Wentzel-Kramers-Brillouin approach and quantum corrections to classical dynamics in the Josephson problem

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We apply a many-body Wentzel-Kramers-Brillouin (WKB) approach to determine the leading quantum corrections to the semiclassical dynamics of the Josephson model, describing interacting bosons able to tunnel between two localized states. The semiclassical dynamics is known to divide between regular oscillations and self-trapped oscillations where the sign of the imbalance remains fixed. In both cases, the WKB wave functions are matched to Airy functions, yielding a modified Bohr-Sommerfeld quantization condition. At the critical energy dividing normal and self-trapped oscillations, the WKB wave functions should instead be matched to parabolic cylinder functions, leading to a quantization formula that is not just the Bohr-Sommerfeld formula, and recovering the known logarithmic quantum break times at this energy. This work thus provides another illustration of the usefulness of the WKB approach in certain many-body problems.

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I. INTRODUCTION

The collective dynamics of a large number of interacting quantum systems can often be described semiclassically, as mean-field approximations of the dynamics of such systems become more accurate with increasing system size. The great progress in trapping and manipulating cold atoms and in producing strong coupling between confined photon modes and matter degrees of freedom have led to an increasing variety of systems in which it becomes possible to study many-body dynamics in isolated systems, and to investigate the extent to which semiclassical descriptions are applicable. In many cases, the classical dynamics can show or has shown interesting collective oscillations; examples include interconversion between fermionic atoms and molecules [1–3], dynamical superradiance in coupled light-matter systems [4,5], optomechanical oscillations [6], and the topic of this article, Josephson oscillations between atoms trapped in different wells [7–11].

Josephson oscillations [12] of atoms between two trapped condensates are well described by the Hamiltonian

$$H = U(a^\dagger a - b^\dagger b)^2 + J(a^\dagger b + b^\dagger a). \tag{1}$$

The semiclassical dynamics of this system have been studied in the context of the Lipkin-Meshkov-Glick Hamiltonian: By introducing the quantum operators $S_z = (b^\dagger b - a^\dagger a)/2$, $S_x = (a^\dagger b + b^\dagger a)/2$, and $S_y = i(a^\dagger b - b^\dagger a)/2$, the Hamiltonian becomes

$$H = 4US_z^2 + 2JS_x. \tag{2}$$

The semiclassical dynamics divide into two regimes: small regular oscillations between the wells for low energies and self-trapped oscillations, where the imbalance remains of a fixed sign for high energies [13]. Specifically, if we replace the operators by c numbers, the canonical equations of motion reduce to the single expression

$$\dot{S_z}^2 = J^2N^2 - E^2 + (2UE - J^2)(2S_z)^2 - U^2(2S_z)^4. \tag{3}$$

Here we have used conservation of energy and particle number $N^2/4 = |S|^2$. If $J^2N^2 - E^2 < 0$ and $2UE - J^2 > 0$ then the region of allowed S_z divides into two, excluding small population imbalances (small values of S_z). This means that for $2NU > J$, there are two classes of oscillations; those with energies $E > NJ$ are self-trapped oscillations [7,8,13] where the sign of imbalance remains fixed, and those with $E < NJ$ where the sign of the imbalance varies periodically. Both regimes have been observed experimentally [9,10]. The Lipkin-Meshkov-Glick Hamiltonian, Eq. (2), has been studied in a coherent-state representation semiclassically [14] and with quantum corrections [15]. It has also been studied for small numbers of particles where quantum corrections may play a larger role [16].

Instead of the spin representation, this article maps the Josephson problem onto a discrete Schrödinger equation. As this is a one-dimensional problem, it can be approximately solved by the Wentzel-Kramers-Brillouin (WKB) approach [17]. Expanding for large system size N, the WKB wave functions can be found for the allowed and forbidden regions of Fock space. At the boundaries, these are matched to exact solutions of the Schrödinger equation, which often yields Airy functions. In such cases, the spectrum is given by the usual Bohr-Sommerfeld quantization condition on the action $A(E_n) = \int p dx = 2\pi(n + 1/2)$ at a particular energy level E_n. Using this phase-space quantization approach, the Josephson problem has been studied in [18,19]. One notable feature of the validity of semiclassics for the Josephson problem is that the quantum break times—the characteristic time at which quantum and classical dynamics start to differ—may grow only logarithmically, rather than algebraically, with N when near the energy dividing self-trapped and non-self-trapped oscillations. This has been seen both by using a cumulant expansion [20,21] and by phase-space quantization [22]. The existence of an associated logarithmic divergence in the density of states has been discussed in [23] for the related model of a single particle moving in an infinite tight-binding lattice with an overall harmonic trap.

The aim of this article is to apply the many-body WKB approach as discussed in Ref. [24] to the Josephson problem of Eq. (1) and to use this to find the quantum break times

for critical and noncritical levels. We solve the Schrödinger equation for large system size N to next-to-leading order. The WKB wave functions in the allowed and forbidden regions are then matched to appropriate special functions at the boundaries. For most energies, the appropriate special function is the Airy function, which then recovers the Bohr-Sommerfeld quantization condition derived in [18,19]. However, near the critical energy separating self-trapped and non-self-trapped oscillations, the relevant special functions are instead parabolic cylinder functions, and these lead to a different quantization condition that goes beyond the Bohr-Sommerfeld formula of [18,19]. Even away from the critical level, the full WKB analysis yields information beyond Bohr-Sommerfeld quantization: The allowed region divides into a regime where matching to a purely decaying Airy function is appropriate and one where the decaying solution is also highly oscillatory. While this behavior modifies the nature of the wave functions, its effect on the quantization condition is only a jump in the Maslov index. As such, it leaves the density of states unaffected, and so it was not crucial to previous work in the context of Bohr-Sommerfeld quantization.

The approach presented in this article has also been used in Ref. [24] to study the Tavis-Cummings model [25], describing, for example, the dynamics of two-level systems coupled to photons. The many-body WKB approach gives quantum dynamics for this model that is very similar to the classical dynamics for most energy levels and initial conditions. However, there are critical levels for which the matching of the WKB wave functions becomes more complex and gives rise to anharmonicity scaling as the logarithm of system size [26]. In both the Tavis-Cummings model and the Josephson model, these critical levels show logarithmic quantum break times, analogous to the $\ln(\hbar)$ quantum break times found near unstable classical states in single-particle quantum problems [27]. A further class of problems in which this many-body WKB approach may be of use concerns variations of problems such as the Josephson problem and the Tavis-Cummings model, in which parameters are varied as a function of time to give many-body generalizations of Landau-Zener problems [28–33]. Many treatments of this problem have been effectively semiclassical, and the WKB approach may provide a method to determine whether large quantum corrections can ever arise as a result of transitions to and from critical levels. An approach along these lines has been explored in [31].

The remainder of this article is arranged as follows. Section II maps the many-body Hamiltonian to a form amenable to solution by the WKB method and then provides the ingredients necessary to determine this solution. These ingredients are the WKB wave functions in the classically allowed and forbidden regions, given in Sec. II A, and the connection formula holding at the boundaries between these regions, given in Sec. II B. Section III then combines these ingredients to give the resulting quantization conditions, which are specified in three separate ranges of energies, according to whether one is above, near, or below the critical energy level (the energy level dividing self-trapped and non-self-trapped oscillations). From these quantization conditions, Sec. IV then extracts the scaling of the quantum break time as a function of system size and summarizes the results.

II. DISCRETE WKB APPROACH TO JOSEPHSON EQUATION

To use the WKB approach, we need to produce a discrete Schrödinger equation by writing the state of the system in terms of the total number of particles N and the imbalance n:

$$|\Psi\rangle = \sum_{n=-N}^{N} \psi_n \frac{a^{\dagger(N+n)/2} b^{\dagger(N-n)/2}}{\sqrt{[(N+n)/2]![(N-n)/2]!}} |\Omega\rangle, \tag{4}$$

where n is restricted to the same odd/even parity as N. Acting on this state with the many-body Hamiltonian and looking for eigenstates with energy E yields a discrete Schrödinger equation:

$$(E - Un^2)\psi_n = \frac{J}{2}[\sqrt{(N+n)(N-n+2)}\psi_{n-2} + \sqrt{(N+n+2)(N-n)}\psi_{n+2}]. \tag{5}$$

Writing $E = \epsilon JN$, $U = Ju/N$, and $z = n/N$, one can separate the system size dependence from the other parameter dependence and write

$$(\epsilon - uz^2)\psi(z) = \frac{1}{2}\sqrt{(1+z)\left(1 - z + \frac{2}{N}\right)}\psi\left(z - \frac{2}{N}\right) + \frac{1}{2}\sqrt{(1-z)\left(1 + z + \frac{2}{N}\right)}\psi\left(z + \frac{2}{N}\right). \tag{6}$$

In these units, self-trapped states exist only if $u > 1/2$ and $\epsilon > 1$.

A. WKB wave function

For a given energy, one may divide the range of $-1 < z < 1$ into classically allowed and forbidden regions. These are distinguished by oscillating versus decaying wave functions and correspond directly to the regions of $S_z = Nz$ in Eq. (3), which the classical dynamics explores.

1. Allowed region

In the allowed region, the WKB ansatz has the form

$$\psi(z) = b(z)[C_+ e^{i(NW_0 + W_1)} + C_- e^{-i(NW_0 + W_1)}], \tag{7}$$

where W_0 and W_1 are the z-dependent phase terms at leading and next-to-leading order. By substituting this into Eq. (6) and identifying real and imaginary terms of the same order in $1/N$, one finds the definitions

$$\cos[2W_0'(z)] = \frac{\epsilon - uz^2}{\sqrt{1-z^2}}, \tag{8}$$

$$b(z) = [1 - z^2 - (\epsilon - uz^2)^2]^{-1/4}, \tag{9}$$

$$W_1'(z) = \frac{\epsilon - uz^2}{2(1-z^2)\sqrt{1 - z^2 - (\epsilon - uz^2)^2}}. \tag{10}$$

[Note that the sign written for W_1' assumes that the solution of Eq. (8) is taken such that $\sin(2W_0') \geqslant 0$.] For this solution to be valid, it is clearly necessary that $|\epsilon - uz^2| \leqslant \sqrt{1-z^2}$, which defines the classically allowed region shown in Fig. 1.

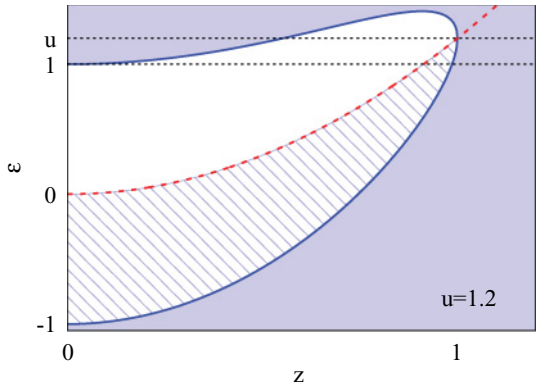

FIG. 1. (Color online) Boundaries between allowed and forbidden regions and between regions which need an extra factor of $(-1)^{n/2}$. The solid-shaded region is classically forbidden, and the hatched region ($\epsilon < uz^2$) requires an extra factor of $(-1)^{n/2}$ to allow matching to the connection formula at the boundary. Plotted for $u = 1.2$.

Within the classically allowed region, there is also a division between $\epsilon > uz^2$, for which $0 \leqslant 2W_0' < \pi/2$, and $\epsilon < uz^2$, for which $\pi/2 < 2W_0' \leqslant \pi$. While this distinction is unimportant within the allowed region, it is necessary later to distinguish these regions so that the connection formulas at the boundary of the allowed region can be written in terms of smooth wave functions. If $\epsilon > uz^2$, then at the boundary $W_0' \to 0$, so the wave function is smooth. For $\epsilon < uz^2$, $W_0' \to \pi/2$, which indicates that $NW_0 = iNx\pi/2$, which is rapidly varying. This rapid variation can be removed in such cases by instead defining $\psi_n \to \psi_n(-1)^{n/2}$. Since only values of n with the same parity as N exist, this corresponds to terms having alternating signs. After this transformation, Eqs. (8)–(10) are modified by replacing $(\epsilon - uz^2) \to -(\epsilon - uz^2)$. The regions where this transformation is necessary are indicated in Fig. 1 by hatching. We use the notation $\tilde{W}_0, \tilde{W}_1$ for the phases calculated with this additional minus sign. When part of the allowed region does require this transformation and part does not (i.e., for $0 < \epsilon < u$), it is necessary to consider carefully the connection between these regions; this is discussed further in Sec. II B 3.

2. *Forbidden region*

In the forbidden region, the wave function exponentially decays, so the WKB ansatz becomes

$$\psi(z) = b(z)[C_+ e^{(N\Omega_0+\Omega_1)} + C_- e^{-(N\Omega_0+\Omega_1)}]. \tag{11}$$

In this case, there is no strict distinction between the terms to be incorporated in $b(z)$ and $\Omega_1(z)$—both describe the real part at order $1/N$. However, it is convenient for the connection formulas to keep $b(z)$ as in Eq. (9), after which $\Omega_{0,1}$ can be identified by powers of $1/N$ as

$$\cosh(2\Omega_0') = \frac{\epsilon - uz^2}{\sqrt{1-z^2}}, \tag{12}$$

$$\Omega_1' = \frac{-(\epsilon - uz^2)}{2(1-z^2)\sqrt{(\epsilon - uz^2)^2 - 1 + z^2}}. \tag{13}$$

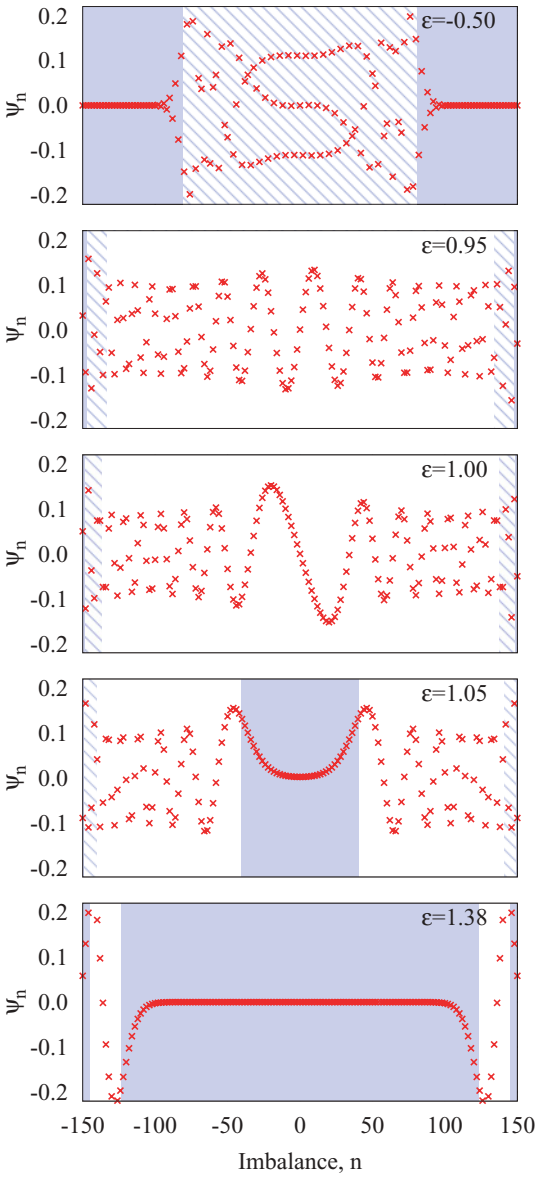

FIG. 2. (Color online) Wave functions at various characteristic energies, showing behavior in allowed and forbidden regions. Background shading is as in Fig. 1. Plotted for $u = 1.2$.

In this case, the sign of Ω_1' assumes $\Omega_0' \geqslant 0$. In the forbidden region at small z, one always has $\epsilon > uz^2$ so no alternating sign factors are needed, but in the forbidden region at large z, a factor $(-1)^{n/2}$ may be needed so that $\cosh(2\Omega_0') > 0$. The distinction between exponential decay and exponential decay with alternating signs is clearly visible in the exact wave functions shown in Fig. 2. In the forbidden region at small z, one will have $|C_-| = |C_+|$ (assuming that Ω is measured from $z = 0$), since the wave functions will be either odd or even functions of z. In the region at large z, only one of $C_\pm$ will be nonzero, describing exponential decay for $z \to \mp\infty$.

3. *Critical level*

At the critical level $\epsilon = 1$, there is a bifurcation, and the WKB wave function in the allowed region must be matched to a different special function for $z \simeq 0$. For this purpose, it is

convenient to write $\epsilon = 1 + \lambda/N$, where λ is small compared to N, and to rewrite the WKB form for the allowed regime making use of this—because the deviation of the energy from 1 is now considered as being of order $1/N$, and its effect is relegated from W_0 to W_1,

$$\cos[2W_0'(z)] = \frac{1-uz^2}{\sqrt{1-z^2}}, \tag{14}$$

$$b(z) = [z^2(2u-1-u^2z^2)]^{-1/4}, \tag{15}$$

$$W_1'(z) = \frac{1-\lambda+(\lambda-u)z^2}{2(1-z^2)|z|\sqrt{2u-1-u^2z^2}}. \tag{16}$$

One can see from Eq. (16) that W_1 will be logarithmically divergent as $z \to 0$. The form of $\psi(z)$ for $z \simeq 0$ will give the natural cutoff to this logarithm, which will depend on the system size N, so exactly at this critical level, the eigenvalue spacing will depend logarithmically on the system size, which in turn leads to logarithmic dependence of the quantum break time on system size.

B. Connection formulas

To connect the WKB ansatz valid in the classically allowed and classically forbidden regions, one needs the solution valid near the boundary. If the wave function is smooth near this point, one may expand Eq. (5) directly in powers of $1/N$ to give a differential equation for the wave function. (Note that in the WKB ansatz, there was no such assumption of smoothness, since we did not require W_0' to be small.) To ensure the wave function is smooth, a factor $(-1)^{n/2}$ may be needed according to whether the boundary is in the hatched or unhatched region of Fig. 1, as discussed previously. Both these cases can be considered together, by replacing $(\epsilon - uz^2) \to |\epsilon - uz^2|$ to take account of the two possible signs. With this change, by expanding Eq. (6) to order $1/N^2$, one finds

$$|\epsilon - uz^2|\psi = \left(\sqrt{1-z^2} + \frac{1}{N\sqrt{1-z^2}}\right)\psi + \frac{2\sqrt{1-z^2}}{N^2} \times \left[\psi'' - \frac{z\psi'}{1-z^2} + (\cdots)\psi\right] + \mathcal{O}(N^{-3}). \tag{17}$$

When considering the leading-order behavior and next-to-leading-order behavior (i.e., classical behavior plus leading-order quantum corrections), the order N^{-2} term involving ψ may be neglected, as there are larger terms involving ψ, whereas the terms involving ψ' and ψ'' should be kept. Rewriting the previous equation, one has

$$0 = \psi'' - \frac{z}{1-z^2}\psi' + \left[\frac{N}{2(1-z^2)} + \frac{N^2}{2}\left(1 - \frac{|\epsilon - uz^2|}{\sqrt{1-z^2}}\right)\right]\psi. \tag{18}$$

1. Regular boundaries—Airy functions

Away from the critical level $\epsilon = 1$, the boundary between the allowed and forbidden region is given by $|\epsilon - uz^2| = \sqrt{1-z^2}$, which has the outer (inner) solutions

$$z_{o,i}^2 = \frac{1}{2u^2}(2u\epsilon - 1 \pm \sqrt{1 - 4u\epsilon + 4u^2}). \tag{19}$$

Near these boundaries, we want to write Eq. (18) in terms of distance ζ from the boundary, $z = \pm z_{o,i} + \zeta$. Since Eq. (18) is even under $z \to -z$, it is clear that the equations at $z = z_{o,i} + \zeta$ and $z = -z_{o,i} + \zeta$ are related by $\zeta \to -\zeta$. Therefore, considering the first of these, we may write Eq. (18) as

$$0 = \psi'' - \alpha_{o,i}\psi' + [\beta_{o,i} + f_{o,i}(\zeta)]\psi, \tag{20}$$

with $\alpha_{o,i} = z_{o,i}/(1-z_{o,i}^2)$, $\beta_{o,i} = N/[2(1-z_{o,i}^2)]$, and

$$\begin{aligned} f_{o,i}(\zeta) &\simeq \frac{N^2}{2}\left(1 - \frac{|\epsilon - uz_{o,i}^2 - 2uz_{o,i}\zeta|}{\sqrt{1-z_{o,i}^2-2z_{o,i}\zeta}}\right) \\ &\simeq \frac{N^2}{2}z_{o,i}\zeta\left[\frac{2u(\epsilon - uz_{o,i}^2) - 1}{1-z_{o,i}^2}\right] \\ &= \mp\frac{N^2}{2}\frac{z_{o,i}\zeta\sqrt{1-4u\epsilon+4u^2}}{1-z_{o,i}^2} \stackrel{\text{def}}{=} \gamma_{o,i}\zeta. \end{aligned} \tag{21}$$

In the previous equation, we have assumed that ζ is small, so the sign of $\epsilon - uz^2$ does not change, and the last line has used the form of $z_{o,i}^2$ in Eq. (19), with the $\mp$ signs corresponding to the outer (inner) boundary. With $f(\zeta) = \gamma\zeta$, the solution Eq. (20) can be written using Airy functions:

$$\psi = e^{-\alpha\zeta/2}[C_a Ai(-\gamma^{1/3}\xi) + C_b Bi(-\gamma^{1/3}\xi)], \tag{22}$$

with $\xi = \zeta + \beta/\gamma - \alpha^2/4\gamma$. For the boundaries at z_0, we match to a decaying solution, so $C_b = 0$. Since γ_o is negative, the solutions are oscillatory for $\zeta < 0$ and decaying for $\zeta > 0$ as expected. For the inner boundaries, both exponentially decaying and growing parts are required. Since γ_i is positive, the solutions are oscillatory for $\zeta > 0$ and growing/decaying for $\zeta < 0$.

2. Critical boundary—parabolic cylinder functions

Near $\epsilon = 1$, the inner boundary becomes an extremum at $z = 0$, and the form of Eq. (18) is different from that in Sec. II B 1. To study energies near this level, we write $\epsilon = 1 + \lambda/N$ as in Sec. II A 3. Since $\epsilon = 1, z = 0$ never requires a factor $(-1)^{n/2}$ in the wave function, Eq. (18) can always be written near this point as

$$0 = \psi'' - z\psi' + \left[\frac{N}{2}(1-\lambda) + \frac{N^2}{4}(2u-1)z^2\right]\psi. \tag{23}$$

To solve this equation, it is convenient to define

$$\mu = \sqrt{2u-1}, \quad \chi = \frac{1-\lambda}{2\mu}. \tag{24}$$

After removing a Gaussian factor $\psi = e^{z^2/4}f$, this can be recognized as Weber's equation [34]; in terms of $\xi = ze^{-i\pi/4}\sqrt{N\mu}$ one has $0 = f'' + (i\chi - \xi^2/4)f$ with solutions in terms of parabolic cylinder functions. The general solution can be written as

$$\begin{aligned} \psi = e^{z^2/4}[&\alpha D_{i\chi-1/2}(e^{-i\pi/4}\sqrt{N\mu}z) \\ &+ \beta D_{i\chi-1/2}(e^{3i\pi/4}\sqrt{N\mu}z)]. \end{aligned} \tag{25}$$

The Gaussian prefactor only contributes at order $1/N$, so in matching the asymptotics of $NW_0, N\Omega_0$ to ψ, this prefactor can be dropped. It is clear that for $z \to -z$, this expression

changes as $\alpha \leftrightarrow \beta$. The asymptotic expansion of this expression for large x given in Ref. [34] can be written

$$\psi = \left[\alpha \exp\left(\frac{\pi\chi}{4} + i\frac{\pi}{8}\right) + \beta \exp\left(-\frac{3\pi\chi}{4} - i\frac{3\pi}{8}\right)\right] \times \exp\left(i\frac{N\mu z^2}{4} + \left[i\chi - \frac{1}{2}\right]\ln[z\sqrt{N\mu}]\right) + \beta\frac{\sqrt{2\pi}}{\Gamma[1/2 - i\chi]}\exp\left(-\frac{\pi\chi}{4} + i\frac{\pi}{8}\right) \times \exp\left(-i\frac{N\mu z^2}{4} + \left[-i\chi - \frac{1}{2}\right]\ln[z\sqrt{N\mu}]\right). \quad (26)$$

As noted in Sec. II A3, the phase has a logarithmic divergence at small z, and the form of the wave function given here provides the cutoff for the logarithm, $1/\sqrt{N\mu}$, which depends on system size.

3. *Connection within allowed region*

In addition to connection formulas at the boundaries of allowed and forbidden, for energies in the range $1 < \epsilon < u$, it is necessary to connect solutions with and without the extra factor of $(-1)^{n/2}$ in the middle of the allowed region.

In such cases, we may write

$$\psi = \begin{cases} [C_+ e^{iW} + C_- e^{-iW}] & |z| < z_s \\ (-1)^{n/2}[\tilde{C}_+ e^{i\tilde{W}} + \tilde{C}_- e^{-i\tilde{W}}] & |z| > z_s \end{cases}, \quad (27)$$

where $z_s = \sqrt{\epsilon/u}$ is the point at which the sign of $\epsilon - uz^2$ changes. The question is how to relate $C_\pm$ to $\tilde{C}_\pm$ at the boundary. To fully define the the phases $W, \tilde{W}$, one must specify the limits of integration. It is convenient to choose

$$W(z) = \int_{z_i}^{z} dz[NW_0' + W_1'], \quad \tilde{W}(z) = \int_{z_o}^{z} dz[N\tilde{W}_0' + \tilde{W}_1'] \quad (28)$$

and to then define the phases at z_s as $\Delta W_{\rm in} = W(z_s)$, $\Delta\tilde{W}_{\rm out} = -\tilde{W}(z_s)$. To determine the connection, we focus on the phase due to W_0'; by expanding for $z = z_s + \zeta$, one has

$$\cos(2W_0') = \frac{-2u\sqrt{\epsilon}\zeta}{\sqrt{u-\epsilon}} = -\cos(2\tilde{W}_0'), \quad (29)$$

and by integrating one has

$$W(z_s + \zeta) = +\Delta W_{\rm in} + N\left(\frac{\pi}{4}\zeta + \frac{\sqrt{\epsilon u}}{\sqrt{1-\epsilon/u}}\frac{\zeta^2}{2}\right), \quad (30)$$

$$\tilde{W}(z_s + \zeta) = -\Delta\tilde{W}_{\rm out} + N\left(\frac{\pi}{4}\zeta - \frac{\sqrt{\epsilon u}}{\sqrt{1-\epsilon/u}}\frac{\zeta^2}{2}\right). \quad (31)$$

To match the ζ dependence near $\zeta = 0$, one can write the prefactor $(-1)^{n/2} = \exp[i\pi N(z_s + \zeta)/2]$, which gives the matching conditions

$$C_- = \tilde{C}_+ \exp\left(i\Delta W_{\rm in} - i\Delta\tilde{W}_{\rm out} - i\frac{\pi N}{2}z_s\right), \quad (32)$$

and the complex conjugate for the other pair.

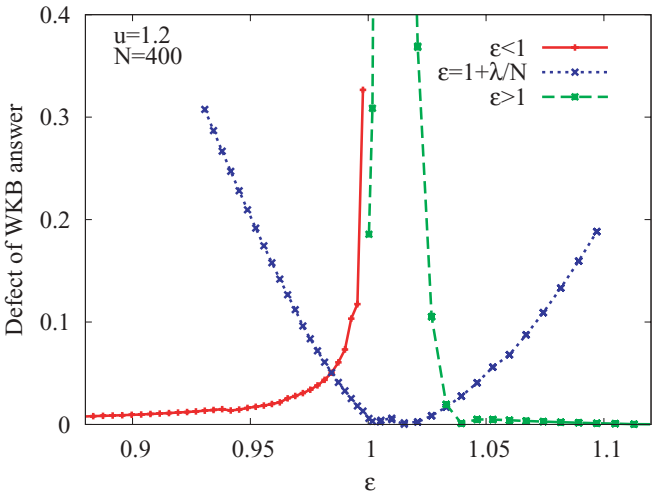

FIG. 3. (Color online) Accuracy of WKB-derived quantization condition. Each line corresponds to difference of appropriately defined phase difference and nearest integer multiple of π, using quantization conditions appropriate to energies below, near, and at the critical level. Plotted for $u = 1.2$.

III. BOUNDARY MATCHING AND QUANTIZATION CONDITIONS

By connecting the wave functions in the allowed and forbidden regions to the Airy or parabolic cylinder functions at the boundaries, one can derive conditions relating the coefficients $C_\pm$ in these various regions. In order to simultaneously satisfy all the equations, it is necessary to have certain conditions on the WKB phases; this provides quantization conditions on the allowed energies. This section derives these quantization conditions in the three cases: $\epsilon < 1$, $\epsilon > 1$, and $\epsilon = 1 + \lambda/N$. In all cases, the quantization condition can be reduced to a form $f(\epsilon) = m\pi$, where m is an integer. Figure 3 illustrates the accuracy of these WKB quantization methods by plotting $\min_m[f(\epsilon_n) - m\pi]$, where ϵ_n is an exact eigenvalue—that is, determining the extent to which the exact eigenvalues obey the WKB quantization condition.

A. Below critical level

We consider separately the cases $\epsilon < 0$ and $0 < \epsilon < 1$. For $\epsilon < 0$, the quantization condition is simple as the factor $(-1)^{n/2}$ is needed throughout the allowed region. Let us define the limits of integration for $W(z)$ such that

$$\tilde{W}(z) = \int_0^z dz(N\tilde{W}_0' + \tilde{W}_1'), \quad \Delta\tilde{W} = \tilde{W}(z_o). \quad (33)$$

Then, by expanding the equation for $\tilde{W}_0'$ [i.e., Eq. (14) with an extra minus sign] near $z = z_0 + \zeta$ and comparing the result to the last line of Eq. (21), one can show that

$$1 - 2(\tilde{W}_0')^2 = 1 - \left[\frac{2u(\epsilon - uz_o^2) - 1}{1 - z_o^2}\right] z_o\zeta = 1 - \frac{2\gamma_o}{N^2}\zeta. \quad (34)$$

Remembering that $\gamma_o < 0$, integrating the expression for $\tilde{W}_0'$ gives

$$\tilde{W}(z_o + \zeta) = \Delta\tilde{W} - \tfrac{2}{3}|\gamma_o|^{1/2}|\zeta|^{3/2}, \quad (\zeta < 0). \tag{35}$$

In order to match $\psi = (-1)^{n/2}[\tilde{C}_+ \exp(i\tilde{W}) + \tilde{C}_- \exp(-i\tilde{W})]$ to the expansion of the Airy function [34] appearing in Eq. (22),

$$\psi \propto Ai[|\gamma_o|^{1/3}(\zeta + \cdots)] \simeq \sin\left(\frac{\pi}{4} + \frac{2}{3}|\gamma_o|^{1/2}|\zeta|^{3/2}\right) \tag{36}$$

gives the condition $(\tilde{C}_+/\tilde{C}_-)\exp(2i\Delta\tilde{W}) = i$. Repeating this procedure at $z = -z_o + \zeta$, one has instead

$$\tilde{W}(-z_o + \zeta) = -\Delta\tilde{W} + \frac{2}{3}|\gamma_o|^{1/2}|\zeta|^{3/2}, \quad (\zeta > 0), \tag{37}$$

and similar matching gives $(\tilde{C}_+/\tilde{C}_-)\exp(-2i\Delta\tilde{W}) = -i$. By combining these, the quantization condition is

$$m\pi = 2\Delta\tilde{W} - \frac{\pi}{2}, \tag{38}$$

which is just Bohr-Sommerfeld quantization, as expected for this simplest case.

In the range $0 < \epsilon < 1$, there are both regions with and without factors $(-1)^{n/2}$; however, since both boundaries have this additional factor, it is possible to get away with Eq. (38), using the integral in Eq. (33), even when $\cos(2\tilde{W}_0') < 0$. To show that this is valid, we discuss the result of taking the connection at $z = z_s$ into account explicitly. Using the results of Sec. II B 3 means that in the connection formulas at the outer boundary, one should replace

$$\tilde{C}_+ e^{i\Delta W} \to C_- e^{i(\Delta\tilde{W}_{\rm out} - \Delta W_{\rm in} + \pi N z_s/2)}, \tag{39}$$

and for $\tilde{C}_-$ the condition is similar, but complex conjugate. Here, the expression $W_{\rm in}$ in Eq. (39) should be understood as taking $z_i \to 0$ in the expressions following Eq. (28), since there is no inner boundary for $\epsilon < 1$. With this replacement, the quantization condition becomes

$$m\pi = 2\left(\Delta\tilde{W}_{\rm out} - \Delta W_{\rm in} + \frac{\pi N}{2} z_s\right) - \frac{\pi}{2}. \tag{40}$$

To relate the integrals $\Delta\tilde{W}_{\rm out}$, $\Delta W_{\rm in}$ to the $\Delta\tilde{W}$ of Eq. (33), we should note two features. First, there is a term $N(\pi/2)z_s$ in the difference of definitions due to the inverse cosine, which is compensated by the last term in brackets in Eq. (40). Other than this, one may see that the various sign factors in front of $\Delta W_{\rm in}$, in the relative definition of $\Delta W_{\rm in}$ versus $\Delta\tilde{W}_{\rm out}$, and in the order of limits of integration in Eq. (28) are such that the remaining parts of the integrals all match so that $\Delta W_{\rm in} - \pi N z_s/2 - \Delta\tilde{W}_{\rm out} = \Delta\tilde{W}$.

B. Above critical level

Above the critical level, there can either be three or five separate regions to consider: a forbidden region in the center and either single allowed regions to the left and right if $\epsilon > u$ or pairs of allowed regions if $\epsilon < u$. We label these regions, from left to right, as $\tilde{L}$, L, F, R, $\tilde{R}$, with the $\tilde{L}$, $\tilde{R}$ regions having $z_s < |z| < z_o$ when $\epsilon < u$, and vanishing otherwise. These labels are used both for the coefficients $\tilde{L}_\pm$, $\tilde{R}_\pm$ and for the phases $\tilde{W}_{\tilde{L}}$, W_L, Ω_F, W_R, $\tilde{W}_{\tilde{R}}$. Focusing on the right-hand side ($z > 0$), it is convenient to define W_R, $\tilde{W}_{\tilde{R}}$ with the same limits as W, $\tilde{W}$ in Eq. (28) and to define $\Omega_F(z)$ as the integral from 0 to z.

Consider first the case $\epsilon < u$. With these definitions, the outer boundary condition simply becomes $\tilde{R}_+/\tilde{R}_- = i$, as the phase in this region is defined so it vanishes at z_o. Using Eq. (32), this translates to a condition

$$\frac{R_+}{R_-}\exp\left(i2\left[\Delta W_{\rm in} - \Delta\tilde{W}_{\rm out} - \frac{\pi N}{2} z_s + \frac{\pi}{2}\right]\right) = i, \tag{41}$$

where the $\pi/2$ term on the left-hand side is associated with an overall minus sign from $i \to 1/i$ on the right-hand side.

In the case $\epsilon > u$, we need only $W(z)$, which we have already defined as the integral from z_i to z. In this case, the phase-matching condition at the outer boundary implies

$$\frac{R_+}{R_-}\exp(i2\Delta W) = i, \quad \Delta W = \int_{z_i}^{z_o} dz(N W_0' + W_1'). \tag{42}$$

Both Eqs. (41) and Eq. (42) can be combined by the statement $(R_+/R_-)\exp(2i\Delta W_{\rm eff}) = i$, with

$$\Delta W_{\rm eff} = \begin{cases} \Delta W_{\rm in} - \Delta\tilde{W}_{\rm out} - \frac{\pi N}{2} z_s + \frac{\pi}{2} & \epsilon < u \\ \Delta W & \epsilon > u \end{cases}. \tag{43}$$

The equivalent analysis at the leftmost boundary gives $(L_+/L_-)\exp(-2i\Delta W_{\rm eff}) = -i$.

The boundary at z_i always involves $R_\pm$ rather than $\tilde{R}_\pm$, and our choice of limits means that the phase is always measured from this boundary. Following the same logic as led to Eq. (37), and noting that γ_i is positive, one has $W(z_i + \zeta) = (2/3)|\gamma_i|^{1/2}|\zeta|^{3/2}$ for $\zeta > 0$. At this boundary, one must match both Airy functions, which gives the matching condition

$$\frac{(C_{R,b} - iC_{R,a})e^{+i\pi/4}}{(C_{R,b} + iC_{R,a})e^{-i\pi/4}} = \frac{R_+}{R_-}, \tag{44}$$

where $C_{R,a}$, $C_{R,b}$ are the coefficients of Airy functions, as in Eq. (22) at the right-hand ($+z_i$) boundary. These should also be matched to the coefficients of the growing/decaying terms in the forbidden region. Defining $\Delta\Omega_F = 2\Omega_F(z_i) = \Omega_F(z_i) - \Omega_F(-z_i)$, one may expand $\Omega_F(z_i + \zeta) = (\Delta\Omega_F/2) - (2/3)|\gamma_i|^{1/2}|\zeta|^{3/2}$ for $\zeta < 0$. Matching this to Airy functions yields

$$\frac{C_{R,b}}{C_{R,a}} = \frac{F_- e^{-\Delta\Omega_F/2}}{F_+ e^{+\Delta\Omega_F/2}}. \tag{45}$$

Equations (44) and (45) can be combined to eliminate the coefficients of Airy functions, giving

$$\frac{F_- e^{-\Delta\Omega_F/2} - iF_+ e^{+\Delta\Omega_F/2}}{F_- e^{-\Delta\Omega_F/2} + iF_+ e^{+\Delta\Omega_F/2}} = \frac{R_+}{R_-} e^{-i\pi/2} = e^{-2i\Delta W_{\rm eff}}. \tag{46}$$

Analysis at the left-hand boundary is very similar:

$$\frac{F_+ e^{-\Delta\Omega_F/2} - iF_- e^{+\Delta\Omega_F/2}}{F_+ e^{-\Delta\Omega_F/2} + iF_- e^{+\Delta\Omega_F/2}} = \frac{L_-}{L_+} e^{-i\pi/2} = e^{-2i\Delta W_{\rm eff}}. \tag{47}$$

Eliminating $F_\pm$ from these equations then yields the quantization condition

$$\frac{1 + \cos(2\Delta W_{\rm eff})}{1 - \cos(2\Delta W_{\rm eff})} = e^{-2\Delta\Omega_F}. \tag{48}$$

Since the quantity $\Delta\Omega_F$ grows linearly with N, the quantum tunneling between allowed regions is hugely suppressed.

Taking this tunneling into account at leading order yields $\cos(2\Delta W_{\rm eff}) \simeq -1 + 2\exp(-2\Delta\Omega_F)$, so that

$$\Delta W_{\rm eff}(\epsilon) \simeq m\pi + \frac{\pi}{2} \pm e^{-\Delta\Omega_F}. \tag{49}$$

Differentiating this gives an expression for the tunneling-induced energy splitting (the inverse of quantum tunneling time for self-trapped states): $\delta\epsilon_{\rm splitting} = 2e^{-\Delta\Omega_F}|d\Delta W_{\rm eff}/d\epsilon|^{-1}$.

C. Near critical level

Near the critical level, we replace the matching to Airy functions and the forbidden region by matching to Eq. (26). As in the previous section, we either have four allowed regions or two depending on whether $\epsilon = 1$ is less than or greater than u. Just as in that case, we may write the boundary conditions at the outer boundaries as

$$\frac{R_+}{R_-}e^{+2i\Delta W_{\rm eff}} = +i, \quad \frac{L_+}{L_-}e^{-2i\Delta W_{\rm eff}} = -i, \tag{50}$$

with $\Delta W_{\rm eff}$ given in Eq. (49) and $z_i \to 0$ so the phase factors vanish at the inner boundary. However, because the expression for W_1 given in Eq. (16) diverges logarithmically at the critical level, it is necessary to slightly modify the definition of $\Delta W_{\rm eff}$, so the lower limit of the integral for W_1' is taken to be $z_i = z_{\rm cutoff}$ rather than zero to avoid the logarithmic divergence.

Expanding Eqs. (14) and (16) near $z = 0$ gives

$$1 - 2(W_0')^2 = 1 - \frac{1}{2}(2u-1)z^2, \quad W_1' = \frac{1-\lambda}{2|z|\sqrt{2u-1}}. \tag{51}$$

Using the definitions of χ, μ given in Eq. (24), these expressions can be integrated to give

$$W_{R(L)}(z) = \pm\frac{N\mu z^2}{4} \pm \chi \ln\left(\frac{|z|}{z_{\rm cutoff}}\right), \tag{52}$$

for $z > 0$ ($z < 0$), respectively. If we choose $z_{\rm cutoff} = 1/\sqrt{N\mu}$, the coefficients $R_\pm$, $L_\pm$ can then be matched to Eq. (26) for $z > 0$ ($z < 0$). This gives

$$R_+ = \alpha\exp\left(\frac{\chi\pi}{4} + i\frac{\pi}{8}\right) + \beta\exp\left(-\frac{3\chi\pi}{4} - i\frac{3\pi}{8}\right), \tag{53}$$

$$R_- = \frac{\beta\sqrt{2\pi}}{\Gamma\left(\frac{1}{2} - i\chi\right)}\exp\left(-\frac{\chi\pi}{4} + i\frac{\pi}{8}\right), \tag{54}$$

$$L_- = \beta\exp\left(\frac{\chi\pi}{4} + i\frac{\pi}{8}\right) + \alpha\exp\left(-\frac{3\chi\pi}{4} - i\frac{3\pi}{8}\right), \tag{55}$$

$$L_+ = \frac{\alpha\sqrt{2\pi}}{\Gamma\left(\frac{1}{2} - i\chi\right)}\exp\left(-\frac{\chi\pi}{4} + i\frac{\pi}{8}\right). \tag{56}$$

By making use of the Weierstrass identity [34] to write $\Gamma(\frac{1}{2} - i\chi)\Gamma(\frac{1}{2} + i\chi) = \pi/\cosh(\chi\pi)$, these expressions can be combined to give the quantization condition

$$m\pi = \mathrm{Arg}\left[\frac{\sqrt{2\pi}}{\Gamma(1/2 + i\chi)}e^{2i\Delta W_{\rm eff}} + e^{-\chi\pi/2}\right] - \frac{\pi}{2}. \tag{57}$$

IV. CONCLUSIONS AND SCALING OF QUANTUM BREAK TIME

We have shown how semiclassical quantization formulas emerge from a many-body WKB approach for the Josephson problem. We now use this to extract the quantum break time; this time is given by the inverse of the anharmonicity of the energy level spacing. At low and high energies, Bohr-Sommerfeld quantization applies with small $1/N$ quantum corrections. In between these two regimes, a critical level with large quantum corrections is seen, where larger deviations from semiclassics occur, leading to quantum break times that scale with the logarithm of the system size. From the quantization formulas in the various regimes, we can straightforwardly extract the anharmonicity of the spectrum. For perfectly regular spacing of energy levels, the dynamics would be periodic, matching the semiclassical dynamics. For the critical level, the energy appears in the quantization condition only through $\lambda = N(\epsilon - 1)$ as a prefactor appearing in W_1 and through $\chi = (1-\lambda)/2\mu$ in the γ function in Eq. (57). The former of these contributions can be written as

$$\frac{d}{d\lambda}\Delta W_{\rm eff} = -\int_{z_{\rm cutoff}=1/\sqrt{N\mu}}^{z_o=\mu/u}\frac{dz}{2z\sqrt{\mu^2 - u^2z^2}} \simeq -\frac{1}{2\mu}\ln\left(\frac{2\mu\sqrt{N\mu}}{u}\right). \tag{58}$$

This logarithmic dependence means that the level spacing near this critical level is $\delta\epsilon = \delta\lambda/N \simeq \pi/[Nd\Delta W_{\rm eff}/d\lambda] \sim 1/[N\ln(N)]$, by making use of Eq. (58). If this logarithm were large, then the solutions of Eq. (57) would require λ to be regularly spaced. However, because of the λ dependence in $\Gamma(1/2 + i\chi)$, there will be some anharmonicity. Since the only parameter controlling the anharmonicity is $d\Delta W_{\rm eff}/d\lambda$, the anharmonicity of level spacing $\delta\delta\epsilon = \epsilon_{n+1} + \epsilon_{n-1} - 2\epsilon_n$ is given by $\delta\delta\epsilon = \delta\delta\lambda/N \sim 1/[N\ln(N)^2]$. In contrast, away from the critical level, level spacing is $1/N$ and anharmonicity $1/N^2$. Thus, the quantum break time scales logarithmically with system size near the critical level and linearly elsewhere. This is exactly the conclusion found by considering next-to-leading-order corrections to the quantum dynamics via a cumulant expansion in Refs. [20,21] and consistent with Ref. [22].

In conclusion, we have shown how the WKB approach for many-body systems, discussed in [24,26], can be applied to the dynamics of the Josephson problem, illustrating that it can be applied to another paradigmatic problem of collective quantum dynamics. This many-body WKB approach is particularly useful in cases where critical energy levels exist. In such cases, the semiclassical description may be inadequate even for mesoscopic systems with up to $\sim 10^6$ particles, yet such numbers of particles make numerical approaches to the full quantum dynamics very expensive. As such, it provides an ideal tool to identify cases where quantum dynamical effects survive in mesocopic systems.

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