[\frac{im}{\hbar(t_2 - t_1)}\xi(X - x_2) + \frac{i}{\hbar}p\xi - \frac{D(t_2 - t_1)}{3\hbar^2}\xi^2 \right]$$

$$\times W_{t_1}(X, p). \tag{7.12}$$

We see from this expression that there is a time scale $\tau_l = \sqrt{2m\hbar/D}$, set by the environment on which localization effects are important. This time scale is the same as that on which the current becomes positive, as we saw earlier.

There are three cases to explore here, the first is where there is no environment, $D = 0$. This is the case covered in Ref. [7]. The second case is the intermediate one, $t_1/\tau_l \gg 1$ but $(t_2 - t_1)/\tau_l \ll 1$. This is the most general case in which we can expect to have environmentally induced decoherence. The final case is where $t_1/\tau_l, (t_2 - t_1)/\tau_l \gg 1$. This is the case of very strong environmental coupling.

For the free case $D = 0$, the integrals over ξ and x_2 in Eq. (7.12) may be carried out to give

$$\Delta = \int_{-\infty}^0 dX \int_{-\infty}^\infty dp\, W_{t_1}(X, p) f\left[\frac{X}{\hbar}\left(\frac{mX}{(t_2 - t_1)} + p \right) \right], \tag{7.13}$$

where

$$f(u) = \frac{1}{\pi}\int_u^\infty dy\, \frac{\sin y}{y}. \tag{7.14}$$

This expression for Δ is now identical to Eq. (5.43) of Ref. [7], and the same conclusions apply. We briefly repeat the analysis here, however, since it turns out to be relevant for the second case.

We note first that $f(+\infty) = 0, f(-\infty) = 1, f(0) = 1/2$. Now we assume that our state at t_1 is of the form

$$W_{t_1}(X, p) = \frac{1}{\pi\hbar}\exp\left[-\frac{(X - X_0 - p_0 t_1/m)^2}{2\sigma^2} - \frac{2\sigma^2}{\hbar^2}(p - p_0)^2 \right], \tag{7.15}$$

and that σ^2 is large, so the state is tightly peaked in momentum. We can therefore integrate out p, setting $p = p_0$ to obtain,

$$\Delta = \int_{-\infty}^0 dX \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(X - X_0 - p_0 t_1/m)^2}{2\sigma^2} \right]$$

$$\times f\left[\frac{X}{\hbar}\left\{ \frac{mX}{(t_2 - t_1)} + p_0 \right\} \right]. \tag{7.16}$$

In [7] it was noted that this type of integral is dominated by values of X for which

$$0 \leqslant X \leqslant -\frac{\pi\hbar}{4|p_0|}, \tag{7.17}$$

provided that $p_0^2(t_2 - t_1)/2m \gg 1$, and in this range the exponential term is approximately constant. We can approximate this integral then by integrating from 0 to $-\pi/4|p_0|$, taking $X = 0$ in the exponential term and approximating

$$f(u) \approx \frac{1}{2} - \frac{u}{\pi} + O(u^3). \tag{7.18}$$

This gives

$$\Delta \approx \sqrt{\frac{\pi}{2}}\frac{\hbar}{8\sigma|p_0|}\exp\left(-\frac{(X_0 + p_0 t_1/m)^2}{2\sigma^2} \right) \ll 1, \tag{7.19}$$

there will therefore be decoherence for Gaussian wave packets tightly peaked in momentum, provided their momentum is such that $p_0^2(t_2 - t_1)/2m \gg \hbar$ [i.e., $E_0(t_2 - t_1) \gg \hbar$]. In [7] it was argued that this conclusion also holds for orthogonal superpositions of Gaussians.

Turning to the intermediate case, since $(t_2 - t_1)/\tau_l \ll 1$ we can set $D = 0$ in Eq. (7.12) and we obtain Eq. (7.13) again. Now, however, $W_{t_1}(X, P)$ is the initial state evolved with an environment, and since $t_1/\tau_l \gg 1$ this should be significant. To proceed we write the Wigner function at time t_1 in terms of the initial state at $t = 0$ using Eq. (3.6)

$$\Delta = \int dX_0 dp_0 W_0(X_0, p_0) F[X_0, p_0], \tag{7.20}$$

$$F[X_0, p_0] = \int_{-\infty}^0 dX \int_{-\infty}^\infty dp\, K(X, p, t_1 | X_0, p_0, 0)$$

$$\times f\left[\frac{X}{\hbar}\left\{ \frac{mX}{(t_2 - t_1)} + p \right\} \right]. \tag{7.21}$$

Because the Wigner function propagator is a Gaussian the analysis is similar to the first case. Recall that we are assuming $t \ll \tau_s$ and so $p_0^2/Dt_1 \gg 1$, this means we can integrate out p, setting $p = p_0$. This gives us

$$F[X_0, p_0] = \int_{-\infty}^0 dX N\sqrt{\frac{\pi}{\alpha}}\exp\left(-\frac{\beta}{4}(X - X_0 - p_0 t_1/m)^2 \right)$$

$$\times f\left[\frac{X}{\hbar}\left\{ \frac{mX}{(t_2 - t_1)} + p_0 \right\} \right], \tag{7.22}$$

where, N, α, β are defined in Eq. (3.8). Now,

$$p_0^2/Dt_1 \gg 1, \tag{7.23}$$

implies that

$$\frac{1}{p_0^2} \ll \frac{1}{\hbar^2\beta}\left(\frac{\tau_l}{t_1} \right)^4, \tag{7.24}$$

so that for $t_1 > \tau_l$, $1/p_0^2 \ll 1/\beta\hbar^2$. This means, again like the case of $D = 0$, that the exponential term is roughly constant compared to f. Since we have $p_0^2(t_2 - t_1)/2m \gg \hbar$, we follow the same procedure as the $D = 0$ case, arriving finally at

$$F[X_0, P_0] \approx \frac{\sqrt{\pi}\hbar}{8|p_0|}\sqrt{\frac{\beta}{4}}\exp\left(-\frac{\beta}{4}(X_0 + p_0 t_1/m)^2 \right). \tag{7.25}$$

The value of Δ now depends on the relationship between the width of the initial state σ and β. However, we can obtain an upper bound by ignoring the effects of the exponential term in F, this gives

$$\Delta \leqslant \frac{\sqrt{\pi}\hbar}{8|p_0|}\sqrt{\frac{\beta}{4}} = \frac{1}{16}\sqrt{\frac{2m\hbar}{p_0^2 t_1}}\left(\frac{\tau_l}{t_1} \right) \ll 1. \tag{7.26}$$

Finally we have the case of strong system-environment coupling. Since $(t_2 - t_1)/\tau_l \gg 1$ the integral over ξ in Eq. (7.12) will be peaked around $\xi = 0$, and we can therefore extend the

limits of integration to $(-\infty, \infty)$. This integral and the one over x_2 may then be carried out and we obtain

$$\Delta = \int_{-\infty}^{0} dX \int_{-\infty}^{\infty} dp \frac{1}{2} \text{Erfc} \left[-\sqrt{\frac{3m^2}{D(t_2 - t_1)^3}} \right.$$

$$\left. \times \left(X + \frac{p(t_2 - t_1)}{m} \right) \right] W_{t_1}(X, p), \qquad (7.27)$$

where Erfc is the complementary error function [29]. For large positive values of the argument we have that

$$\text{Erfc}[u] \approx \frac{e^{-u^2}}{u\sqrt{\pi}} \left[1 + O\left(\frac{1}{u}\right) \right]. \qquad (7.28)$$

Therefore, since the Wigner function is peaked around $p < 0$, Δ will be very small, provided $(t_2 - t_1) \ll \tau_s$. This gives us an *upper* bound on the time interval $[t_1, t_2]$, rather than a lower one. The lower time scale is provided by the condition $t_2 - t_1 \gg \tau_l = \sqrt{2m\hbar/D}$. This lower time scale is compatible with the condition that the current be positive.

Note, however, that this lower limit is state independent. There will be states for which arrival time probabilities decohere on much shorter time scales than this, for example, the simple cases which decohere in the absence of any environment will continue to do so in the presence of an environment, at least until a time $\sim \tau_s$.

The key point is that whether or not one can assign arrival time probabilities in decoherent histories depends on the form of the state at the time it crosses the origin. Environmentally induced decoherence produces mixtures of Gaussian states from generic initial ones, and thus after a short time, arrival time probabilities can be defined whatever the initial state. Crucially, however, it is not necessary for the system to be monitored while it is crossing the origin. The smallest time interval δt, over which we can define a decoherent arrival time probability is therefore set by the energy of the system and not the details of the environment and we must have $E\delta t \gg \hbar$. This is in agreement with Ref. [7], and also with the results of earlier works, concerning the accuracy with which a quantum system may be used as a clock [30].

In conclusion then, for a general state, decoherence of histories requires that we evolve for a time much greater than $\tau_l = \sqrt{2m\hbar/D}$ before the first crossing time. This is because this is the time scale on which quantum correlations disappear and our initial state begins to resemble a mixture of Gaussian states. After this time, we may define arrival time probabilities to an accuracy δt, provided only that $E\delta t \gg \hbar$. States which start as Gaussians may be assigned arrival time probabilities without this initial period of evolution. This is in line with the general result that some coarse-graining is always required to achieve a decoherent set of histories in quantum theory.

VIII. SUMMARY AND DISCUSSION

In this paper we have been concerned with deriving an arrival time distribution for open quantum systems, and comparing this with the classical result. We began by discussing the generalization of the current, which is the classical arrival time distribution, to open quantum systems, and in particular

to the case of quantum Brownian motion. We found that, in general, the inclusion of an environment leads to extra terms appearing in the expressions for the current, compared with the those valid in the unitary case. However, we have shown that in the limit of negligible dissipation these correction terms may be dropped. We then explored the resulting arrival time distribution and showed that it is nonnegative after a time of order τ_l, and that after this time it can be written as the trace of the density matrix times a POVM.

We then turned to the question of deriving this arrival time distribution from the decoherent histories approach to quantum theory. We extended the decoherent histories analysis of the arrival time problem to the case of a particle coupled to an environment. As expected, the inclusion of an environment produces decoherence of arrival time probabilities for a generic initial state. There are, however, some limitations to the permitted class of histories. For a generic state arrival times can only be specified after an initial time $t \gg \tau_l$. Even after this time arrival times cannot be specified with arbitrary precision, coarse-graining over intervals $\delta t \gg \hbar/E$ is required to ensure decoherence. We showed that the decoherence condition is very closely related to a semiclassical approximation for the evolution of the state, and that both conditions are satisfied if the time between projections is sufficiently large. This is a specific case of a more general connection between decoherence and classical behavior.

Our approach has proceeded at two levels. At the heuristic level the simple generalization of Eq. (1.2) to the case of a particle coupled to an environment, Eq. (4.9), is a positive arrival time distribution after a time of order τ_l. This can also be written as the expectation value of a POVM, Eq. (4.14), and thus has the same form as other probabilities in quantum theory. On a more fundamental level, these expressions can be *derived* from the decoherent histories approach to quantum theory, where they are seen to be valid for times much later than τ_l.

Although the arrival time probabilities computed from decoherent histories agree with the heuristic ones when we have decoherence, their derivation in this way represents a significant advance in our understanding. The great difficulty with regarding the current as the arrival time distribution is the arbitrary way in which one accepts these "probabilities" when they are positive, but declines to do so when they are not. Because decoherence is an essential part of the histories formalism, this arbitrariness is replaced with a consistent set of rules governing when probabilities may or may not be assigned. While this may be of no consequence in the setting of a laboratory, it may prove hugely important in the analysis of closed systems, in particular the study of quantum cosmology [31].

Another very interesting result we have presented is that the current becomes strictly positive after a finite time while assignment of probabilities in decoherent histories is only possible asymptotically. In some ways this difference between the heuristic analysis and decoherent histories is to be expected. Indeed, the current represents a linearly positive history [32], and it is essentially the condition of linear positivity that we have proven holds after a time of order τ_l. It is known that linear positivity is a weaker condition than decoherence [33,34], and thus it is not surprising that demanding decoherence leads

to a stricter limit on the assignment of probabilities than the heuristic analysis. It would be interesting to examine what is gained in this context by imposing decoherence rather than linear positivity.

There are similarities here with the relationship between the current and Kijowski's arrival time distribution [35], which may be shown to agree in the classical limit, but not more generally. Indeed because it is manifestly positive, one might expect the arrival time distribution computed from decoherent histories to be more closely related to Kijowski's distribution on shorter time scales. These issues will be discussed elsewhere.

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