

# Rotating three-dimensional solitons in Bose-Einstein condensates with gravitylike attractive nonlocal interaction

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We study formation of rotating three-dimensional high-order solitons (azimuthons) in Bose-Einstein condensate with attractive nonlocal nonlinear interaction. In particular, we demonstrate formation of toroidal rotating solitons and investigate their stability. We show that variational methods allow a very good approximation of such solutions and predict accurately the soliton rotation frequency. We also find that these rotating localized structures are very robust and persist even if the initial condensate conditions are rather far from the exact soliton solutions. Furthermore, the presence of repulsive contact interaction does not prevent the existence of those solutions, but allows one to control their rotation. We conjecture that self-trapped azimuthons are generic for condensates with attractive nonlocal interaction.

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## I. INTRODUCTION

Studies of Bose-Einstein condensates (BEC) belong to one of the fastest developing research directions. The major theoretical progress in this area has been stimulated by the fast experimental advances that enable one to investigate subtle phenomena of fundamental nature [1,2]. In the semiclassical approach the spatial and temporal evolution of the condensates' wave function is commonly described by the Gross-Pitaevskii equation [3] which reflects the interplay between kinetic energy of the condensate and the nonlinearity originating from the interaction potential leading, among others, to the formation of localized structures, bright and dark solitons [4,5]. So far the main theoretical and experimental efforts have been concentrating on condensates with contact (or hard-sphere) bosonic interaction which, in case of attraction, may lead to collapse-like dynamics. Recently, also systems exhibiting a nonlocal, long-range dipolar interaction [6] have attracted a significant attention. This interest has been stimulated by successful condensation of chromium atoms which exhibit an appreciable magnetic dipole moment [7–9]. The presence of spatially nonlocal nonlinear interaction and, at the same time, the ability to control externally the character of local (contact) interactions via the Feshbach resonance techniques offer the unique opportunity to study the effect of nonlocality on the dynamics, stability, and interaction of bright- and dark-matter wave solitons [10–13]. The enhanced stability of localized structures including fundamental, vortex, and rotating solitons in nonlocal nonlinear media (not necessarily BEC) was already pointed out in a number of theoretical works [14–21]. In particular, stable toroidal solitons were presented in [20,21]. However, since the dipole-dipole interaction is spatially anisotropic, an additional trapping potential or a combination of attractive two-particle and repulsive three-particle interaction was necessary. Various trapping arrangements have been proposed to minimize or completely eliminate this anisotropy. In particular, O'Dell *et al.* [22] have suggested using a series of triads of orthogonally polarized laser beams illuminating a cloud of cold atoms along three orthogonal axes so that

the angular dependence of the dipole-dipole nonlinear term is averaged out. The resulting nonlocal interaction potential becomes effectively isotropic of the form  $1/r$ . It was already shown by Turitsyn [23] that such a purely attractive “gravitational” (or Coulomb) interaction potential prevents collapse of nonlinear localized waves and gives rise to the formation of localized states (bright solitons) which could be supported without necessity of using the external trapping potential. If realized experimentally such trapping geometry would enable to study effects akin to gravitational interaction. Few recent works have been dealing with this “gravitational” model of condensate looking, among others, at the stability of localized structures such as fundamental solitons and two-dimensional vortices [24–27].

In this paper we study formation of three-dimensional (3D) high-order solitons in BEC with gravity-like attractive nonlocal nonlinear potential. In particular, we demonstrate formation of toroidal solitons [28,29] and investigate their stability. We show that such BEC supports robust localized structures even if the initial conditions are rather far from the exact soliton solutions. This extraordinary robustness allows us to propose straightforward generation scenarios for rotating solitons. Furthermore, we demonstrate that the presence of repulsive contact interaction does not prevent the existence of those solutions, but allows one to control their rotation by changing the strength of the local repulsive contribution compared to the nonlocal attractive one.

The paper is organized as follows. In Sec. II we introduce briefly a scaled nonlocal Gross-Pitaevskii equation (GPE). In Sec. III we recall general properties of rotating soliton solutions (azimuthons), which are then approximated in Sec. IV by means of a variational approach. Those variational approximations allow us to predict the rotation frequency of the azimuthons which are then confronted with results from rigorous numerical simulations. Finally, self-trapped higher-order 3D rotating solitons are presented in Sec. V, and we show that such nonlocal BECs support robust localized structures.

## II. MODEL

We consider a Bose-Einstein atomic condensate with isotropic interatomic potential consisting of both repulsive contact as well as attractive long-range nonlocal interaction contributions. Following O'Dell *et al.* [22] the attractive long-range interaction which is electro-magnetically induced by the triads of frequency detuned laser beams with the intensity  $I$  can be presented in the “gravitational” form,

$$U_{Gr}(\tilde{\mathbf{r}}) = -\frac{11}{4\pi} \frac{Iq^2\alpha^2}{c\epsilon_0} \frac{1}{\tilde{r}} = -\frac{u}{\tilde{r}}. \quad (1)$$

Here,  $q$  is the modulus of the wave vector,  $\alpha$  the isotropic, dynamic polarizability of the atoms,  $c$  the light velocity, and  $\epsilon_0$  the permittivity of the free space. Then the complete two-body interaction potential is given by

$$V(\tilde{\mathbf{r}}) = \frac{4\pi a\hbar^2}{m} \delta(\tilde{\mathbf{r}}) - \frac{u}{\tilde{r}}, \quad (2)$$

where the first term comes from the contact s-wave scattering,  $a$  is the scattering length, and  $m$  is the atomic mass. The potential can only be written in this form if the mean kinetic energy per particle dominates the  $u/\tilde{r}$ -term, so that the short-range hard-sphere scattering is not affected. This is fulfilled if  $a_B \ll \lambda_B \ll a$ , where  $a_B := \frac{\hbar^2}{mu}$  is the Bohr radius, associated with the interaction, and  $\lambda_B$  is the de Broglie wavelength.

The temporal and spatial dynamics of the condensate wave function  $\tilde{\psi}(\tilde{\mathbf{r}}, \tilde{t})$  is then governed by the following Gross-Pitaevskii equation (GPE):

$$i\hbar\partial_{\tilde{t}}\tilde{\psi} + \frac{\hbar^2}{2m}\Delta_{\tilde{\mathbf{r}}}\tilde{\psi} + u \int \frac{|\tilde{\psi}(\tilde{\mathbf{r}}', \tilde{t})|^2}{|\tilde{\mathbf{r}} - \tilde{\mathbf{r}}'|} d^3\tilde{\mathbf{r}}'\tilde{\psi} - \frac{4\pi a\hbar^2}{m}|\tilde{\psi}|^2\tilde{\psi} = 0. \quad (3)$$

Characteristic values for the physical parameters are  $m = 3.8 \times 10^{-26}$  kg,  $a = 3$  nm, and  $u = 2 \cdot 10^{-13}$  eVnm, and  $\int |\tilde{\psi}|^2 d^3\tilde{\mathbf{r}} = N$  is the number of atoms in the condensate [22]. Using the normalization,

$$\mathbf{r} = \sqrt{\frac{um}{4\pi a\hbar^2}} \tilde{\mathbf{r}} = \frac{1}{R_c} \tilde{\mathbf{r}} \quad (4a)$$

$$t = \frac{\hbar}{2mR_c^2} \tilde{t} = \frac{1}{T_c} \tilde{t} \quad (4b)$$

$$\psi = \sqrt{\frac{R_c^3}{N}} \tilde{\psi}, \quad \int |\psi|^2 d^3\mathbf{r} = 1, \quad (4c)$$

one ends up with the dimensionless GPE,

$$\partial_t \psi = i\Delta\psi + i\rho\Theta\psi \quad (5a)$$

$$\Theta(\mathbf{r}, t) = \int \frac{|\psi(\mathbf{r}', t)|^2}{|\mathbf{r} - \mathbf{r}'|} d^3r' - |\psi|^2. \quad (5b)$$

The normalized nonlinearity  $\Theta$  consists of both local and nonlocal contributions, and  $\rho = 8\pi Na/R_c$  depends on all physical parameters of the system and can be tuned by (e.g., changing the number of atoms  $N$  in the condensate). Interestingly, the ratio between local and nonlocal term is solely determined by the form of the wave function  $\psi$ . We will see later (Sec. IV) that for very broad solitons the local contact interaction  $\sim |\psi|^2$  becomes negligible compared to

the nonlocal one. Then the governing equation of motion is formally equivalent to the so-called Schrödinger-Newton equation, proposed by Penrose in [30,31]. This equation was studied numerically in [32], where in particular “spinning solutions” were considered.

For the parameter values given above we find the characteristic length and timescales  $R_c = 19 \mu\text{m}$  and  $T_c = 0.25$  s, respectively. Then, a typical condensate of, say,  $N = 10\,000$  atoms gives  $\rho \approx 42$ . Hence, in order to claim potential experimental relevance of self-trapped solutions the demonstration of robust time evolution over normalized times  $t \gtrsim 1$  is necessary. Turitsyn [23] found that the ground state of the nonlocal Schrödinger equation with a purely attractive  $1/r$  kernel is stable with respect to collapse using Lyapunoff’s method. Moreover, in [33], linear and global (modulational) stability under small perturbations of solutions to the Hartree-equation was shown. Here, we are interested in the existence and robustness of rotating higher-order states, which we will investigate in the following by means of numerical simulations.

## III. ROTATING SOLITONS

It has been shown earlier that azimuthons (i.e., multi-peak solitons) with angular phase ramp exhibit constant angular rotation and hence can be represented by straightforward generalization of the usual (nonrotating) soliton ansatz by including an additional parameter, the rotation frequency  $\Omega$  [34,35]. We write

$$\psi(r, z, \phi, t) = U(r, z, \phi - \Omega t) e^{iEt}, \quad (6)$$

where  $U$  is the complex amplitude and  $E$  is the normalized chemical potential,  $r = \sqrt{x^2 + y^2}$ , and  $\phi$  denotes the azimuthal angle in the plane  $(x, y)$ . It can be shown that by inserting the above function into the nonlocal GPE (5) one can derive the formal relation for the rotation frequency [36,37],

$$\Omega = -\frac{IL - I'M + XL - X'M}{L^2 - MM'}, \quad (7)$$

where the functionals  $M, M', X, X', L, I, I'$  represent the integrals over the stationary amplitude profiles of the azimuthons given in Appendix A. The first two conserved functionals ( $M$ ) and ( $L$ ) are the norm of the wave function and “angular momentum,” respectively. In the next section, we will compute approximate azimuthon solutions and their rotation frequency employing a certain ansatz for the stationary amplitude profile  $U$ .

## IV. VARIATIONAL APPROACH

In order to get some insight into possible localized states of the Gross-Pitaevskii equation we resort first to the so-called Lagrangian (or variational) approach [38]. Equation (5) can be derived from the following Lagrangian density:

$$\mathcal{L} := \frac{i}{2}(\psi\partial_t\psi^* - \psi^*\partial_t\psi) + |\nabla\psi|^2 - \frac{1}{2}\rho|\psi|^2\Theta(\mathbf{r}, t). \quad (8)$$

It has been shown before that rotating solitons or “azimuthons” are associated with nontrivial phase and amplitude structure [39]. In two-dimensional optical problems the simplest case represents the state falling between optical vortex

(ring-like pattern with  $2\pi$  angular phase shift) and optical dipole in the form of two out-of-phase intensity peaks [39,40]. In three dimensions, a reasonable ansatz for corresponding localized solutions is

$$\psi(r,z,\varphi,t) := Ar \exp\left(-\frac{r^2+z^2}{2\sigma^2}\right) e^{iEt} \times [\cos(\varphi - \Omega t) + ip \sin(\varphi - \Omega t)], \quad (9)$$

where parameter  $p$  varies between zero and unity. For  $p = 0$ , Eq. (9) describes a dipole structure consisting of two out-of-phase lobes, while for  $p = 1$  it is a 3D vortex [i.e., toroid-like structure with zero in the center and azimuthal (in the  $(x,y)$  plane)] phase ramp of  $2\pi$ . Using the ansatz Eq. (9), one can easily find that

$$IL - I'M = 0, \quad (10)$$

which shows that only the nonlinear terms contribute to the frequency  $\Omega$  [vide formula Eq. (7)]. Without loss of generality, we restrict our analysis to  $0 \leq p \leq 1$ ; the results for the opposite angular phase shift ( $-1 \leq p \leq 0$ ) can be obtained by mirroring the coordinate system.

After inserting the solution Eq. (9) into the Lagrangian density  $\mathcal{L}$ , and integrating over the whole 3D space we obtain the Lagrangian  $L$  which is the function of variational parameters  $\sigma$  and  $A$  only. Looking for the extrema of  $L$  leads to a set of algebraic relations among the variational variables (see also Appendix B). Using  $\int |\psi|^2 d^3\mathbf{r} = 1$  we can eliminate the amplitude  $A$ , and the width  $\sigma$  of the azimuthon is given by

$$\sigma = \frac{300\sqrt{2\pi}C_3 + 3\sqrt{20000\pi^2C_3^2 + 5\rho^2C_1C_2}}{2\sqrt{\pi}\rho C_1}, \quad (11)$$

with  $C_1 = 49p^4 + 86p^2 + 49$ ,  $C_2 = 3p^4 + 2p^2 + 3$ , and  $C_3 = (1 + p^2)^2$ . Obviously, the width  $\sigma$  is minimal but nonzero for  $\rho \rightarrow \infty$ . Hence, the localized solution exists only if its width is greater than the critical value  $\sigma_{\text{cr}}(p)$ ,

$$\sigma_{\text{cr}} = \sqrt{\frac{45C_2}{4\pi C_1}} = \frac{3}{2}\sqrt{\frac{5}{\pi}}\sqrt{\frac{3p^4 + 2p^2 + 3}{49p^4 + 86p^2 + 49}}. \quad (12)$$

This threshold is a direct consequence of competition between nonlocal and local interaction potentials. While the former is attractive and thus leads to spatial localization, the latter, which is repulsive, tends to counteract it. For small  $\sigma$  the kinetic energy term is large and can be compensated only if the particle density is high enough. Further on, the chemical potential  $E$  is given by

$$E = \frac{15(4\pi\sigma^2C_1 - 5C_2)}{2\sigma^2(4\pi\sigma^2C_1 - 45C_2)}, \quad (13)$$

and according to Eq. (7) the rotation frequency  $\Omega$  reads

$$\Omega = \frac{60p\sqrt{C_3}(4\sigma^2\pi - 5)}{\sigma^2(4\pi\sigma^2C_1 - 45C_2)}. \quad (14)$$

Interestingly, the latter expression is not sign definite, which means that we can expect both co- and counter-rotating azimuthons with respect to the phase. In particular, the azimuthon with the ‘‘stationary’’ width  $\sigma_s = \sqrt{5/4\pi} \approx 0.63$  should have no angular velocity. Again, this effect is due to competition between nonlocal and local contribution to  $\Omega$  for small  $\sigma$ . The

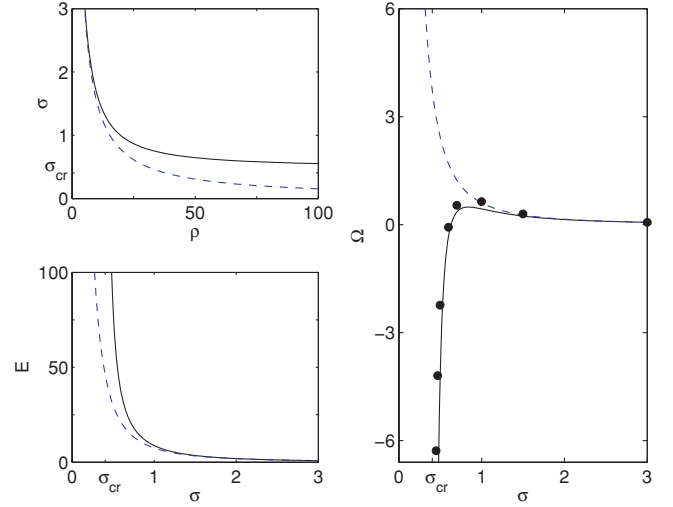


FIG. 1. (Color online) The left panels show the dependencies of the azimuthon width  $\sigma$  on the parameter  $\rho = 8\pi Na/R_c$  (top) and the chemical potential  $E$  on the width  $\sigma$  (bottom). Black curves show results from the variational approach; dashed blue curves represent the asymptotic limits for  $\sigma \rightarrow \infty$  (without repulsive contact interaction). The right panel shows the rotation frequency  $\Omega$  as a function of the width  $\sigma$ . Black dots denote results obtained from numerical simulations of the GPE (5). All plots are for  $p = 0.7$ .

nonlocal attractive interaction leads to a positive contribution to  $\Omega$ , the repulsive local interaction to a negative one. The formal expression for  $\Omega_\infty$  without repulsion can be obtained by the outlined variational procedure or by asymptotic expansion ( $\sigma \rightarrow \infty$ ) up to  $O(1/\sigma^2)$ ,  $\Omega_\infty = 60p\sqrt{C_3}/C_1\sigma^2$ . As also expected from simple scaling analysis, this quantity is sign definite with respect to  $\sigma$ .

The solid black curves in Fig. 1 summarize our findings from the variational approach. For each value of the parameter  $\rho = 8\pi Na/R_c$  and  $0 \leq p \leq 1$ , we find exactly one azimuthon with width  $\sigma(\rho, p)$ , chemical potential  $E(\sigma(\rho, p))$ , and rotation frequency  $\Omega(\sigma(\rho, p))$ . The dashed blue curves in Fig. 1 depict the asymptotic behaviors for large width  $\sigma \rightarrow \infty$ , when repulsive contact interaction becomes negligible. The most pronounced effect occurs in the angular velocity, shown in the right panel in Fig. 1. We can see that the repulsive local interaction kicks in for  $\sigma \lesssim 1.5$ , thus for  $\rho \gtrsim 12$ . Moreover, for large width  $\sigma$  one finds  $E \sim 1/\sigma^2$  and  $\sigma \sim 1/\rho$ . The fact that we find a localized solution for any nonzero value of  $\rho$  is a well-known property of very long-range kernels, such as the Coulomb potential in three dimensions [33].

## V. NUMERICAL RESULTS

In this section, the predictions of the variational approach will be confronted with direct numerical simulations. The approximate solitons resulting from the variational approach will be used as initial conditions to our 3D simulation code to compute their time evolution. In general, we find stable evolution; in particular, the characteristic shape of the initial conditions is preserved. For rotating azimuthons, the angular velocities will be measured and compared to the ones obtained in the previous section.

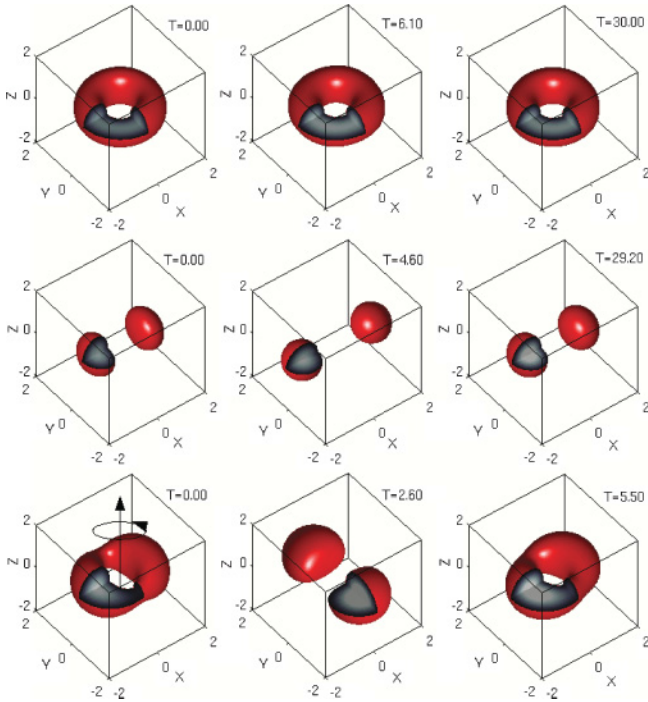


FIG. 2. (Color online) Dynamics of the 3D stable solitons in gravity-like BEC. Iso-surfaces of the normalized density  $|\psi|^2$  are depicted for different evolution times; the interior density distribution is represented in grayscales. The initial variational parameters used are  $\sigma = 1$  and  $p = 1$  (torus,  $\rho = 19.4$ ; iso-density surface at  $|\psi|^2 = 0.039$ ) for the upper row;  $p = 0$  (dipole,  $\rho = 19.7$ ; iso-density surface at  $|\psi|^2 = 0.071$ ) for the middle one; and finally,  $p = 0.7$  (azimuthon,  $\rho = 19.4$ ; iso-density surface at  $|\psi|^2 = 0.044$ ). The sense of the rotation ( $\Omega = 0.64$ ) is indicated by the arrow.

In Fig. 2 we illustrate the temporal evolution of three-dimensional solitons for the “gravitational” potential [i.e., solutions to Eq. (5)]. This first two rows present the classical stationary soliton solutions torus and dipole, respectively. Due to imperfections of the initial conditions obtained from the variational approach we observe slight oscillations upon evolution, in particular, for the dipole solutions (second row). Those oscillations are not present if we use numerically *exact* solutions (obtained from an iterative solver [41]) as initial conditions (not shown). In the last row of Fig. 2, we show the evolution of an azimuthon ( $p = 0.7, \sigma = 1$ ). The rotation of the amplitude profile is clearly visible. Again we observe radial oscillations due to the imperfect initial condition, but the solution is robust.

Figure 3 shows the dependence of the azimuthon rotation frequency as a function of the modulation parameter  $p$ . Solid lines represent predictions from the variational model; black dots represent rotation frequency obtained from numerical simulations. As expected from two-dimensional nonlocal models [37,42], the modulus of  $\Omega$  increases with  $p$ . Our variational calculations predict that for small width  $\sigma$ , when repulsive interaction comes into play, the sense of azimuthon rotation changes. In particular, we found a “stationary” width  $\sigma_s$  where the rotation frequency  $\Omega$  vanishes. Indeed, full model simulations confirm this property, since the first row in Figs. 4 and 3(a) show a very slow rotation with opposite orientation, so

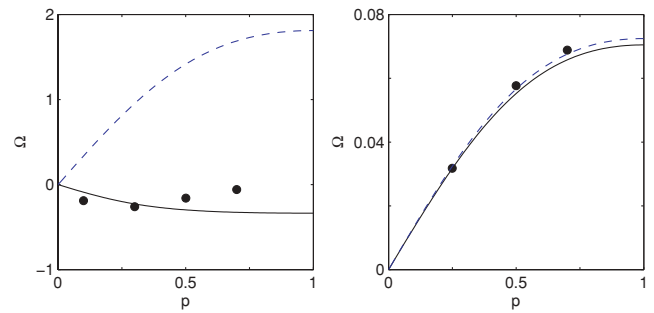


FIG. 3. (Color online) Azimuthon rotation frequency  $\Omega$  versus modulation parameter  $p$  in gravity-like BEC for  $\sigma = 0.6 \approx \sigma_s$  ( $\rho = 48.0 \dots 65.4$ , left panel) and  $\sigma = 3$  ( $\rho = 5.2 \dots 5.5$ , right panel). Black curves show results from the variational approach; dashed blue curves represent the asymptotic limits for  $\sigma \rightarrow \infty$  (without repulsive contact interaction). Black dots denote results obtained from numerical simulations of the GPE (5).

that the numerical stationary width is between 0.6 and 0.61. Furthermore, we observe that very narrow azimuthons ( $\sigma \rightarrow \sigma_{cr}$ ) have negative  $\Omega$  and rotate very fast (see Fig. 1). This may be interesting for potential realizations, since the duration of BEC experiments is restricted to typically several hundreds of milliseconds. However, for azimuthons very close to  $\sigma_{cr}$  the ansatz function (9) becomes less appropriate and using variational initial conditions leads to very strong oscillations upon evolution, up to the point where it is no longer possible to identify properly the rotation frequency  $\Omega$ .

Finally, we will discuss possible strategies to actually generate azimuthons from more simple initial configurations of the condensate. It turns out that apart from their extraordinary robustness, azimuthons are also strong nonlinear attractors in parameter space, which makes their excitation feasible. In general, 3D tori or vortex rings could be generated by either imprinting the vortex phase distribution onto a condensate [43,44], or by stirring the condensate with an external localized laser beam [45,46]. Both of these methods have been shown to excite either single or multiple vortices in a condensate.

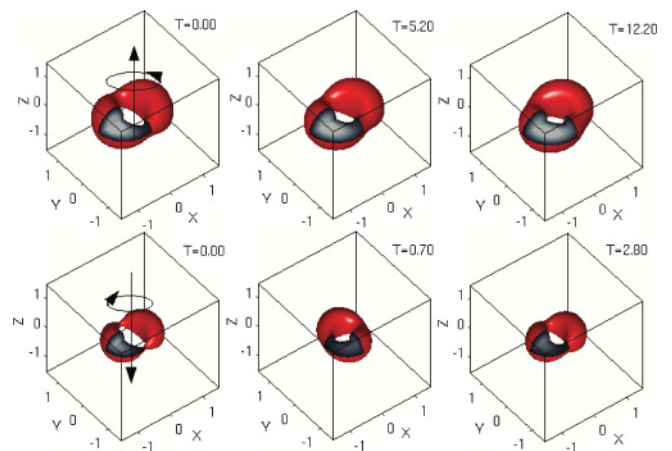


FIG. 4. (Color online) The upper row shows iso-density surfaces at  $|\psi|^2 = 0.16$  for the very slow rotating ( $\Omega \approx 0$ ) azimuthon with  $p = 0.7$  and  $\sigma = 0.61$  ( $\rho = 47.5$ ). The lower row shows a fast counter-rotating ( $\Omega = -2.2$ ) azimuthon with  $p = 0.7$ ,  $\sigma = 0.5$  ( $\rho = 93.9$ ), and iso-density surface at  $|\psi|^2 = 0.34$ . Same plot style as in Fig. 2.

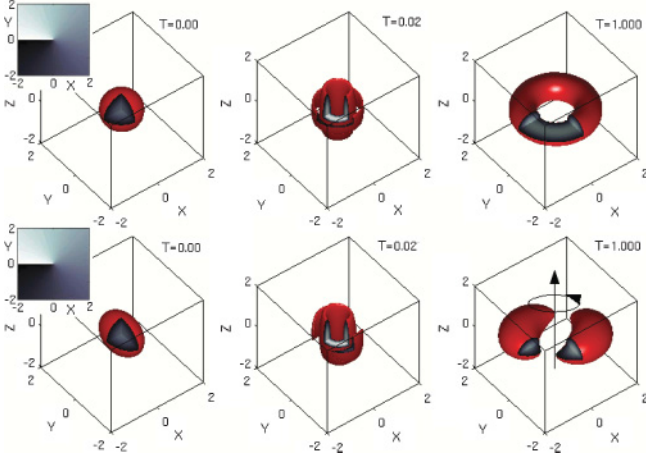


FIG. 5. (Color online) The upper row shows iso-density surfaces at  $|\psi|^2 = 0.040$  of the time evolution of a sphere ( $\rho = 18.8$ ). The imprinted vortex phase is shown in the inset; a stable torus is formed. The lower row shows the formation of a rotating dipole azimuthon by using initially an ellipsoid ( $\rho = 19.6$ ; iso-density surface at  $|\psi|^2 = 0.039$ ). Same plot style as in Fig. 2.

Here, we will focus on the former method of phase imprint. To this end, we assume a condensate initially trapped in a spherical or ellipsoidal harmonic trap. Then, we switch off the trap, switch on the nonlocal attractive interaction, and imprint a vortex phase at the same time. Figure 5 shows snapshots of the resulting time evolutions. The spherical condensate (first row) develops immediately a minimum with zero density at center and radial modulations, which disappear and a toroidal structure close to the soliton is formed at  $t \approx 0.1$  (not shown). The subsequent evolution is an oscillation around the stable stationary state. When we use an initially ellipsoidal condensate and imprint the vortex phase, a similar dynamics follows, but due to the spatial asymmetry a dipole azimuthon is formed at  $t \approx 0.1$ , which then starts to rotate (see lower row in Fig. 5).

## VI. CONCLUSION

We have studied the formation of rotating localized structures in Bose Einstein condensate with gravity-like attractive nonlocal interaction. We successfully used variational techniques to investigate their dynamics and showed numerically that such localized structures are indeed robust objects which persist over long evolution times even if the initial conditions significantly differ from the exact soliton solutions. It turned out that it is possible to create such higher-order structures from very basic trapped condensates via phase imprint.

For rotating solitons (azimuthons), we derived analytical expressions for the angular velocity, in excellent agreement with rigorous 3D numerical simulations. Furthermore, we show that properties of the azimuthons depend on a single parameter  $\rho \sim N\sqrt{a/mu}$  only. This makes it possible to control the rotation frequency by, for example, tuning the local contact interaction ( $\sim a$ ), which is routinely done by Feshbach resonance techniques. In particular, we can change the sense of rotation with respect to the phase, and we can find nonrotating azimuthons. We also identify parameter regions

with particularly fast rotation, which may be important for potential experimental observation of such solutions.

Last but not least, by using different nonlocal kernel functions we checked that rotating soliton solutions are generic structures in nonlocal GPEs. Hence, we conjecture that the phenomena observed in this paper are rather universal and apply for a general class of attractive nonlocal interaction potentials.

## ACKNOWLEDGMENTS

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## APPENDIX A: FUNCTIONALS

The functionals  $M, M', X, X', L, I, I'$  in Eq. (7) represent the following integrals over the stationary amplitude profiles of the azimuthons:

$$M = \int |U|^2 d^3\mathbf{r} = 1, \quad (\text{A1a})$$

$$L = -i \int U^* \partial_\phi U d^3\mathbf{r}, \quad (\text{A1b})$$

$$I = \int U^* \Delta U d^3\mathbf{r}, \quad (\text{A1c})$$

$$X = \rho \int \Theta(\mathbf{r}) |U(\mathbf{r})|^2 d^3\mathbf{r}, \quad (\text{A1d})$$

$$M' = \int |\partial_\phi U|^2 d^3\mathbf{r}, \quad (\text{A1e})$$

$$I' = i \int \partial_\phi U^* \Delta U d^3\mathbf{r}, \quad (\text{A1f})$$

$$X' = i\rho \int \Theta(\mathbf{r}) U(\partial_\phi U^*) d^3\mathbf{r}. \quad (\text{A1g})$$

## APPENDIX B: CONVOLUTION

The convolution term in the Lagrangian (8) can be calculated analytically by, for instance, rewriting the integrand in terms of spherical harmonics  $Y_{ij}$ . Then,  $|\psi(\mathbf{r})|^2 = \sum y_{ij} Y_{ij}$ , where  $y_{ij}$  denote the coefficients of the spherical harmonics, and the convolution integral can be easily calculated leading to the following result:

$$\begin{aligned} & \frac{1}{2} \int |\psi(\mathbf{r})|^2 \int \frac{|\psi(\mathbf{r}')|^2}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' d^3\mathbf{r} \\ &= \frac{1}{2} \int_0^\infty \int_0^r 4\pi r A^4 r'^2 \exp\left(-\frac{r'^2 + r^2}{\sigma^2}\right) \\ & \quad \times \left[ y_{00}^2 + \frac{1}{5} y_{20}^2 \left(\frac{r'}{r}\right)^2 + \frac{2}{5} y_{2\pm 2}^2 \left(\frac{r'}{r}\right)^2 \right] r'^2 dr' r^2 dr \\ &+ \frac{1}{2} \int_0^\infty \int_r^\infty 4\pi r' A^4 r^2 \exp\left(-\frac{r'^2 + r^2}{\sigma^2}\right) \\ & \quad \times \left[ y_{00}^2 + \frac{1}{5} y_{20}^2 \left(\frac{r}{r'}\right)^2 + \frac{2}{5} y_{2\pm 2}^2 \left(\frac{r}{r'}\right)^2 \right] r'^2 dr' r^2 dr \\ &= \frac{49 + 86p^2 + 49p^4}{240} \sigma^9 \pi^{\frac{5}{2}} A^4 \frac{1}{\sqrt{2}}. \end{aligned}$$

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