Correlations and superfluidity of a one-dimensional Bose gas in a quasiperiodic potential

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We consider the correlations and superfluid properties of a Bose gas in an external potential. Using a Bogoliubov scheme, we obtain expressions for the correlation function and the superfluid density in an arbitrary external potential. These expressions are applied to a one-dimensional system at zero temperature subject to a quasiperiodic modulation. The critical parameters for the Bose glass transition are obtained using two different criteria and the results are compared. The Lifshitz glass is seen to be the limiting case for vanishing interactions.

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I. INTRODUCTION

Research on the problem of a Bose gas in a disordered potential gained momentum with the realization of well-controlled disordered potentials for ultracold bosonic atoms, which inspired a surge in theoretical and experimental activity [1–7], and the subsequent observation of Anderson localization by two groups [8,9]. The quest is now on to better understand the effect of disorder on an interacting quantum system.

The phase diagram of an interacting Bose gas subject to a disordered potential was outlined in, for example, Refs. [2–7,10,11]. While the exact picture of the phase diagram depends on the employed potential, the qualitative behavior appears to be the same for all kinds of disorder realizations. In the absence of disorder, the gas is in a superfluid phase, which in a two- or three-dimensional Bose system at zero temperature is identical to a Bose-Einstein condensate (BEC) and in one dimension is a quasicondensate with suppressed density fluctuations and algebraically decaying correlations. Upon raising the disordered potential, the system can enter a new quantum phase: the Bose glass state, also called a fragmented BEC. The Bose glass state lacks long-range phase coherence, having exponentially decaying correlations and zero superfluid density, but it is compressible. In the weakly interacting limit, it has been conjectured that the Bose glass goes over into a Lifshitz (or Anderson) glass [4,6], where the density profile is that of a superposition of the lowest-lying single-particle states, which are exponentially localized. Finally, a gas of noninteracting bosons will have all the particles occupying the lowest of the single-particle states.

Experimentally, two basic types of disordered potential for ultracold atoms have been realized, both by optical means: speckle and quasiperiodic. The first type of disorder uses laser speckle, the two-dimensional diffraction pattern of a laser beam passing through a roughened plate. Speckle potentials for ultracold atoms were first used in Ref. [12]. A quasiperiodic potential in one dimension is created simply by superposing two standing waves with different wavelengths; if the wavelengths differ by an irrational factor—the most popular being the golden ratio—the resulting intensity is a quasiperiodic function of the coordinate [1]. In higher dimensions, quasiperiodic potentials can be created as the interference pattern of a number of beams meeting at judiciously chosen angles [13].

The speckle potential is intended to achieve the effect of a one-dimensional random pattern. This situation has been studied extensively in the literature. For a weakly interacting (repulsive) system, Albanese and Frölich [14] showed that the Hamiltonian admits exponentially localized solution in the presence of arbitrarily small disorder. On the other hand, an interacting one-dimensional (1D) Bose gas acts as a Luttinger liquid. Using this approach, Giamarchi and Schultz [15] showed that for a strong-enough interaction the system appears again in a localized phase. In order to match with the known results in the weakly interacting limit, they speculated the existence of two localized phases. Later on, the existence of those two sides of the glass transition were shown numerically in Refs. [16,17].

In the quasiperiodic case the situation is less clear. While it is established that—for no interaction—the system is exponentially localized only when the external potential is strong enough [18], in the presence of a repulsive interaction the exact behavior of the localization is still not known, and only recent results [19] show that a quasiperiodic potential can inhibit the diffusion of a wave packet. In this kind of potential the phase diagram has been studied by Refs. [5,6]. These works are done under the assumption that the secondary lattice is small compared to the primary one, and the system can be mapped into a Bose Hubbard (BH) model. However, very little is known when the secondary lattice strength is comparable to the primary one. In this case the BH model cannot be used, and one has to address the problem in the continuum. Moreover, if there exists an underlying periodic potential to which the disorder is added, there is also a Mott insulating phase for strong interactions if the filling is integer. The latter feature is not discussed much in the present article because it is not correctly described by the approximation that we intend to use.

We see from the preceding discussion that the correlation properties constitute a key to understanding the behavior of the Bose gas; the superfluid properties constitute another. Both of these can be calculated in the Bogoliubov scheme, where the Gross-Pitaevskii equation gives the (quasi-) condensate density and superfluid velocity and the Bogoliubov equations yield the excitation spectrum and corrections to the density. The Bose glass transition in a Gaussian correlated potential was explored recently for strong interparticle potentials by Fontanesi *et al.* in Ref. [7]. The Bogoliubov scheme is usually derived starting from the assumption that the system is Bose-Einstein condensed and there exists a condensate wave function $\Phi(\mathbf{r})$, defined either as the expectation value of the bosonic field operator or as the wave function for the single-particle mode that is occupied by a macroscopic fraction of the particles. Clearly, this assumption would appear to preclude the description of both quasicondensate and Bose glass, but, as we see, it turns out that the Bogoliubov scheme can, in fact, describe those states as well. The explanation is that BEC is convenient, but not a necessary requirement, for deriving the Bogoliubov equations; it suffices to make the weaker assumption that the quantum fluctuations in the density and gradient of the phase are small. Several articles have presented different sketches of the derivation of Bogoliubov theory for a quasicondensate [20–25]. In this article we put up what we believe to be a complete, simple, and consistent derivation of the Bogolioubov equations in a Bose gas using a minimum of assumptions in order to be able to analyze the quasicondensate-Bose glass transition in one dimension. Using the same formalism we derive an expression for the superfluid density in order to better understand the transition.

In this work we consider the experimentally relevant case of a 1D Bose gas in a quasiperiodic potential and examine a wide parameter regime, allowing for a discussion of the Bose glass transition and the conjectured Lifshitz glass. It is important to stress that the Bogoliubov approximation has to be used with some care in 1D systems. This problem has been treated in detail for a uniform gas by Lieb and Liniger [26], where it is shown that the Bogoliubov perturbation theory agrees with the exact answer for $g/n < \hbar^2/2m$. Moreover, our numerical scheme is consistent only if the fluctuation in the phase and in the density are small, and this clearly cannot be the case upon increasing indefinitely the interaction within the particles. For this reason, due to the intrinsic limit of the Bogoliubov approach, we expect our results to be reliable only for weak interaction and high density. Within these limits, the application of the Bogoliubov recipe has been proven to be theoretically sound, and it is known to give the same correlation function of the Luttinger liquid theory, as shown in Ref. [27]. We wish to compare the different predictions for the phase transition obtained from analyzing the behavior of the correlation function and of the superfluid density.

As we see, our analysis suggests the existence of a glassy phase for small interparticle interaction. The plot of this phase appears as a series of peaks with little overlap. According to the language used in Ref. [4], this phase is called "fragmentd BEC," or Bose glass; in the same reference, the Anderson glass appears when the overlap between the peaks is negligible. We maintain these definitions through our article.

This article is organized as follows. In Sec. II we present a derivation of Bogoliubov theory and in particular an expression for a correlation function. In Sec. III, an expression for the superfluid density is derived. Section IV presents numerical results for the superfluid-Bose glass phase transition in a 1D Bose gas. Finally, in Sec. V, we summarize and conclude.

II. CORRELATION FUNCTION

Inspired by the works of Ho and Ma [22] and Xia and Silbey [25], we use a path integral formalism, considering a system of bosons described by the Euclidean action

$$\mathcal{S}[\psi^*,\psi] = \int d\tau\, d\mathbf{r}\, \psi^*(\mathbf{r},\tau)\Big[-\hbar\,\partial_\tau + \frac{\hbar^2}{2m}\nabla^2 - U(\mathbf{r}) + \mu - \frac{g}{2}\,|\psi(\mathbf{r},\tau)|^2\Big]\psi(\mathbf{r},\tau), \tag{1}$$

where ψ is the scalar boson field. Assuming that the quantum fluctuations of density and gradient of the phase are small, we expand ψ as

$$\begin{aligned}\psi(\mathbf{r},\tau) &= e^{i\,\theta(\mathbf{r},\tau)}\sqrt{n_0(\mathbf{r})+\delta n(\mathbf{r},\tau)}\\ &\approx e^{i\,\theta(\mathbf{r},\tau)}\sqrt{n_0(\mathbf{r})}\left[1+\frac{1}{2}\frac{\delta n(\mathbf{r},\tau)}{n_0(\mathbf{r})}-\frac{1}{8}\frac{\delta n^2(\mathbf{r},\tau)}{n_0(\mathbf{r})^2}\right]\\ &= \psi_0(\mathbf{r},\tau)+\delta\psi(\mathbf{r},\tau).\end{aligned} \tag{2}$$

The equation of motion for ψ_0 is found by means of the variational principle $\delta\mathcal{S}/\delta n_0^* = 0$, and the result is the Gross-Pitaevskii equation for n_0 [28]:

$$-\frac{\hbar^2}{2m}\nabla^2\sqrt{n_0(\mathbf{r})} + U(\mathbf{r})\sqrt{n_0(\mathbf{r})} + g n_0(\mathbf{r})^{\frac{3}{2}} = \mu\sqrt{n_0(\mathbf{r})}\,. \tag{3}$$

This amounts to treating $n_0(\mathbf{r})$ as a scalar field representing the (quasi-) condensate density and δn and θ as perturbations. In a 3D system, and in two dimensions at zero temperature, the condensate wave function Φ would in the absence of currents be equal to the square root of the classical density, $\sqrt{n_0}$. In principle, one could describe a state with a current by assuming a slowly varying phase θ_0 that is also treated classically, but we refrain from doing so for convenience.

Ignoring terms of order higher than θ^2 and δn^2, the action becomes

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2\,. \tag{4}$$

Here, $\mathcal{S}_0$ contains only the classical density n_0 and is minimized by the equation of motion for n_0, $\mathcal{S}_1$ must vanish for n_0 to be a stationary solution, while $\mathcal{S}_2$ is

$$\mathcal{S}_2 = \frac{1}{2}\int d\tau\, d\mathbf{r}\begin{bmatrix}\frac{\delta n(\mathbf{r},\tau)}{\sqrt{2\,n_0(\mathbf{r})}}\\ i\sqrt{2\,n_0(\mathbf{r})}\,\theta(\mathbf{r},\tau)\end{bmatrix}^* \times S\begin{bmatrix}\frac{\delta n(\mathbf{r},\tau)}{\sqrt{2\,n_0(\mathbf{r})}}\\ i\sqrt{2\,n_0(\mathbf{r})}\,\theta(\mathbf{r},\tau)\end{bmatrix}, \tag{5}$$

with

$$\begin{aligned}S &= \begin{pmatrix}-\frac{\hbar^2\nabla^2}{2m}+U+3\,gn_0-\mu & -\hbar\,\partial_\tau\\ -\hbar\,\partial_\tau & -\frac{\hbar^2\nabla^2}{2m}+U+gn_0-\mu\end{pmatrix}\\ &= -\hbar\partial_\tau\,\sigma_1 + \begin{pmatrix}H_3 & 0\\ 0 & H_1\end{pmatrix},\end{aligned} \tag{6}$$

where we have implicitly defined the scalar operators H_3 and H_1. In order to find the correlators between δn and θ, we need to find the function G that inverts S,

$$\left[-\hbar\partial_\tau\,\sigma_1 + \begin{pmatrix} H_3 & 0 \\ 0 & H_1 \end{pmatrix}\right] G(\mathbf{r},\tau,\mathbf{r}',\tau') = \delta(\mathbf{r}-\mathbf{r}'). \quad (7)$$

Of course, what we have derived so far is mathematically identical to the well-known Bogoliubov theory, as we now show. Introducing the transformation T as

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (8)$$

we obtain

$$T\,\mathcal{L}\,T^{-1} = \begin{pmatrix} H_3 & 0 \\ 0 & H_1 \end{pmatrix}, \quad (9)$$

with

$$\mathcal{L} = \begin{pmatrix} -\frac{\hbar^2}{2m}\nabla^2 + U + 2\,gn_0 - \mu & n_0\,g \\ -n_0\,g & \frac{\hbar^2}{2m}\nabla^2 - U - 2\,gn_0 + \mu \end{pmatrix}. \quad (10)$$

The diagonalization of $\mathcal{L}$ leads to the Bogoliubov equations for the Bogoliubov amplitudes $u_j(\mathbf{r})$ and $v_j(\mathbf{r})$,

$$\mathcal{L}\begin{pmatrix} u_j \\ v_j \end{pmatrix} = E_j\begin{pmatrix} u_j \\ v_j \end{pmatrix}. \quad (11)$$

Moreover, the Green function for the action $S = i\hbar\,\partial_t + \mathcal{L}$ is already known [29] and equal to

$$G(\mathbf{r},\mathbf{r}',\omega) = -\sum_{j\neq 0}\hbar\left[\frac{1}{\hbar\omega - E_j}\begin{pmatrix} u_j \\ v_j \end{pmatrix}\begin{pmatrix} u'_j \\ v'_j \end{pmatrix}^\dagger - \frac{1}{\hbar\omega + E_j}\begin{pmatrix} v^*_j \\ u^*_j \end{pmatrix}\begin{pmatrix} v'^*_j \\ u'^*_j \end{pmatrix}^\dagger\right]. \quad (12)$$

Defining χ_j as

$$\chi_j = \begin{pmatrix} \chi^1_j \\ \chi^2_j \end{pmatrix} = T\begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} \dfrac{\delta n_j}{\sqrt{2}\sqrt{n_0}} \\ i\sqrt{2\,n_0}\,\theta_j \end{pmatrix}, \quad (13)$$

and correspondingly

$$\tilde{\chi}_j = T\begin{pmatrix} v^*_j \\ u^*_j \end{pmatrix}, \quad (14)$$

we can apply the transformation T to Eq. (7) and eventually find

$$\begin{aligned} G(\mathbf{r},\tau,\mathbf{r}',\tau^+) &= \sum_{\omega_n}\frac{e^{i\omega_n\eta}}{\beta}G(\mathbf{r},\mathbf{r},\omega_n) \\ &= -\sum_{j\neq 0}\sum_{\omega_n}\frac{e^{i\omega_n\eta}}{\beta}\left[\frac{1}{i\hbar\omega_n - E_j}\chi_j\chi'^\dagger_j - \frac{1}{i\hbar\omega_n + E_j}\tilde{\chi}_j\tilde{\chi}'^\dagger_j\right] \\ &= \sum_{j\neq 0}\chi_j\chi'^\dagger_j N(E_j) + \tilde{\chi}_j\tilde{\chi}'^\dagger_j[N(E_j)+1] \\ &= \begin{pmatrix} \frac{1}{2\sqrt{n_0 n'_0}}\langle\delta n\delta n'\rangle & -\sqrt{\frac{n'_0}{n_0}}\langle\delta n i\theta'\rangle \\ \sqrt{\frac{n_0}{n'_0}}\langle i\theta\delta n\rangle & 2\sqrt{n_0 n'_0}\langle\theta\theta'\rangle \end{pmatrix}, \end{aligned} \quad (15)$$

with ω_n being the Matsubara frequencies, η a positive infinitesimal, and $N(E_j) = 1/(e^{\beta E_j} - 1)$ the Bose-Einstein distribution function. For convenience, we have defined the notation

$$\begin{aligned} \theta &= \theta(\mathbf{r},\tau), \\ \theta' &= \theta(\mathbf{r}',\tau^+), \\ \delta n &= \delta n(\mathbf{r},\tau), \\ \delta n' &= \delta n(\mathbf{r}',\tau^+). \end{aligned} \quad (16)$$

Furthermore, let us define

$$\Delta\theta = \theta' - \theta. \quad (17)$$

Using Eq. (2), the one-body correlation function becomes

$$\begin{aligned} \langle\psi^*(\mathbf{r})\psi(\mathbf{r}')\rangle &= \langle\sqrt{n_0+\delta n}e^{-i\theta}e^{i\theta'}\sqrt{n'_0+\delta n'}\rangle \\ &= \langle\sqrt{n_0+\delta n}e^{i\,\Delta\theta}\sqrt{n'_0+\delta n'}\rangle \\ &= \sqrt{n_0 n'_0}\Big\langle e^{i\Delta\theta} + \frac{1}{2}\frac{\delta n}{n_0}e^{i\Delta\theta} + \frac{1}{2}e^{i\,\Delta\theta}\frac{\delta n'}{n'_0} \\ &\quad - \frac{1}{8}\left(\frac{\delta n}{n_0}\right)^2 e^{i\,\Delta\theta} - \frac{1}{8}e^{i\Delta\theta}\left(\frac{\delta n'}{n'_0}\right)^2 \\ &\quad - \frac{1}{4}\frac{\delta n}{n_0}e^{i\Delta\theta}\frac{\delta n'}{n'_0}\Big\rangle. \end{aligned} \quad (18)$$

This expression can be evaluated using Wick's theorem. To lowest order in δn and θ, one finds [24]

$$\begin{aligned} \langle e^{i\,\Delta\theta}\rangle &= e^{-\frac{1}{2}\langle(\Delta\theta)^2\rangle}, \\ \left\langle\frac{\delta n}{n_0}e^{i\,\theta}\right\rangle &= e^{-\frac{1}{2}\langle(\Delta\theta)^2\rangle}\left\langle\frac{\delta n}{n_0}\,i\,\Delta\theta\right\rangle, \\ \left\langle\left(\frac{\delta n}{n_0}\right)^2 e^{i\,\Delta\theta}\right\rangle &\approx e^{-\frac{1}{2}\langle(\Delta\theta)^2\rangle}\left\langle\left(\frac{\delta n}{n_0}\right)^2\right\rangle, \\ \left\langle\frac{\delta n}{n_0}e^{i\,\Delta\theta}\frac{\delta n'}{n'_0}\right\rangle &\approx e^{-\frac{1}{2}\langle(\Delta\theta)^2\rangle}\left\langle\frac{\delta n}{n_0}\frac{\delta n'}{n'_0}\right\rangle, \end{aligned} \quad (19)$$

so that, eventually, to second order [22],

$$\begin{aligned} \langle\psi^*(\mathbf{r})\,\psi(\mathbf{r}')\rangle &= \sqrt{n_0\,n'_0}\,e^{-\frac{1}{2}\langle(\Delta\theta)^2\rangle}\Big\{1 + \frac{1}{2}\Big(\left\langle\frac{\delta n}{n_0}\,i\,\Delta\theta\right\rangle \\ &\quad + \left\langle i\,\Delta\theta\,\frac{\delta n'}{n'_0}\right\rangle\Big) + \frac{1}{4}\left\langle\frac{\delta n}{n_0}\frac{\delta n'}{n'_0}\right\rangle \\ &\quad - \frac{1}{8}\left[\left\langle\left(\frac{\delta n}{n_0}\right)^2\right\rangle + \left\langle\left(\frac{\delta n'}{n'_0}\right)^2\right\rangle\right]\Big\}. \end{aligned} \quad (20)$$

Since both δn and θ are small, the expression between square brackets can be thought as a first-order expansion of an exponential. Using Eqs. (15) and (13), the following expression is obtained:

$$\begin{aligned}\ln g_1(\mathbf{r},\mathbf{r}') &= \ln\langle\psi^*(\mathbf{r})\,\psi(\mathbf{r}')\rangle - \ln\sqrt{n\,n'}\\ &= -\frac{1}{2}\sum_{j\neq 0}\left\{\left|\frac{v_j}{\sqrt{n}}-\frac{v_j'}{\sqrt{n'}}\right|^2\right.\\ &\quad\left. + N_j\left[\left|\frac{u_j}{\sqrt{n}}-\frac{u_j'}{\sqrt{n'}}\right|^2+\left|\frac{v_j}{\sqrt{n}}-\frac{v_j'}{\sqrt{n'}}\right|^2\right]\right\}\\ &\quad + i\sum_{j\neq 0}\left[\frac{1}{2}\,I(\mathbf{r},\mathbf{r}')+N_j I(\mathbf{r},\mathbf{r}')\right], \end{aligned} \tag{21}$$

where we used the shorthand $N_j = N(E_j)$. The quantity $i\,I(\mathbf{r},\mathbf{r}')$ is purely imaginary and equal to

$$\begin{aligned} i\,I(\mathbf{r},\mathbf{r}') &= \left(\frac{v_j^*}{\sqrt{n}}\frac{v_j'}{\sqrt{n'}}-\frac{v_j'^*}{\sqrt{n'}}\frac{v_j}{\sqrt{n}}\right)+\left(\frac{u_j'^*}{\sqrt{n'}}\frac{u_j}{\sqrt{n}}-\frac{u_j^*}{\sqrt{n}}\frac{u_j'}{\sqrt{n}}\right)\\ &\quad+\left(\frac{u_j'^*}{\sqrt{n'}}\frac{v_j'}{\sqrt{n'}}-\frac{v_j'^*}{\sqrt{n'}}\frac{u_j'}{\sqrt{n'}}\right)+\left(\frac{v_j^*}{\sqrt{n}}\frac{u_j}{\sqrt{n}}-\frac{u_j^*}{\sqrt{n}}\frac{v_j}{\sqrt{n}}\right). \end{aligned} \tag{22}$$

If the excitation energies and the condensate wave function are real, then the Bogoliubov amplitudes u_j and v_j can also be chosen as real. The imaginary part in Eq. (22) therefore vanishes, and the resulting expression coincides with that found by Mora and Castin using a different formalism [24]. In particular, for the uniform case the expressions for the Bogoliubov functions are known, leading to

$$\begin{aligned}\ln g_1 &= -\frac{1}{4}\frac{1}{n_0}\sum_{k\neq 0}\left[1-\cos\left(\frac{k}{2}|\mathbf{r}-\mathbf{r}'|\right)\right]^2\\ &\quad\times[|v_k|^2+N_k(|u_k|^2+|v_k|^2)]. \end{aligned} \tag{23}$$

In one dimension, Eq. (21) is an expression for the correlation in a Bose system which is ultraviolet and infrared convergent. Even at $T=0$ the sum has a finite value without assuming a cutoff or a modification in the interparticle potential.

III. SUPERFLUID DENSITY

According to the two-fluid model [30,31], the mass density of a quantum fluid can be divided into a superfluid part $\rho_s(\mathbf{r})$ and a normal one $\rho_n(\mathbf{r})$, the total mass current being $J(\mathbf{r}) = \rho_n(\mathbf{r})\,v_s(\mathbf{r}) + \rho_n(\mathbf{r})\,v_n(\mathbf{r})$. While the superfluid density is in general different from the condensate density, the superfluid velocity *is* the condensate velocity. In particular, upon imposing a phase twist on the condensate wave function, the superfluid part will be proportional to the additional kinetic energy.

It is in this sense that the superfluid density can be defined as a response to a twist of the order parameter [32,33], by means of rewriting Eq. (2) as

$$\psi(\mathbf{r}) = e^{i\,\theta(\mathbf{r})}\sqrt{n_0}\left(e^{i\,\mathbf{k}_0\cdot\mathbf{r}}+\frac{1}{2}\frac{\delta n}{n_0}-\frac{1}{8}\frac{\delta n^2}{n_0^2}\right), \tag{24}$$

with $\mathbf{k}_0 = \Theta\,\hat{e}_0/L$, L being the length of the system in the direction of the unit vector $\hat{e}_0$, and Θ a small twist angle. For convenience, let us take the order parameter normalized to unity; the superfluid density—in the direction of $\hat{e}_0$—is then defined by the thermodynamic limit of

$$\begin{aligned}\rho_s &= \frac{2L^2m^2N}{\hbar^2\Theta^2V}[F^\Theta(\mu,T)-F^0(\mu,T)]\\ &= \frac{2m^2N}{\hbar^2k_0^2V}[F^\Theta(\mu,T)-F^0(\mu,T)]. \end{aligned} \tag{25}$$

The substitution (24) results in the twisted action

$$\begin{aligned}\mathcal{S}^\Theta &= \mathcal{S}+\int d\mathbf{r}d\tau\,\frac{\hbar^2}{2\,m}\left[k_0^2\,n_0(\mathbf{r},\tau)-2ik_0\delta n(\mathbf{r},\tau)\nabla i\theta(\mathbf{r},\tau)\right]\\ &= \mathcal{S}+k_0\int d\tau\,V(\tau), \end{aligned} \tag{26}$$

where in the second line we ignored higher-order terms in the fluctuating fields δn and θ. If the system is large enough ($L \gg 1$), then $V(\tau)$ can be seen as a perturbation; moreover, let us assume that the symmetry of the problem imposes that the odd terms in k_0 vanish in F^Θ, since the cases with k_0 and $-k_0$ lead to the same physical situations. The new free energy can be computed using the linked cluster theorem (see, for example [34])

$$\begin{aligned}F^\Theta &= F^0+\sum_{n=1}^{\infty}\frac{1}{n!}\frac{k_0^n}{\beta\hbar^n}\int d\tau^1\ldots d\tau^n\langle V(\tau^1)\ldots V(\tau^n)\rangle_{\text{connected}}\\ &= F^0+\frac{\hbar^2}{2\,m}k_0^2-\frac{\hbar^4}{2\,m^2}k_0^2\frac{1}{\beta}\int d\tau\,d\tau'[\langle\delta n\,\delta n'\rangle\langle\nabla\theta\,\nabla\theta'\rangle\\ &\quad+\langle\delta n\,\nabla\theta'\rangle\langle\nabla\theta\,\delta n'\rangle]+O\left(k_0^4\right)\\ &= F^0+\frac{\hbar^2}{2\,m}k_0^2\frac{V}{N}\rho_s+O\left(k_0^4\right). \end{aligned} \tag{27}$$

Here, the notation $\langle\ldots\rangle_{\text{connected}}$ stands, as usual, for a diagrammatic expansion where only connected diagrams are retained. The superfluid density can be found in terms of the Green functions (15), defining

$$\begin{aligned}\tilde{G}(r_1,r_2,r_3,r_4,\tau,\tau') &= G_{11}(r_1,\tau,r_3,\tau')\,G_{00}(r_2,\tau,r_4,\tau')\\ &\quad+G_{10}(r_1,\tau,r_4,\tau')\,G_{01}(r_2,\tau,r_3,\tau'), \end{aligned} \tag{28}$$

so that, by partial integration of Eq. (27),

$$\begin{aligned}\rho_s &= \frac{N}{V}-\lim_{r_1\to r}\lim_{r_2\to r'}\frac{\hbar^2}{m\,\beta}\frac{N}{V}\int d\mathbf{r}\,d\mathbf{r}'\,\nabla_1\,\nabla_2\int d\tau\,d\tau'\tilde{G}(r,r',r_1,r_2,\tau,\tau')\\ &\quad-\lim_{r_1\to r}\frac{\hbar^2}{m\,\beta}\frac{N}{V}\int d\mathbf{r}\,d\mathbf{r}'\left(\frac{\nabla\sqrt{n_0(\mathbf{r}')}}{\sqrt{n_0(\mathbf{r}')}}\right)\,\nabla_1\int d\tau\,d\tau'\tilde{G}(r,r',r_1,r',\tau,\tau') \end{aligned}$$

$$- \lim_{r_2 \to r'} \frac{\hbar^2}{m\,\beta} \frac{N}{V} \int d\mathbf{r}\, d\mathbf{r}' \left(\frac{\nabla \sqrt{n_0(\mathbf{r})}}{\sqrt{n_0(\mathbf{r})}} \right) \nabla_2 \int d\tau\, d\tau' \tilde{G}(r,r',r,r_2,\tau,\tau')$$
$$- \frac{\hbar^2}{m\,\beta} \frac{N}{V} \int d\mathbf{r}\, d\mathbf{r}' \left(\frac{\nabla \sqrt{n_0(\mathbf{r})}}{\sqrt{n_0(\mathbf{r})}} \right) \left(\frac{\nabla \sqrt{n_0(\mathbf{r}')}}{\sqrt{n_0(\mathbf{r}')}} \right) \int d\tau\, d\tau' \tilde{G}(r,r',r,r',\tau,\tau'). \quad (29)$$

The evaluation of the averaged operators is done by summing over the Matsubara frequencies in expressions like

$$\begin{aligned}
&\frac{1}{\beta} \int_0^\beta d\tau \int_0^\beta d\tau' G_{\alpha\beta}(\mathbf{r}_1,\tau,\mathbf{r}_2,\tau')\, G_{\gamma\delta}(\mathbf{r}_3,\tau,\mathbf{r}_4,\tau') \\
&= \frac{1}{\beta^3} \int d\tau\, d\tau' \sum_{m,n} \exp\big(i\,(\omega_n + \omega_m)\,(\tau - \tau')\big)\, G_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2,\omega_n)\, G_{\gamma\delta}(\mathbf{r}_3,\mathbf{r}_4,\omega_m) \\
&= \frac{1}{\beta} \sum_n G_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2,\omega_n)\, G_{\gamma\delta}(\mathbf{r}_3,\mathbf{r}_4,-\omega_n) \\
&= \sum_{i \neq j} \Bigg[\frac{N(E_i) + N(E_j) + 1}{E_i + E_j} \chi_i^\alpha(\mathbf{r}_1)\, \chi_i^{\beta\dagger}(\mathbf{r}_2) \chi_j^\gamma(\mathbf{r}_3)\, \chi_j^{\delta\dagger}(\mathbf{r}_4) - \frac{N(E_i) + N(E_j)}{E_i - E_j} \chi_i^\alpha(\mathbf{r}_1)\, \chi_i^{\beta\dagger}(\mathbf{r}_2) \tilde{\chi}_j^\gamma(\mathbf{r}_3)\, \tilde{\chi}_j^{\delta\dagger}(\mathbf{r}_4) \\
&\quad + \frac{N(E_i) - N(E_j)}{E_i - E_j} \tilde{\chi}_i^\alpha(\mathbf{r}_1)\, \tilde{\chi}_i^\beta(\mathbf{r}_2) \chi_j^\gamma(\mathbf{r}_3)\, \chi_j^{\delta\dagger}(\mathbf{r}_4) - \frac{N(E_i) + N(E_j) + 1}{E_i + E_j} \tilde{\chi}_i^\alpha(\mathbf{r}_1)\, \tilde{\chi}_i^{\beta\dagger}(\mathbf{r}_2) \tilde{\chi}_j^\gamma(\mathbf{r}_3)\, \tilde{\chi}_j^{\delta\dagger}(\mathbf{r}_4) \Bigg] \\
&\quad + \sum_i \frac{2\,N(E_i) + 1}{2\,E_i} \left[\chi_i^\alpha(\mathbf{r}_1)\, \chi_i^{\beta\dagger}(\mathbf{r}_2) \chi_i^\gamma(\mathbf{r}_3)\, \chi_i^{\delta\dagger}(\mathbf{r}_4) + \tilde{\chi}_i^\alpha(\mathbf{r}_1)\, \tilde{\chi}_i^{\beta\dagger}(\mathbf{r}_2) \tilde{\chi}_i^\gamma(\mathbf{r}_3)\, \tilde{\chi}_i^{\delta\dagger}(\mathbf{r}_4) \right] \\
&\quad + \beta \sum_i N(E_i)\big(N(E_i) + 1\big) \left[\tilde{\chi}_i^\alpha(\mathbf{r}_1)\, \tilde{\chi}_i^{\beta\dagger}(\mathbf{r}_2) \chi_i^\gamma(\mathbf{r}_3)\, \chi_i^{\delta\dagger}(\mathbf{r}_4) + \chi_i^\alpha(\mathbf{r}_1)\, \chi_i^{\beta\dagger}(\mathbf{r}_2) \tilde{\chi}_i^\gamma(\mathbf{r}_3)\, \tilde{\chi}_i^{\delta\dagger}(\mathbf{r}_4) \right],
\end{aligned} \quad (30)$$

with χ_i and $\tilde{\chi}_i$ defined in Eqs. (13) and (14). In particular, for the uniform case, all the terms proportional to $\nabla\sqrt{n_0}$ vanish; moreover, since the expressions for the Bogoliubov functions are known analytically, it is possible to see that only the last term of (30) survives, each term of the sum between square parentheses giving a contribution of $k^2/2$. In one dimension one obtains

$$\rho_s = \frac{N}{V} - \frac{\hbar^2\,N}{m\,V} \sum_{k \neq 0} k^2 \frac{\partial N(E_k)}{\partial E_k}, \quad (31)$$

which is the well-known Landau result [31].

IV. BOSE GLASS TRANSITION

We now apply our expressions for the correlation function and the superfluid density to a system at $T = 0$. The external potential is quasiperiodic, obtained as a sum of two potentials whose periods are incommensurate with each other,

$$U(x) = V_1 \cos\left(\frac{2\pi}{d}\,x\right) + V_2 \cos\left(\frac{2\pi}{\lambda d}\,x\right). \quad (32)$$

For our specific realization we have chosen λ to be the golden ratio, approximated as a fraction of two consecutive Fibonacci numbers. In the following we consider two systems with length $L = 377\,d$ and $L = 89\,d$; the value of λ is, respectively, $\lambda = 377/233$ and $\lambda = 89/55$.

For $V_2 = 0$, upon increasing the value of V_1, the system ceases to be superfluid and enters a Mott insulating phase [10,35,36]. As stated in the Introduction, this phase cannot be seen using the Bogoliubov ansatz we employ for the excitations, and our method should not be applied when the physics of the system is affected by the presence of the Mott phase. However, when the second lattice is turned on a new quantum phase, the Bose glass, becomes possible. According to Ref. [10], when V_2 is greater than the interaction energy, the Mott phase disappears from the phase diagram of the gas, and the only insulating phase is the glass phase. We believe that in this situation of strong disorder our approach can be quantitatively correct, and for definiteness we have chosen the weights of the two potentials to be equal ($V_1 = V_2 = V$).

Taking d as the unit of length, the Gross-Pitaevskii equation can be rewritten as

$$\left[-\frac{1}{2} \frac{d^2}{dx^2} + gN\,|\Phi(x)|^2 + U(x) \right] \Phi(x) = \mu\,\Phi(x), \quad (33)$$

where the energy is measured in units of $E_d = \hbar^2/m\,d^2$, and the norm of the quasicondensate wave function Φ is set to 1. For each value of the parameters V and g, we have found the ground state of Eq. (33) using an imaginary time evolution with the split-operator method, along with periodic boundary conditions. Using the ground-state wave function the Bogoliubov excitations are found by means of a direct diagonalization of the Bogoliubov equations (11). In all our calculations the Laplacian term is represented using the Fourier transform, while the derivative in the expression for superfluid density (29) is approximated using a finite difference expression.

The Bogoliubov analysis is expected to be relevant for values of the interparticle interaction $g/n \ll 1$. This means that our work is realistic for

$$gn \ll n^2; \quad (34)$$

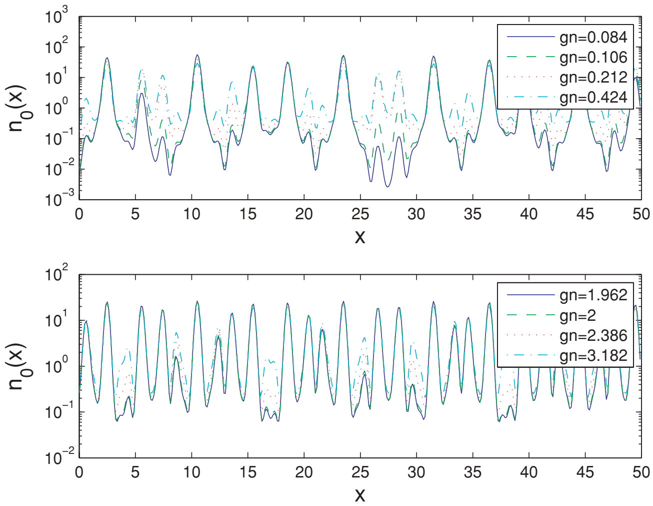

FIG. 1. (Color online) Density profiles for some values of gn for $V = 5\,E_d$ (top) and $V = 9\,E_d$ (bottom); gn is expressed in units of $E_d = \hbar^2/md^2$ and x in units of d. Upon decreasing the interaction strength, the condensate breaks up in sets of spikes with little overlap.

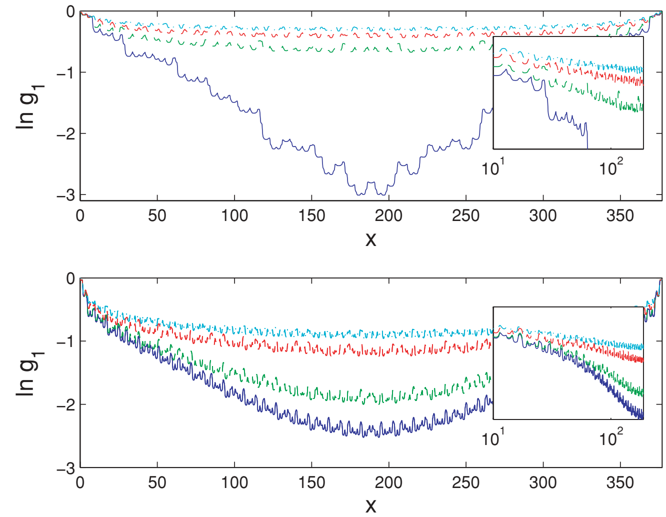

FIG. 2. (Color online) Exponent of the correlation function (35) for the same values of gn as in Fig 1, for $V = 5\,E_d$ (top) and $V = 9\,E_d$ (bottom); gn is expressed in units of $E_d = \hbar^2/m\,d^2$ and x in units of d. The insets show the same plot with a log scale on the horizontal axis. In the main panels it is seen that for lower gn the values of ln g_1 (35) follow a linear behavior. For higher gn the plot appears to depend logarithmically on x.

that is, the approximation we are using starts to be meaningful for systems that contain a few particles per site.

We first solve for the ground state in a system with $L = 377\,d$, using a numerical grid of 3016 points. Choosing a mean density $n = N/L = 4$, we show in Fig. 1 $n_0(x) = N\,|\Phi(x)|^2$ for the two cases $V = 5\,E_d$ and $V = 9\,E_d$. Upon decreasing g, the condensate density develops dips that become more and more pronounced. Indeed, in some regions the condensate seems to be broken up into several pieces that hardly overlap. As we see, this has consequences for the behavior of the correlation function.

Using the same mean density $n = N/L = 4$, Fig. 2 plots the exponent of the correlation function

$$\ln\, g_1(0,x) = \ln\,\langle\psi(0)\,\psi(x)\,\rangle - 1/2\,\ln\, n_0(0)n_0(x). \quad (35)$$

While for higher gn the values of ln g_1 follow a logarithmic behavior (insets in Fig. 2), leading to a power-law decay in the correlation function, for lower gn the function shows a linear fall, therefore giving an exponential decay. Since our system is finite, the transition to a different decay behavior is gradual, but as a quantitative measure we record the point where a linear fit for ln g_1 gives a smaller sum of the residuals than a logarithmic fit. The solid line in Fig. 3 shows the critical value of V as a function of the interaction strength, according to an interpolation of ln g_1 between $x = 10\,d$ and $x = 150\,d$.

The change in the decay behavior becomes evident at the values of g when the quasicondensate seems to be broken up into a set of spikes with little overlap. In this situation the lowest-energy excited modes have a phase flip character, since it costs little energy to change the relative phase of two barely overlapping zones; the energy of the lowest excitation decreases upon decreasing g. Eventually, this value becomes so low that reaches the limit of our numerical precision ($\sim 10^{-15}\,E_d$). At this point, we may say that the condensate has become completely disjointed. This boundary is indicated by the dashed line in Fig. 3. It is seen that it occurs close to the Bose glass transition, but it does not coincide with the transition found using the other criteria. In the lower left corner of the phase diagram, for small values of the interparticle

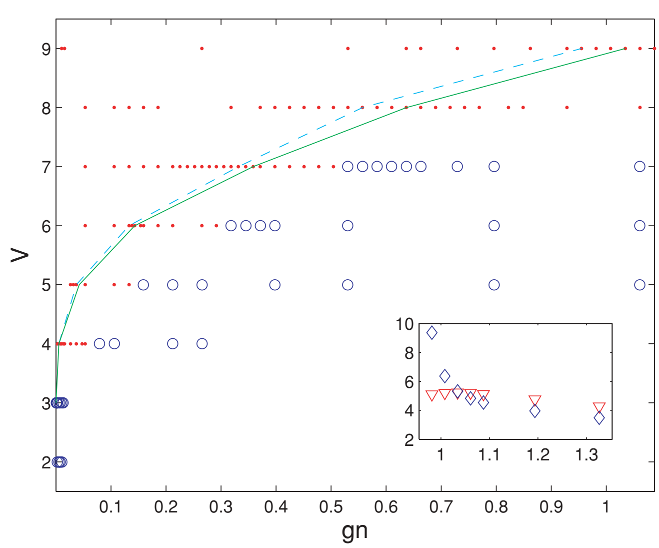

FIG. 3. (Color online) Phase diagram of the Bose glass transition. The open circles correspond to a state with nonvanishing superfluid component, while for the red dots the system has only a normal part. The solid line indicates when the linear interpolation of ln g_1 [Eq. (35)] gives a smaller error than a logarithmic fit; that is, at the left side of the solid line the correlation function decays exponentially. The dashed line indicates the point at which the smallest excitation energies cannot be resolved numerically. The inset shows the sum of the residuals of the fits (in arbitrary units) as a function of gn, for $V = 9\,E_d$: the triangles refer to a linear regression, the diamonds to a logarithmic interpolation. V and gn are expressed in units of $E_d = \hbar^2/m\,d^2$.

potential, detecting the transition can be numerically challenging, since at the point where the correlation function changes its behavior the excitation spectrum is already at the boundaries of the numerical precision.

Figure 3 also shows when the superfluid density, as given by Eq. (29), drops to zero (red points). Apparently, the superfluid formula systematically overestimates the critical interaction strength for the transition to an exponential decay. Indeed, Eq. (29) gives the superfluid density only in the thermodynamic limit and it might be that the finite size of the system is part of the explanation for the disagreement. However, we believe there is another reason for this discrepancy: the way we applied the phase twist to the condensate part in Eq. (24). For a nonuniform system, a phase twist in the boundary conditions should give a velocity that depends on the coordinate as a function of the condensate density in each point. The corrections to (29) should be more relevant for small g, when the variations in density are bigger.

The interpretation of the phase diagram in Fig. 3 is the following: at the left side of the line there is an exponentially decaying correlation function, along with a vanishing superfluidity; we believe that at this point our Bogoliubov scheme detects the Bose glass phase, as described in Sec. I. The Bose glass phase is seen to disappear completely below a finite value for the disorder strength V (approximately $V = 3$), in contrast to the findings of Ref. [7]. This is the main difference with the results of Fontanesi *et al.* Besides finite size effects, this behavior seems to be characteristic of the quasiperiodic potential. As stated in Sec. I, for vanishing interaction the system is localized only when the external potential strength is higher than a critical value. When a repulsive interaction is considered, this property survives in the glassy phase.

In Fig. 4, we show the behavior of ρ_s in an enlarged phase diagram. Because of the computational burden, this phase diagram was calculated for a smaller system with $L = 89\,d$ and a grid of 1424 points, and therefore the Bose glass boundary is slightly shifted compared to that in Fig. 3. Monte Carlo simulations [16,17] have previously shown a re-entrant phase transition upon increasing g. While we do not see this re-entrance, the behavior of the superfluidity is suggestive, as it shows a nonmonotonous dependence of the normal phase with respect to the interaction and the external potential strength. Indeed, for a quasiperiodic potential Ref. [5] shows that the re-entrant phase is limited to a small portion of the phase space, and it is compatible with a superfluid phase that extends to higher interaction strengths. Within the limit of the Bogoliubov approximation, our work agrees with this conclusion.

Finally, Fig. 5 and the inset of Fig. 4 show the inverse participation ratio of the quasicondensate wave function Φ,

$$P = \frac{1}{L}\frac{[\int dx|\Phi(x)|^2]^2}{\int dx|\Phi(x)|^4}, \qquad (36)$$

for the cases $L = 377\,d$ and $L = 89\,d$. Since the computation of P is much faster than the diagonalization of the Bogoliubov equations, Fig. 5 presents the behavior of the system for a finer resolution in the values of gn and V than Fig. 3. As we can see, the plots do not show any distinct transition. The glassy phase

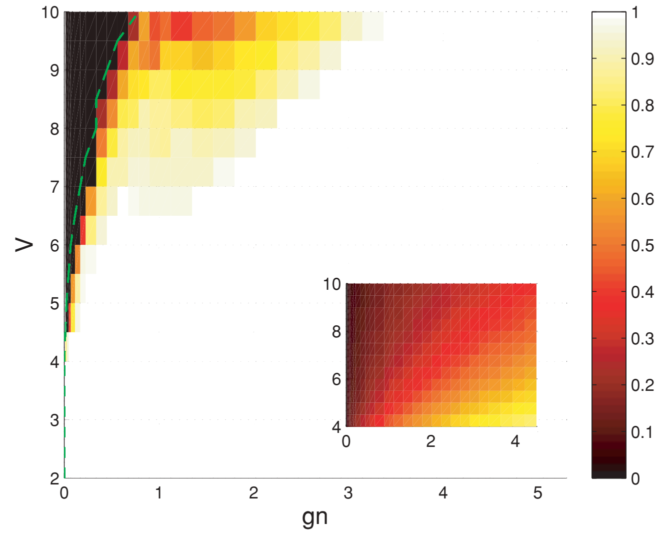

FIG. 4. (Color online) Surface plot of the superfluid density (29) (normalized to the total density n) as a function of V and gn, for $L = 89\,d$. The dashed line is the boundary that we estimate for the validity of our numerical calculations (see text). The inset shows the inverse participation ratio P (36) of the quasicondensate wave function. Both V and gn are measured in units of $E_d = \hbar^2/m\,d^2$.

does not appear to be related to a radical change in the density profile. Moreover, as stated in Sec. I, a new phase, the Lifshitz glass, has been conjectured to exist for weak interparticle potential, where the peaks in the density profile do not overlap. We argue that such a phase should have a distinctive signal in the quantity P. However, the plots of the inverse participation ratio show only a smooth decrease. We conclude that—within our approximation—the Lifshitz glass does not seem to exist as a phase in its own right, but only as the limit of vanishing interaction.

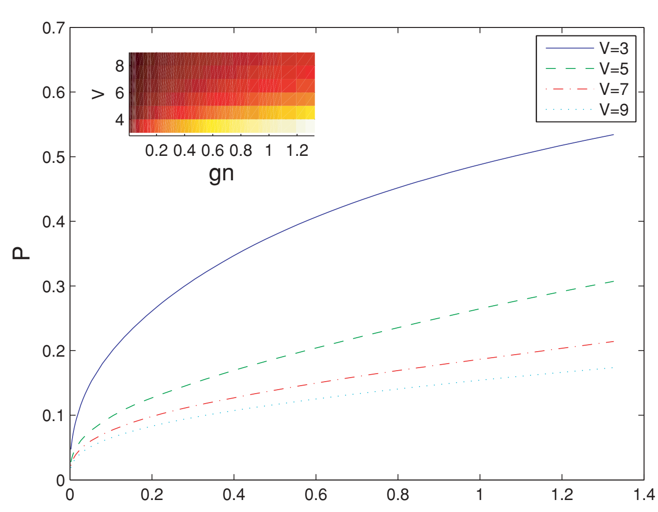

FIG. 5. (Color online) Inverse participation ratio P (36) for $L = 377\,d$ as a function of gn, for some values of the external potential strength V. The inset shows a surface plot of P as a function of V and gn. All the energies are measured in units of $E_d = \hbar^2/m\,d^2$.

V. CONCLUSIONS

Using a path integral formalism, we have obtained—within the Bogoliubov approximation—expressions for the correlation function and the superfluid density in a Bose gas for an arbitrary external potential at zero or finite temperature. Applying these expressions to a 1D system at $T = 0$, a quasiperiodic external potential was seen to cause a transition to a phase where the correlation decays exponentially and the superfluid density vanishes. We believe this is the Bose glass phase. A comparison with the experiment of Fallani *et al.* [1] is not straightforward, because they employed a quasiperiodic potential with different relative weights [i.e., $V_1 \neq V_2$ in Eq. (32)], and the proximity to the Mott insulating phase in the experiment makes the Bogoliubov approximation questionable. However, they spotted signs of a Bose glass transition when the peak height of the external potential varies between $2V \approx 16\, E_d$ and $2V \approx 18\, E_d$, with $gn \approx 1\, E_d$, which is where the transition takes place in our study for the bigger system ($L = 377\, d$).

We have not found a perfect match between the results for the superfluid density and the correlation behavior for the location of the Bose glass transition. We would like to point out that the formula for the superfluid density (29) assumes that the velocity v_s of the condensate does not depend on the position, and in this sense it is an approximation even within the Bogoliubov approach. For this reason we plan to improve the expression for ρ_s in the future.

Finally, we notice that our expression for the correlation function and the superfluid density are valid also at a finite temperature, and they can be used to study the Bose glass transition for $T \neq 0$. This relevant issue will be considered in a future work.

Note added in proof. A recently published article by Fontanesi, Wouters and Savona [PRA **81**, 053603 (2010)] extends the analysis done in Ref. [7].

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