Coherence versus interferometric resolution

Alfredo Luis*

Departamento de Óptica, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain

(Received 19 March 2010; published 30 June 2010)

We examine the relation between second-order coherence and resolution in the interferometric detection of phase shifts. While for classical thermal light resolution and second-order coherence are synonymous, we show that for quantum light beams reaching optimum precision second-order coherence and resolution become antithetical.

DOI: 10.1103/PhysRevA.81.065802 PACS number(s): 42.50.St, 42.25.Kb, 42.25.Hz, 42.50.Ar

I. INTRODUCTION

Coherence is a key concept in optics derived from the statistical nature of real light beams [1–3]. Coherence is usually understood as the principal requisite for good quality interference fringes. In this work we show from a quantum metrological perspective that this is not always the case, so that optimum interference and second-order coherence may become antithetical.

More specifically, we focus on interference as a practical procedure to detect and measure minute phase changes. For classical thermal-chaotic light resolution and second-order coherence are proportional both in the classical and quantum domains. This may be expected and traced back to some well-known previous results [1–3]. However, we show that increasing coherence degrades resolution for quantum field states reaching optimum precision, such as squeezed light [4–6]. Some previous works have also noticed differences between quantum and classical visibility [7].

For the sake of illustration we consider the most simple two-beam interferometric schemes, such as the Young interferometer and the 50% lossless beam splitter illustrated in Fig. 1, producing the interference of two harmonic scalar electromagnetic waves with complex amplitudes $E_{1,2}$. In the quantum domain $E_{1,2}$ become complex amplitude operators satisfying the commutation relations $[E_j, E_j^\dagger] = 1$. Throughout we consider the spatial-frequency representation. Although the light beams examined may have large bandwidths, for definiteness we focus on a single spectral component (of random complex amplitude) selected by a suitable filtering.

In Sec. II we recall the definition of coherence and resolution. These are applied then to typical classical light in Sec. III and quantum squeezed light improving resolution in Sec. IV. The results are further compared in Sec. V.

II. COHERENCE AND RESOLUTION

The complex second-order degree of coherence is [1]

$$\mu = \frac{\langle E_1 E_2^\dagger \rangle}{\sqrt{\langle E_1^\dagger E_1 \rangle \langle E_2^\dagger E_2 \rangle}}, \tag{2.1}$$

where the angle brackets denote the corresponding averages. The intensity or photon number in the interference region is

$$\langle I_\phi \rangle \propto \langle E_1^\dagger E_1 \rangle + \langle E_2^\dagger E_2 \rangle + \langle E_1 E_2^\dagger \rangle e^{i\phi} + \langle E_1^\dagger E_2 \rangle e^{-i\phi}, \tag{2.2}$$

where ϕ is the phase difference acquired within the interferometer. To obtain simpler formulas we consider balanced detection by the subtraction of the intensities corresponding to relative phases ϕ differing by π, $\langle M_\phi \rangle \propto \langle I_\phi \rangle - \langle I_{\phi+\pi} \rangle$ (this naturally occurs at the two outputs of a lossless beam splitter), so that

$$M_\phi = E_1 E_2^\dagger e^{i\phi} + E_1^\dagger E_2 e^{-i\phi}. \tag{2.3}$$

A minute change $\delta\phi \ll 1$ of the phase difference produces the variation of the balanced intensity from $\langle M_\phi \rangle$ to $\langle M_{\phi+\delta\phi} \rangle$ with

$$\langle M_{\phi+\delta\phi} \rangle \simeq \langle M_\phi \rangle + \frac{d\langle M_\phi \rangle}{d\phi} \delta\phi. \tag{2.4}$$

The meaningful detection of $\delta\phi$ depends on the relation between the intensity change $|\langle M_{\phi+\delta\phi} \rangle - \langle M_\phi \rangle|$ and the noise of the measured observable $\Delta M_\phi = \sqrt{\langle M_\phi^2 \rangle - \langle M_\phi \rangle^2}$. This relation can be properly assessed by the signal-to-noise (S/N) ratio

$$\frac{\mathrm{S}}{\mathrm{N}} = \frac{|\langle M_{\phi+\delta\phi} \rangle - \langle M_\phi \rangle|}{\Delta M_\phi} = \frac{1}{\Delta M_\phi} \left| \frac{d\langle M_\phi \rangle}{d\phi} \right| \delta\phi. \tag{2.5}$$

More specifically, the unity signal-to-noise criterion defines the minimum phase shift that can be detected $\delta\phi_{\min}$ as the one producing a signal thats equals the noise

$$\frac{\mathrm{S}}{\mathrm{N}} = 1 \to \delta\phi_{\min} = \frac{\Delta M_\phi}{\left| \frac{d\langle M_\phi \rangle}{d\phi} \right|}. \tag{2.6}$$

Optimum phase-shift detection requires that $\delta\phi_{\min}$ should be as small as possible.

For definiteness, let us consider $\phi = 0$ as our working point, with phases defined so that $\langle M_0 \rangle = 0$; this is to say $\mathrm{Re}\langle E_1 E_2^\dagger \rangle = 0$ (for the following examples to be considered any choice of phases leads to the same result). In such a case $|d\langle M_\phi \rangle / d\phi|$ takes its maximum value

$$\left| \frac{d\langle M_\phi \rangle}{d\phi} \right|_{\phi=0} = 2|\langle E_1 E_2^\dagger \rangle|. \tag{2.7}$$

*alluis@fis.ucm.es; URL: http://www.ucm.es/info/gioq

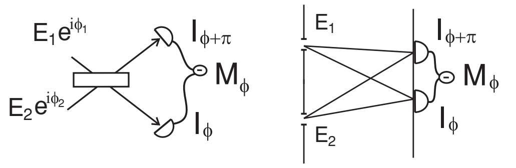

FIG. 1. Beam mixer and Young interferometer producing the interference of two waves with complex amplitudes $E_{1,2}$.

After computing $(\Delta M_0)^2 = \langle M_0^2 \rangle$ we get

$$(\delta\phi_{\min})^2 = \frac{2\langle E_1^\dagger E_1 E_2^\dagger E_2\rangle + 2\mathrm{Re}\langle E_1^2 E_2^{\dagger 2}\rangle + \langle E_1^\dagger E_1\rangle + \langle E_2^\dagger E_2\rangle}{2|\langle E_1 E_2^\dagger\rangle|^2}. \tag{2.8}$$

We can appreciate that the resolution is not given exclusively by second-order coherence, but it also depends on fourth-order correlations. The overall result of this combination is not trivial and strongly depends on the statistics of the light state experiencing the phase shift. More specifically, we will show that $\delta\phi_{\min}$ and μ can display either the same or fully opposite behaviors. To this end we particularize this general approach to two different Gaussian wave fields, thermal-chaotic and squeezed.

Finally, we note that $\delta\phi_{\min}$ provides a widely used simple performance criterion. However, its theoretical support is rather fragile and in some meaningful occasions it may fail. Because of this, all conclusions will be tested by more sound and complete criteria based on Fisher information as recalled in the Appendix.

III. COHERENCE VERSUS RESOLUTION FOR THERMAL-CHAOTIC LIGHT

This is typical classical light where $E_{1,2}$ are Gaussian statistically circular variables of zero mean. By using the Gaussian moment and optical equivalence theorems [1] and taking into account $\mathrm{Re}\langle E_1 E_2^\dagger\rangle = 0$ we get

$$(\delta\phi_{\min})^2 = \frac{(1-|\mu|^2)\bar{n}_1\bar{n}_2 + \bar{n}_1 + \bar{n}_2}{2\bar{n}_1\bar{n}_2|\mu|^2}, \tag{3.1}$$

where $\bar{n}_j = \langle E_j^\dagger E_j\rangle$. Thus, for thermal light we get that second-order coherence and resolution are synonymous so that increasing coherence implies increasing resolution. This is no longer true for other field states as revealed in Sec. IV.

The result in Eq. (3.1) is confirmed by the quantum Fisher information, and also by the Fisher information, because of Eq. (A4). For simplicity we consider the case of equal mean photon numbers $\bar{n}_1 = \bar{n}_2 = \bar{n}$. In order to use Eq. (A2) we perform the mode transformation $A_\pm = (E_1 \pm iE_2)/\sqrt{2}$, so that $\langle A_+ A_-^\dagger\rangle = 0$ and the field state in modes $A_\pm$ factorizes as the product of two thermal states $\rho_+\rho_-$ with mean photon numbers $\langle A_\pm^\dagger A_\pm\rangle = \bar{n}(1 \pm |\mu|)$. The eigenvectors of $\rho_\pm$ are the number states $|n_\pm\rangle$ in modes $A_\pm$ with eigenvalues $(1-\xi_\pm)\xi_\pm^{n_\pm}$, where $\xi_\pm = \bar{n}(1\pm|\mu|)/[1+\bar{n}(1\pm|\mu|)]$. The generator of the phase shift in Eq. (A2) is $G = (E_1^\dagger E_1 - E_2^\dagger E_2)/2 = (A_+^\dagger A_- + A_-^\dagger A_+)/2$. All this leads to the following quantum Fisher information and phase uncertainty:

$$F_Q = \frac{2\bar{n}|\mu|^2}{1+\bar{n}(1-|\mu|^2)}, \quad (\Delta\phi)^2 \geqslant \frac{\bar{n}(1-|\mu|^2)+1}{2\bar{n}|\mu|^2}. \tag{3.2}$$

It can be appreciated that $\Delta\phi$ is fully equivalent to the result obtained with the signal-to-noise ratio in Eq. (3.1) for $\bar{n}_1 = \bar{n}_2 = \bar{n}$. After Eq. (A4) we get also that the measurement is close to optimum in the sense of approaching the limit established by the quantum Fisher information. Maximum resolution is obtained for $|\mu| = 1$ with $\Delta\phi \geqslant 1/\sqrt{2\bar{n}}$. This is essentially the standard quantum limit, or shot-noise limit, which is the maximum signal-to-noise ratio that can be achieved with classical light [5,6,8].

IV. COHERENCE VERSUS RESOLUTION FOR SQUEEZED LIGHT

Next we consider two examples that show second-order coherence and resolution become antithetical for typical examples of quantum light that improve the resolution beyond the standard quantum limit.

A. Single-mode approximation

To simplify formulas let us assume that $E_1 = \alpha$ is a real, deterministic nonfluctuating variable. In such a case we have simply

$$\langle M_0\rangle = \alpha\langle X\rangle, \quad \langle M_0^2\rangle = \alpha^2\langle X^2\rangle, \quad \left.\frac{d\langle M_\phi\rangle}{d\phi}\right|_{\phi=0} = \alpha\langle Y\rangle, \tag{4.1}$$

where X, Y are twice the real and imaginary parts of E_2

$$X = E_2 + E_2^\dagger, \quad Y = i(E_2^\dagger - E_2), \tag{4.2}$$

and $\langle X\rangle = 0$ so that $\langle M_0\rangle = 0$. Thus we get

$$\delta\phi_{\min} = \frac{\Delta X}{|\langle Y\rangle|}, \tag{4.3}$$

while the second-order coherence is

$$|\mu|^2 = \frac{\langle Y\rangle^2}{\langle X^2\rangle + \langle Y^2\rangle - 2} = \frac{\langle Y\rangle^2}{\langle Y\rangle^2 + (\Delta X)^2 + (\Delta Y)^2 - 2}. \tag{4.4}$$

The factor -2 in the denominator is of quantum origin because of the commutator $[X,Y] = 2i$ involved in the computation of $\langle E_2^\dagger E_2\rangle$ [1,4].

These relations show that μ and $\delta\phi_{\min}$ depend on the amount of fluctuations ΔX, ΔY, in agreement with the statistical nature of these parameters. The fluctuations of X and Y are not independent, but related by the Heisenberg uncertainty relations [4]

$$\Delta X \Delta Y \geqslant 1, \quad (\Delta X)^2 + (\Delta Y)^2 \geqslant 2, \tag{4.5}$$

derived from the commutator $[X,Y] = 2i$ [1,4]. From now on we assume that the light is in a minimum uncertainty state, $\Delta X \Delta Y = 1$, so that quantum fluctuations are reduced to a minimum.

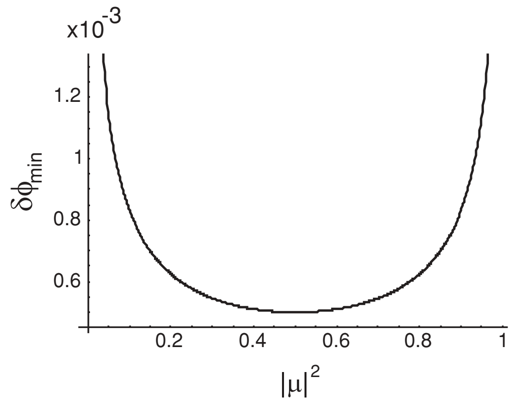

FIG. 2. Plot of $\delta\phi_{\min}$ as a function of $|\mu|^2$ for $\bar{n} = 10^3$.

The goal is to express $\delta\phi_{\min}$ in terms of μ and the mean number of photons in mode E_2,

$$\begin{aligned} \bar{n} &= \langle E_2^\dagger E_2 \rangle = \tfrac{1}{4}(\langle X^2 \rangle + \langle Y^2 \rangle - 2) \\ &= \tfrac{1}{4}[\langle Y \rangle^2 + (\Delta X)^2 + (\Delta Y)^2 - 2]. \end{aligned} \tag{4.6}$$

Thus from Eqs. (4.4), (4.5), and (4.6) we have

$$\begin{aligned} (\Delta X)^2 &= 2(1-|\mu|^2)\bar{n} + 1 - 2\sqrt{(1-|\mu|^2)^2\bar{n}^2 + (1-|\mu|^2)\bar{n}}, \\ \langle Y \rangle^2 &= 4|\mu|^2\bar{n}. \end{aligned} \tag{4.7}$$

In Fig. 2 we have represented $\delta\phi_{\min}$ as a function of $|\mu|^2$ for $\bar{n} = 10^3$. For $\bar{n}(1-|\mu|^2) \gg 1$ the following approximation holds:

$$\delta\phi_{\min} \simeq \frac{1}{4\bar{n}|\mu|\sqrt{1-|\mu|^2}}. \tag{4.8}$$

Figure 2 and Eq. (4.8) show that the optimum signal-to-noise ratio is obtained for rather low second-order coherence $|\mu|^2 \simeq 1/2$ with $\delta\phi_{\min} = 1/(2\bar{n})$ so that increasing coherence beyond this point degrades resolution. Optimum performance arises provided that $\langle Y \rangle \simeq \Delta Y \gg 1 \gg \Delta X$. This is quadrature squeezed light. On the other hand, for unsqueezed light $\Delta X = \Delta Y = 1$, the second-order coherence reaches its maximum value $|\mu|^2 = 1$ and $\delta\phi_{\min} = 1/(2\sqrt{\bar{n}})$, which is consistent since these are the coherent states [1,4].

These conclusions are supported by the quantum Fisher information. Since the probe is in a pure state $F_Q = 4(\Delta G)^2$ and $G = E_2^\dagger E_2$ we get for $\Delta Y \gg 1$

$$F_Q \simeq 8\bar{n}^2(1-|\mu|^4), \tag{4.9}$$

so that the maximum resolution is obtained for minimum coherence $|\mu| = 0$, which holds for the squeezed vacuum $\langle E_2 \rangle = 0$. This case has been extensively studied [9]. After Eq. (A4) we get that the measurement is close to be optimum since Eqs. (4.8) and (4.9) provide similar results for $|\mu|^2 \simeq 1/2$. Moreover, for $\phi \to 0$ the Fisher information and the signal-to-noise ratio coincide because the statistics is Gaussian and $d(\Delta X)/d\phi|_{\phi=0}$.

B. Two-mode example

For the sake of completeness we consider a truly two-mode scheme where both modes are treated quantum mechanically. The paradigmatic example is provided by the Caves arrangement [5], where the field in modes $E_{1,2}$ is the output state of a symmetric beam splitter illuminated by a coherent state of a mean number of photons $\bar{n}_c$ and a squeezed vacuum with mean number of photons $\bar{n}_s$. In this case it holds that

$$|\mu| = \frac{|\bar{n}_c - \bar{n}_s|}{\bar{n}_c + \bar{n}_s}. \tag{4.10}$$

On the other hand, since the field state is pure and the generator is $G = (E_1^\dagger E_1 - E_2^\dagger E_2)/2$, the quantum Fisher information becomes, for large squeezing,

$$F_Q = \tfrac{1}{2}\bar{n}(1-|\mu|) + \bar{n}^2(1-|\mu|^2), \tag{4.11}$$

where $\bar{n} = \bar{n}_c + \bar{n}_s$. This agrees with the Fisher information in Ref. [10]. Therefore, replacing vacuum by squeezed vacuum unavoidably degrades the coherence at the same time that resolution is improved. Maximum resolution is obtained again for minimum coherence $|\mu| = 0$.

V. CONCLUSIONS

We have shown that second-order coherence and resolution are synonymous for classical thermal light, while they become antonymous for quantum light reaching maximum interferometric resolution. This paradox is relevant since it arises for the states allowing the most precise interferometric measurements. For example, we have shown that squeezing improves resolution but degrades second-order coherence. This agrees with previous results showing that squeezing reduces the degree of polarization [11] and that $\mu = 0$ holds for most light states leading to optimum resolution [12,13]. This suggests suitable generalizations of coherence beyond second order reconciling coherence and resolution [13].

ACKNOWLEDGMENTS

This work has been supported by project No. FIS2008-01267 of the Spanish Dirección General de Investigación del Ministerio de Ciencia e Innovación, and by project QUITEMAD S2009-ESP-1594 of the Consejería de Educación de la Comunidad de Madrid.

APPENDIX: FISHER INFORMATION

A key point of quantum metrology is to estimate the phase uncertainty $\Delta\phi$ of a phase-shift detection arrangement, since this allows us to compare the performance of different schemes. A powerful approach considers a Bayesian strategy to get a probability distribution $P(\phi|m)$ representing our knowledge about ϕ after the outcome m in the measurement of M. This is proportional to the probability $P(m|\phi)$ of obtaining m when the phase is ϕ [14–17]. In particular, the minimum variance of any unbiased and efficient estimator is given by the Cramér-Rao lower bound [14–16]

$$\Delta\phi \geqslant \frac{1}{\sqrt{F}}, \quad F = \sum_m \frac{1}{P(m|\phi)} \left(\frac{\partial P(m|\phi)}{\partial \phi} \right)^2, \tag{A1}$$

where F is the Fisher information. The Cramér-Rao lower bound coincides with the signal-to-noise ratio (2.6) when $P(m|\phi)$ is Gaussian and $d(\Delta M)/d\phi = 0$.

It can be also interesting to use assessments of $\Delta\phi$ independent of M, representing the optimum results that may be obtained with any measurement. This can be conveniently given by the quantum Fisher information

$$F_Q = 2\sum_{k,\ell} \frac{(p_k - p_\ell)^2}{p_k + p_\ell} |\langle \psi_k | G | \psi_\ell \rangle|^2, \tag{A2}$$

where p_k, $|\psi_k\rangle$ are the eigenvalues and eigenvectors, respectively, of the probe state ρ, and G is the generator of the transformation imprinting the phase shift ϕ on the probe state $\rho_\phi = \exp(i\phi G)\rho\exp(-i\phi G)$. We have $F_Q \leqslant 4(\Delta G)^2$, where the equality holds for pure states. Since F_Q is the maximum F over all possible measurements $F \leqslant F_Q$ there is a quantum Cramér-Rao bound [14,16]

$$\Delta\phi \geqslant \frac{1}{\sqrt{F_Q}}. \tag{A3}$$

The above performance measures satisfy the following chain of inequalities [18]:

$$\frac{\Delta M}{\left|\frac{\partial \langle M \rangle_\phi}{\partial \phi}\right|} \geqslant \frac{1}{\sqrt{F}} \geqslant \frac{1}{\sqrt{F_Q}} \geqslant \frac{1}{2\Delta G}. \tag{A4}$$

In particular this implies that when the signal-to-noise ratio and the quantum Fisher information coincide we get $F = F_Q$ and the measurement is optimum.

[1] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, England, 1995).
[2] J. W. Goodman, *Statistical Optics* (Wiley, New York, 1985).
[3] E. Wolf, *Introduction to the Theory of Coherence and Polarization* (Cambridge University Press, Cambridge, England, 2007).
[4] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, England, 1997); H.-A. Bachor, *A Guide to Experiments in Quantum Optics* (Wiley, Weinheim, Germany, 1998).
[5] C. M. Caves, Phys. Rev. D **23**, 1693 (1981).
[6] R. Loudon and P. L. Knight, J. Mod. Opt. **34**, 709 (1987); M. C. Teich and B. E. A. Saleh, Quantum Opt. **1**, 153 (1989).
[7] K. J. Resch, K. L. Pregnell, R. Prevedel, A. Gilchrist, G. J. Pryde, J. L. O'Brien, and A. G. White, Phys. Rev. Lett. **98**, 223601 (2007); R. Okamoto, H. F. Hofmann, T. Nagata, J. L. O'Brien, K. Sasaki, and S. Takeuchi, New J. Phys. **10**, 073033 (2008); I. Afek, O. Ambar, and Y. Silberberg, Phys. Rev. Lett. **104**, 123602 (2010).
[8] Z. Y. Ou, Phys. Rev. A **55**, 2598 (1997).
[9] E. Bagan, A. Monras, and R. Muñoz-Tapia, Phys. Rev. A **78**, 043829 (2008); M. Aspachs, J. Calsamiglia, R. Muñoz-Tapia, and E. Bagan, *ibid.* **79**, 033834 (2009); S. Olivares and M. G. A. Paris, J. Phys. B: At. Mol. Opt. Phys. **42**, 055506 (2009).
[10] L. Pezzé and A. Smerzi, Phys. Rev. Lett. **100**, 073601 (2008).
[11] A. Luis, Opt. Commun. **273**, 173 (2007).
[12] A. Rivas and A. Luis, Phys. Rev. A **77**, 022105 (2008).
[13] A. Luis, Phys. Rev. A **79**, 053855 (2009).
[14] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).
[15] S. L. Braunstein, A. S. Lane, and C. M. Caves, Phys. Rev. Lett. **69**, 2153 (1992); S. L. Braunstein, *ibid.* **69**, 3598 (1992); B. R. Frieden, *Physics from Fisher Information: A. Unification* (Cambridge University Press, Cambridge, England, 1998); Z. Hradil and J. Řeháček, Phys. Lett. A **334**, 267 (2005).
[16] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976); A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[17] S. Luo and Q. Zhang, Phys. Rev. A **69**, 032106 (2004).
[18] H. Uys and P. Meystre, Phys. Rev. A **76**, 013804 (2007).