Tsallis entropy and entanglement constraints in multiqubit systems

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We show that the restricted shareability and distribution of multiqubit entanglement can be characterized by Tsallis-q entropy. We first provide a class of bipartite entanglement measures named Tsallis-q entanglement, and provide its analytic formula in two-qubit systems for $1 \leqslant q \leqslant 4$. For $2 \leqslant q \leqslant 3$, we show a monogamy inequality of multiqubit entanglement in terms of Tsallis-q entanglement, and we also provide a polygamy inequality using Tsallis-q entropy for $1 \leqslant q \leqslant 2$ and $3 \leqslant q \leqslant 4$.

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I. INTRODUCTION

Whereas classical correlations can be freely shared among parties in multiparty systems, quantum correlation, especially quantum entanglement, is known to have some restriction in its shareability and distribution. For example, in a tripartite system consisting of parties A, B, and C, let us assume A is maximally entangled with both B and C simultaneously. Because maximal entanglement can be used to teleport an arbitrary unknown quantum state [1], A can teleport an unknown state ρ to B and C by using the simultaneous maximal entanglement. Now, each B and C has an identical copy of ρ, and this means cloning an unknown state ρ, which is impossible by the *no-cloning* theorem [2]. In other words, the assumption of simultaneous maximal entanglement of A with B and C is quantum mechanically forbidden.

This restricted shareability of quantum entanglement is known as the *Monogamy of Entanglement* (MoE) [3], and it was also shown to play an important role in many applications of quantum information processing. For instance, in quantum cryptography, MoE can be used to restrict the possible correlation between authorized users and the eavesdropper, which is the basic concept of the security proof [4].

For three-qubit systems, MoE was first characterized in forms of a mathematical inequality using *concurrence* [5] as the bipartite entanglement measure. This characterization is known as the *CKW inequality* named after its establishers, Coffman, Kundu, and Wootters [6], and it was also generalized for multiqubit systems later [7].

MoE in multiqubit systems is mathematically well characterized in terms of concurrence, it is, however, also known that the CKW-type characterization for MoE is not generally true for other entanglement measures such as *Entanglement of Formation* (EoF) [8]: Even in multiqubit systems, there exists a counterexample that violates the CKW-type inequality in terms of EoF.

As bipartite entanglement measures, both concurrence and EoF of a bipartite pure state $|\psi\rangle_{AB}$ quantify the uncertainty of the subsystem $\rho_A = \mathrm{tr}_B|\psi\rangle_{AB}\langle\psi|$. For the case when $|\psi\rangle_{AB}$ is a two-qubit state, the uncertainty of ρ_A is completely determined by a single parameter. Furthermore, the extension of concurrence and that of Eof for a mixed state ρ_{AB} are based on the same method of *convex-roof extension*, which minimizes the average entanglement over all possible pure-state decompositions of ρ_{AB}. In other words, concurrence and EoF for two-qubit states are essentially equivalent based on the same concept, the uncertainty of the subsystem. Moreover, it was also shown that these two measures are related by a monotone-increasing convex function [5].

However, these two equivalent measures for two-qubit systems show very different properties in multipartite systems in characterizing MoE, and this exposes the importance of having proper entanglement measures to characterize MoE even in multiqubit systems. Moreover, for the study of general MoE in multipartite higher-dimensional quantum systems, having a proper bipartite entanglement measure is one of the most important and necessary things that must precede.

As generalizations of the von Neumann entropy, there are two representative classes of entropies quantifying the uncertainty of quantum systems: One is quantum Rényi entropy [9,10], and the other is quantum Tsallis entropy [11,12]. Although both of them are one-parameter classes by a nonnegative real parameter q having von Neumann entropy as a special case when $q \to 1$, the Tsallis entropy, however, shows quite distinct properties from the Rényi entropy. As a function of probability distributions, the Tsallis entropy is concave for all $q > 0$ whereas the Rényi entropy is concave only for $0 < q \leqslant 1$. This concavity of the Tsallis entropy plays an important role in quantifying quantum entanglement because it assures the property of *entanglement monotone* [13]. In other words, the concavity of the Tsallis entropy assures the construction of an entanglement measure based on the Tsallis-q entropy for all $q > 0$, which does not increase under *local operations and classical communication* (LOCC). Furthermore, it is also known that the Tsallis entropy is Lesche-stable for all $q > 0$, whereas the Rényi entropy is not [14].

The Tsallis entropy has been widely used in many areas of quantum information theory such as the conditions for separability of quantum states [15–17] and the characterization of classical statistical correlations inherent in quantum states [18]. There are also discussions about using the nonextensive statistical mechanics to describe quantum entanglement [19].

For the characterization of MoE in multipartite quantum systems, it was recently shown that the Rényi entropy can be used for the CKW-type characterization of multiqubit monogamy. Using a bipartite entanglement measure based on the Rényi-q entropy, a CKW-type monogamy inequality was proposed for all $q \geqslant 2$ [20].

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Here, we show that the Tsallis entropy can characterize MoE in multiqubit systems for a selective choice of the parameter q. Using quantum Tsallis entropy of order q (or Tsallis-q entropy), we first provide a one-parameter class of bipartite entanglement measures, *Tsallis-q entanglement*, and provide its analytic formula for arbitrary two-qubit states when $1 \leqslant q \leqslant 4$. This class contains EoF as a special case when $q \to 1$. Furthermore, we show the monogamy inequality of multiqubit systems in terms of the Tsallis-q entanglement for $2 \leqslant q \leqslant 3$. For $1 \leqslant q \leqslant 2$ or $3 \leqslant q \leqslant 4$, we also provide a polygamy inequality of multiqubit entanglement using the Tsallis-q entropy.

This paper is organized as follows. In Sec. II A, we recall the definition of Tsallis-q entropy, and define Tsallis-q entanglement and its dual quantity for bipartite quantum states. In Sec. II B, we provide an analytic formula of Tsallis-q entanglement for arbitrary two-qubit states when $1 \leqslant q \leqslant 4$. In Sec. III, we derive a monogamy inequality of multiqubit entanglement in terms of Tsallis-q entanglement for $2 \leqslant q \leqslant 3$. We also provide a polygamy inequality of multiqubit entanglement for $1 \leqslant q \leqslant 2$ or $3 \leqslant q \leqslant 4$. Finally, we summarize our results in Sec. IV.

II. TSALLIS-q ENTANGLEMENT

A. Definition

For any quantum state ρ, its Tsallis-q entropy is defined as

$$T_q(\rho) = \frac{1}{q-1}(1 - \mathrm{tr}\rho^q), \tag{1}$$

for any $q > 0$ and $q \neq 1$. For the case when α tends to 1, $T_q(\rho)$ converges to the von Neumann entropy, that is,

$$\lim_{q\to 1} T_q(\rho) = -\mathrm{tr}\rho \log \rho = S(\rho). \tag{2}$$

In other words, Tsallis-q entropy has a singularity at $q = 1$, and it can be replaced by the von Neumann entropy. Throughout this paper we will just consider $T_1(\rho) = S(\rho)$ for any quantum state ρ.

For a bipartite pure state $|\psi\rangle_{AB}$ and each $q > 0$, Tsallis-q entanglement is

$$\mathcal{T}_q(|\psi\rangle_{AB}) := T_q(\rho_A), \tag{3}$$

where $\rho_A = \mathrm{tr}_B|\psi\rangle_{AB}\langle\psi|$ is the reduced density matrix onto subsystem A. For a mixed state ρ_{AB}, we define its Tsallis-q entanglement via convex-roof extension, that is,

$$\mathcal{T}_q(\rho_{AB}) := \min \sum_i p_i \mathcal{T}_q(|\psi_i\rangle_{AB}), \tag{4}$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i|\psi_i\rangle_{AB}\langle\psi_i|$.

As a dual quantity to Tsallis-q entanglement, we also define *Tsallis-q entanglement of Assistance* (TEoA) as

$$\mathcal{T}_q^a(\rho_{AB}) := \max \sum_i p_i \mathcal{T}_q(|\psi_i\rangle_{AB}), \tag{5}$$

where the maximum is taken over all possible pure state decompositions of ρ_{AB}.

Because Tsallis-q entropy converges to von Neumann entropy when q tends to 1, we have

$$\lim_{q\to 1} \mathcal{T}_q(\rho_{AB}) = E_{\mathrm{f}}(\rho_{AB}), \tag{6}$$

where $E_{\mathrm{f}}(\rho_{AB})$ is the EoF of ρ_{AB} defined as [8]

$$E_{\mathrm{f}}(\rho_{AB}) = \min \sum_i p_i S(\rho_A^i). \tag{7}$$

Here, the minimization is taken over all possible pure state decompositions of ρ_{AB}, such that,

$$\rho_{AB} = \sum_i p_i |\phi^i\rangle_{AB}\langle\phi^i|, \tag{8}$$

with $\mathrm{tr}_B|\phi^i\rangle_{AB}\langle\phi^i| = \rho_A^i$. In other words, Tsallis-q entanglement is one-parameter generalization of EoF, and the singularity of $\mathcal{T}_q(\rho_{AB})$ at $q = 1$ can be replaced by $E_{\mathrm{f}}(\rho_{AB})$.

Similarly, we have

$$\lim_{q\to 1} \mathcal{T}_q^a(\rho_{AB}) = E^a(\rho_{AB}), \tag{9}$$

where $E^a(\rho_{AB})$ is the *Entanglement of Assistance* (EoA) of ρ_{AB} defined as [21]

$$E^a(\rho_{AB}) = \max \sum_i p_i S(\rho_A^i). \tag{10}$$

Here, the maximum is taken over all possible pure state decompositions of ρ_{AB}, such that,

$$\rho_{AB} = \sum_i p_i |\phi^i\rangle_{AB}\langle\phi^i|, \tag{11}$$

with $\mathrm{tr}_B|\phi^i\rangle_{AB}\langle\phi^i| = \rho_A^i$.

B. Analytic formula for two-qubit states

Before we provide an analytic formula for Tsallis-q entanglement in two-qubit systems, let us first recall the definition of concurrence and its functional relation with EoF in two-qubit systems.

For any bipartite pure state $|\psi\rangle_{AB}$, its concurrence, $\mathcal{C}(|\psi\rangle_{AB})$ is defined as [5]

$$\mathcal{C}(|\psi\rangle_{AB}) = \sqrt{2(1 - \mathrm{tr}\rho_A^2)}, \tag{12}$$

where $\rho_A = \mathrm{tr}_B(|\psi\rangle_{AB}\langle\psi|)$. For a mixed state ρ_{AB}, its concurrence is defined as

$$\mathcal{C}(\rho_{AB}) = \min \sum_k p_k \mathcal{C}(|\psi_k\rangle_{AB}), \tag{13}$$

where the minimum is taken over all possible pure state decompositions, $\rho_{AB} = \sum_k p_k|\psi_k\rangle_{AB}\langle\psi_k|$.

For two-qubit systems, concurrence is known to have an analytic formula [5]; for any two-qubit state ρ_{AB},

$$\mathcal{C}(\rho_{AB}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{14}$$

where λ_i's are the eigenvalues, in decreasing order, of $\sqrt{\sqrt{\rho_{AB}}\tilde{\rho}_{AB}\sqrt{\rho_{AB}}}$ and $\tilde{\rho}_{AB} = \sigma_y \otimes \sigma_y \rho_{AB}^* \sigma_y \otimes \sigma_y$ with the Pauli operator σ_y. Furthermore, the relation between concurrence and EoF of a two-qubit mixed state ρ_{AB} (or a pure state $|\psi\rangle_{AB} \in \mathbb{C}^2 \otimes \mathbb{C}^d$, $d \geqslant 2$), can be given as a monotone increasing, convex function [5], such that

$$E_{\rm f}(\rho_{AB}) = \mathcal{E}(\mathcal{C}(\rho_{AB})), \tag{15}$$

where

$$\mathcal{E}(x) = H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1-x^2}\right), \quad \text{for} \quad 0 \leqslant x \leqslant 1, \tag{16}$$

with the binary entropy function $H(t) = -[t\log t + (1-t)\log(1-t)]$. In other words, the analytic formula of concurrence as well as its functional relation with EoF lead us to an analytic formula for EoF in two-qubit systems.

For any $2 \otimes d$ pure state $|\psi\rangle_{AB}$ (especially a two-qubit pure state) with its Schmidt decomposition $|\psi\rangle_{AB} = \sqrt{\lambda_0}|00\rangle_{AB} + \sqrt{\lambda_1}|11\rangle_{AB}$, its Tsallis-$q$ entanglement is

$$\mathcal{T}_q(|\psi\rangle_{AB}) = T_q(\rho_A) = \frac{1}{q-1}\left(1 - \lambda_0^q - \lambda_1^q\right). \tag{17}$$

Because the concurrence of $|\psi\rangle_{AB}$ is

$$\mathcal{C}(|\psi\rangle_{AB}) = \sqrt{2\left(1 - \mathrm{tr}\rho_A^2\right)} = \sqrt{\lambda_0\lambda_1}, \tag{18}$$

it can be easily verified that

$$\mathcal{T}_q(|\psi\rangle_{AB}) = g_q(\mathcal{C}(|\psi\rangle_{AB})), \tag{19}$$

where $g_q(x)$ is an analytic function defined as

$$g_q(x) := \frac{1}{q-1}\left[1 - \left(\frac{1+\sqrt{1-x^2}}{2}\right)^q - \left(\frac{1-\sqrt{1-x^2}}{2}\right)^q\right], \tag{20}$$

on $0 \leqslant x \leqslant 1$. In other words, for any $2 \otimes d$ pure state $|\psi\rangle_{AB}$, we have a functional relation between its concurrence and Tsallis-q entanglement for each $q > 0$. Note that $g_q(x)$ converges to the function $\mathcal{E}(x)$ in Eq. (16) for the case when q tends to 1.

It was shown that there exists an optimal decomposition for the concurrence of a two-qubit mixed state such that every pure state concurrence in the decomposition has the same value [5]: For any two-qubit state ρ_{AB}, there exists a pure state decomposition $\rho_{AB} = \sum_i p_i |\phi_i\rangle_{AB}\langle\phi_i|$ such that

$$\mathcal{C}(\rho_{AB}) = \sum_i p_i \mathcal{C}(|\phi_i\rangle_{AB}), \tag{21}$$

and

$$\mathcal{C}(|\phi_i\rangle_{AB}) = \mathcal{C}(\rho_{AB}), \tag{22}$$

for each i. Based on this, one possible sufficient condition for the relation in Eq. (19) to be also true for two-qubit mixed states is that the function $g_q(x)$ is monotonically increasing and convex [22]. In other words, we have

$$\mathcal{T}_q(\rho_{AB}) = g_q(\mathcal{C}(\rho_{AB})), \tag{23}$$

for any two-qubit mixed state ρ_{AB} provided that $g_q(x)$ is monotonically increasing and convex. Moreover, for the range of q where $g_q(x)$ is monotonically increasing and convex, Eq. (23) also implies an analytic formula of Tsallis-q entanglement for any two-qubit state.

Now, let us consider the monotonicity and convexity of $g_q(x)$ in Eq. (20). Because $g_q(x)$ is an analytic function on $0 \leqslant x \leqslant 1$, its monotonicity and convexity follow from the nonnegativity of its first and second derivatives.

By taking the first derivative of $g_q(x)$, we have

$$\frac{dg_q(x)}{dx} = \frac{qx[(1+\sqrt{1-x^2})^{q-1} - (1-\sqrt{1-x^2})^{q-1}]}{2^q(q-1)\sqrt{1-x^2}}, \tag{24}$$

which is always nonnegative on $0 \leqslant x \leqslant 1$ for $q > 0$. It is also direct to check that Eq. (24) is strictly positive for $0 < x < 1$. In other words, $g_q(x)$ is a strictly monotone-increasing function for any $q > 0$.

For the second derivative of $g_q(x)$, we have

$$\begin{aligned}\frac{d^2g_q(x)}{dx^2} &= \alpha\Bigg[\frac{(1+\sqrt{1-x^2})^{q-2}}{1-x^2}\left(\frac{1+\sqrt{1-x^2}}{\sqrt{1-x^2}} - x^2(q-1)\right)\\ &\quad - \frac{(1-\sqrt{1-x^2})^{q-2}}{1-x^2}\left(\frac{1-\sqrt{1-x^2}}{\sqrt{1-x^2}} + x^2(q-1)\right)\Bigg],\end{aligned} \tag{25}$$

where $\alpha = \frac{q}{2^q(q-1)}$. Here, we first prove that $g_q(x)$ is not convex for $q \geqslant 5$ by showing the existence of x_0 between 0 and 1 such that $\frac{d^2g_q(x_0)}{dx^2}$ is negative. To see this, first note that the second term of the right-hand side in Eq. (25) is always negative for $0 < x < 1$ if $q > 1$. Thus, it suffices to show that the first term of the right-hand side in Eq. (25) is nonpositive at $x_0 \in (0,1)$ for $q \geqslant 5$. Furthermore, the only factor of the first term that can be negative is

$$\left(\frac{1+\sqrt{1-x^2}}{\sqrt{1-x^2}} - x^2(q-1)\right), \tag{26}$$

since both α and $\frac{(1+\sqrt{1-x^2})^{q-2}}{1-x^2}$ are always positive at $x \in (0,1)$ if $q > 1$. By defining a function such that

$$h(x) = \frac{1-\sqrt{1-x^2}}{x^2\sqrt{1-x^2}} + 1, \tag{27}$$

the nonpositivity of Eq. (26) is equivalent to

$$q \geqslant h(x). \tag{28}$$

Since $h(x)$ is an analytic function on $0 < x < 1$, it is direct to verify that it has a critical point at $x_0 = \frac{\sqrt{3}}{2}$ with $g_q(x_0) = 5$, which is the global minimum. In other words, for $q \geqslant 5$, there always exists $x_0 \in (0,1)$ making Eq. (26) nonpositive, and thus $g_q(x)$ is not convex for this region of q.

For the region of $q < 5$, let us first consider the function $g_q(x)$ of the integer value q, that is $q = 1, 2, 3,$ and 4. If $q \to 1$, $g_q(x)$ converges to $\mathcal{E}(x)$ in Eq. (16), which is already known to be convex on $0 \leqslant x \leqslant 1$. Furthermore, we have

$$g_2(x) = \frac{x^2}{2}, \quad g_3(x) = \frac{3x^2}{8}, \quad g_4(x) = \frac{8x^2 - x^4}{24}, \tag{29}$$

which are convex polynomials on $0 \leqslant x \leqslant 1$.

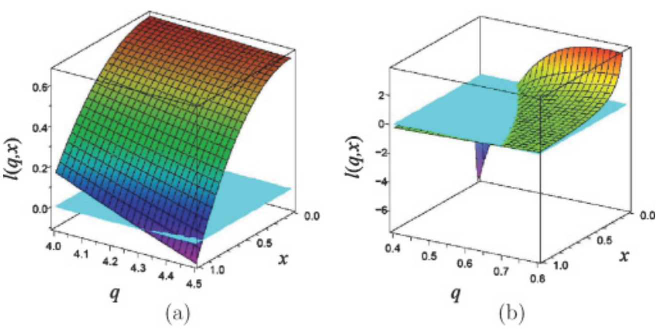

FIG. 1. (Color online) The function values of $l(q,x) = \frac{d^2 g_q(x)}{dx^2}$ for $4 \leqslant q \leqslant 4.5$ and $0.4 \leqslant q \leqslant 0.8$ are illustrated in pictures (a) and (b), respectively.

In fact, if we consider $\frac{d^2 g_q(x)}{dx^2}$ in Eq. (25) as a function of x and q

$$l(x,q) = \frac{d^2 g_q(x)}{dx^2}, \tag{30}$$

defined on the domain $\mathcal{D} = \{(x,q)|0 \leqslant x \leqslant 1, 1 \leqslant q \leqslant 4\}$, it is tedious but also straightforward to check that $l(x,q)$ does not have any vanishing gradient in the interior of $\mathcal{D}$, and its function value on the boundary of $\mathcal{D}$ is always nonnegative. Because $l(x,q)$ is analytic in the interior of $\mathcal{D}$, and continuous on the boundary, $l(x,q)$ is nonnegative through whole the domain $\mathcal{D}$, and this implies the convexity of $g_q(x)$ for $1 \leqslant q \leqslant 4$. Thus, we have the following theorem.

Theorem 1. For $1 \leqslant q \leqslant 4$,

$$g_q(x) = \frac{1}{q-1}\left[1 - \left(\frac{1+\sqrt{1-x^2}}{2}\right)^q - \left(\frac{1-\sqrt{1-x^2}}{2}\right)^q\right], \tag{31}$$

is a monotonically increasing convex function on $0 \leqslant x \leqslant 1$. Furthermore, for this range of q, any two-qubit state ρ_{AB} has an analytic formula for its Tsallis-q entanglement such that $\mathcal{T}_q(\rho_{AB}) = g_q(\mathcal{C}(\rho_{AB}))$ where $\mathcal{C}(\rho_{AB})$ is the concurrence of ρ_{AB}.

Due to the continuity of $g_q(x)$ with respect to q, we can always assure the convexity of $g_q(x)$ for some region of q slightly less than 1 or larger than 4. Furthermore, the continuity of $l(x,q)$ in Eq. (30) also assures the existence of q_0 between 4 and 5, at which the convexity of $g_q(x)$ starts being violated. However, it is generally hard to get an algebraic solution of such q_0 since $\frac{d^2 g_q(x)}{dx^2}$ in Eq. (25) is not an algebraic function with respect to q. Here, we have a numerical way of calculation to test various values of x and q, and it is illustrated in Fig. 1.

According to Fig. 1, $g_q(x)$ is convex for the region $0.7 \leqslant q \leqslant 4.2$, and thus the analytic formula of Tsallis-q entanglement for two-qubit states in Eq. (23) can also be claimed for this region of q.

III. MULTIQUBIT ENTANGLEMENT CONSTRAINT IN TERMS OF TSALLIS-q ENTANGLEMENT

Using concurrence as the bipartite entanglement measure, the monogamous property of a multiqubit pure state $|\psi\rangle_{A_1A_2\cdots A_n}$ was shown to have a mathematical characterization as,

$$\mathcal{C}^2_{A_1(A_2\cdots A_n)} \geqslant \mathcal{C}^2_{A_1A_2} + \cdots + \mathcal{C}^2_{A_1A_n}, \tag{32}$$

where $\mathcal{C}_{A_1(A_2\cdots A_n)} = \mathcal{C}(|\psi\rangle_{A_1(A_2\cdots A_n)})$ is the concurrence of $|\psi\rangle_{A_1A_2\cdots A_n}$ with respect to the bipartite cut between A_1 and the others, and $\mathcal{C}_{A_1A_i} = \mathcal{C}(\rho_{A_1A_i})$ is the concurrence of the reduced density matrix $\rho_{A_1A_i}$ for $i = 2, \ldots, n$ [6,7].

As a dual value to concurrence, *Concurrence of Assistance* (CoA) [23] of a bipartite state ρ_{AB} is defined as

$$\mathcal{C}^a(\rho_{AB}) = \max \sum_k p_k \mathcal{C}(|\psi_k\rangle_{AB}), \tag{33}$$

where the maximum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_k p_k |\psi_k\rangle_{AB}\langle\psi_k|$. Furthermore, it was also shown that there exists a *polygamy* (or dual monogamy) relation of multiqubit entanglement in terms of CoA [24]: For any multiqubit pure state $|\psi\rangle_{A_1\cdots A_n}$, we have

$$\mathcal{C}^2_{A_1(A_2\cdots A_n)} \leqslant \left(\mathcal{C}^a_{A_1A_2}\right)^2 + \cdots + \left(\mathcal{C}^a_{A_1A_n}\right)^2, \tag{34}$$

where $\mathcal{C}^a_{A_1A_i}$ is the CoA of the reduced density matrix $\rho_{A_1A_i}$ for $i = 2, \ldots, n$.

Here, we show that this monogamous and polygamous property of multiqubit entanglement can also be characterized in terms of Tsallis-q entanglement and TEoA. Before this, we provide an important property of the function $g_q(x)$ in Eq. (20) for the proof of multiqubit monogamy and polygamy relations.

For each $q > 0$, let us define a two-variable function $m_q(x,y)$,

$$m_q(x,y) := g_q(\sqrt{x^2+y^2}) - g_q(x) - g_q(y), \tag{35}$$

on the domain $\mathcal{D} = \{(x,y)|0 \leqslant x, y, x^2+y^2 \leqslant 1\}$. Since $m_q(x,y)$ is continuous on the domain $\mathcal{D}$ and analytic in the interior, its maximum or minimum values can arise only at the critical points or on the boundary of $\mathcal{D}$. By taking the first-order partial derivatives of $m_q(x,y)$, we have its gradient

$$\nabla m_p(x,y) = \left(\frac{\partial m_p(x,y)}{\partial x}, \frac{\partial m_p(x,y)}{\partial y}\right), \tag{36}$$

where

$$\begin{aligned}
\frac{\partial m_q(x,y)}{\partial x} &= \alpha x\left[\frac{(1+\sqrt{1-x^2-y^2})^{q-1} - (1+\sqrt{1-x^2-y^2})^{q-1}}{\sqrt{1-x^2-y^2}}\right. \\
&\quad \left. - \frac{(1+\sqrt{1-x^2})^{q-1} - (1+\sqrt{1-x^2})^{q-1}}{\sqrt{1-x^2}}\right] \\
\frac{\partial m_q(x,y)}{\partial y} &= \alpha y\left[\frac{(1+\sqrt{1-x^2-y^2})^{q-1} - (1+\sqrt{1-x^2-y^2})^{q-1}}{\sqrt{1-x^2-y^2}}\right. \\
&\quad \left. - \frac{(1+\sqrt{1-y^2})^{q-1} - (1+\sqrt{1-y^2})^{q-1}}{\sqrt{1-y^2}}\right],
\end{aligned} \tag{37}$$

with $\alpha = \frac{q}{2^q(q-1)}$.

Suppose there exists (x_0,y_0) in the interior of $\mathcal{D}$ (that is, $0<x_0,y_0,x_0^2+y_0^2<1$) such that $\nabla m_p(x_0,y_0)=0$. From Eq. (37), it is straightforward to verify that $\nabla m_p(x_0,y_0)=0$ is equivalent to

$$n_q(x_0)=n_q(y_0), \tag{38}$$

for an analytic function

$$n_q(t)=\frac{(1+\sqrt{1-t^2})^{q-1}-(1+\sqrt{1-t^2})^{q-1}}{\sqrt{1-t^2}}, \tag{39}$$

on $0<t<1$. Furthermore, it is straightforward to see that $\frac{dn_q(t)}{dt}<0$ for $q>1$. In other words, $n_q(t)$ is a strictly monotone-decreasing function with respect to t for $q>1$; therefore Eq. (38) implies $x_0=y_0$. However, from Eq. (37), $\frac{\partial m_q(x_0,y_0)}{\partial x}=0$ together with $x_0=y_0$ imply that $n_q(\sqrt{2}x_0)=n_q(x_0)$, which contradicts to the strict monotonicity of $n_q(t)$. Thus $m_q(x,y)$ has no vanishing gradient in the interior of $\mathcal{D}$.

Now, let us consider the function values of $m_q(x,y)$ on the boundary of $\mathcal{D}$. If $x=0$ or $y=0$, it is clear that $m_q(x,y)=0$. For the case when $x^2+y^2=1$, $m_q(x,y)=0$ becomes a single variable function

$$\begin{aligned} b_q(x)=&\beta[(1+\sqrt{1-x^2})^q+(1-\sqrt{1-x^2})^q]\\ &+\beta[c(1+x)^q+(1-x)^q-2-2^q], \end{aligned} \tag{40}$$

with $\beta=\frac{1}{(q-1)2^q}$, which is an analytic function on $0\leqslant x\leqslant 1$. For the case when $q=2$ or 3, it is clear form Eq. (29) that $m_q(x,y)=0$, and thus $b_q(x)=0$. If q is neither 2 nor 3, $b_q(x)$ has only one critical point at $x=\frac{1}{\sqrt{2}}$ for any $q>1$. Because $b_q(0)=b_q(1)=0$, which are the function values at the boundary, the signs of the function values of $b_q(x)$ are totally determined by that of $b_q(\frac{1}{\sqrt{2}})$, which is the function value at the critical point. Now, we have

$$\begin{aligned} b_q\left(\frac{1}{\sqrt{2}}\right)=&\frac{2}{(q-1)2^q}\left[\left(1+\frac{1}{\sqrt{2}}\right)^q+\left(1-\frac{1}{\sqrt{2}}\right)^q\right]\\ &-\frac{1}{(q-1)2^q}(2+2^q), \end{aligned} \tag{41}$$

whose function value with respect to q is illustrated in Fig. 2.

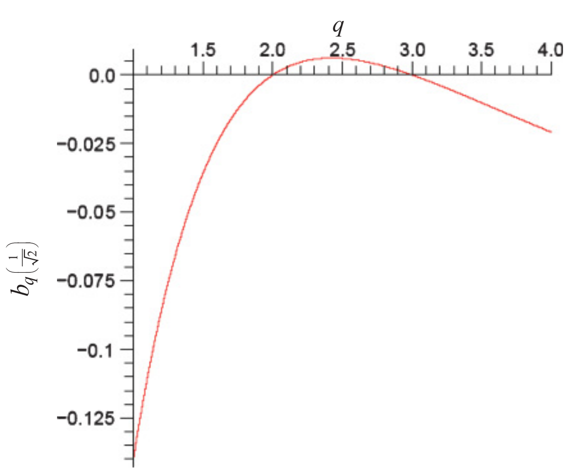

FIG. 2. (Color online) The function values of $b_q(\frac{1}{\sqrt{2}})$ for $1<q\leqslant 4$.

In other words, the function $m_q(x,y)$ in Eq. (35) has no vanishing gradient in the domain $\mathcal{D}$ for $q>1$, and its function values at the boundary of $\mathcal{D}$ is always nonpositive for $1\leqslant q<2$ and $3<q\leqslant 4$, whereas $m_q(x,y)$ is always nonnegative for $2<q<3$. Thus, we have

$$g_q(\sqrt{x^2+y^2})\leqslant g_q(x)+g_q(y), \tag{42}$$

for $1<q<2$ and $3<q<4$, and

$$g_q(\sqrt{x^2+y^2})\geqslant g_q(x)+g_q(y), \tag{43}$$

for $2<q<3$. For the case when $q=2$ or 3, we have

$$g_q(\sqrt{x^2+y^2})=g_q(x)+g_q(y). \tag{44}$$

Now, we are ready to have the following theorem, which is the monogamy inequality of multiqubit entanglement in terms of Tsallis-q entanglement.

Theorem 2. For a multiqubit state $\rho_{A_1\cdots A_n}$ and $2\leqslant q\leqslant 3$, we have

$$\mathcal{T}_q\big(\rho_{A_1(A_2\cdots A_n)}\big)\geqslant \mathcal{T}_q\big(\rho_{A_1A_2}\big)+\cdots+\mathcal{T}_q\big(\rho_{A_1A_n}\big), \tag{45}$$

where $\mathcal{T}_q(\rho_{A_1(A_2\cdots A_n)})$ is the Tsallis-q entanglement of $\rho_{A_1(A_2\cdots A_n)}$ with respect to the bipartite cut between A_1 and $A_2\cdots A_n$, and $\mathcal{T}_q(\rho_{A_1A_i})$ is the Tsallis-q entanglement of the reduced density matrix $\rho_{A_1A_i}$ for $i=2,\ldots,n$.

Proof 1. For the case when $q=2$ or 3, Eq. (29) implies

$$\mathcal{T}_2(\rho_{AB})=\frac{{\mathcal{C}_{AB}}^2}{2},\quad \mathcal{T}_3(\rho_{AB})=\frac{3}{2}{\mathcal{C}_{AB}}^2, \tag{46}$$

for any two-qubit mixed state or $2\otimes d$ pure state ρ_{AB} and its concurrence $\mathcal{C}_{AB}$. Thus, the monogamy inequality in Eq. (45) follows from Eqs. (32) and (46).

For $2<q<3$, we first prove the theorem for n-qubit pure state $|\psi\rangle_{A_1\cdots A_n}$. Note that Eq. (32) is equivalent to

$$\mathcal{C}_{A_1(A_2\cdots A_n)}\geqslant\sqrt{\mathcal{C}^2_{A_1A_2}+\cdots+\mathcal{C}^2_{A_1A_n}}, \tag{47}$$

for any n-qubit pure state $|\psi\rangle_{A_1(A_2\cdots A_n)}$. Thus, from Eq. (43) together with Eq. (47), we have

$$\begin{aligned} \mathcal{T}_q\big(|\psi\rangle_{A_1(A_2\cdots A_n)}\big)&=g_q\left(\mathcal{C}_{A_1(A_2\cdots A_n)}\right)\\ &\geqslant g_q\Big(\sqrt{\mathcal{C}^2_{A_1A_2}+\cdots+\mathcal{C}^2_{A_1A_n}}\Big)\\ &\geqslant g_q\big(\mathcal{C}_{A_1A_2}\big)\\ &\quad+g_q\Big(\sqrt{\mathcal{C}^2_{A_1A_3}+\cdots+\mathcal{C}^2_{A_1A_n}}\Big)\\ &\quad\vdots\\ &\geqslant g_q\big(\mathcal{C}_{A_1A_2}\big)+\cdots+g_q\big(\mathcal{C}_{A_1A_n}\big)\\ &=\mathcal{T}_q\big(\rho_{A_1A_2}\big)+\cdots+\mathcal{T}_q\big(\rho_{A_1A_n}\big), \end{aligned} \tag{48}$$

where the first equality is by the functional relation between the concurrence and the Tsallis-q entanglement for $2\otimes d$ pure states, the first inequality is by the monotonicity of $g_q(x)$, the other inequalities are by iterative use of Eq. (43), and the last equality is by Theorem 1.

For an n-qubit mixed state $\rho_{A_1(A_2\cdots A_n)}$, let $\rho_{A_1(A_2\cdots A_n)}=\sum_j p_j|\psi_j\rangle_{A_1(A_2\cdots A_n)}\langle\psi_j|$ be an optimal decomposition such that $\mathcal{T}_q(\rho_{A_1(A_2\cdots A_n)})=\sum_j p_j\mathcal{T}_q(|\psi_j\rangle_{A_1(A_2\cdots A_n)})$.

Because each $|\psi_j\rangle_{A_1(A_2\cdots A_n)}$ in the decomposition is an n-qubit pure state, we have

$$\begin{aligned}\mathcal{T}_q(\rho_{A_1(A_2\cdots A_n)}) &= \sum_j p_j \mathcal{T}_q(|\psi_j\rangle_{A_1(A_2\cdots A_n)})\\ &\geqslant \sum_j p_j\left(\mathcal{T}_q(\rho^j_{A_1A_2})+\cdots+\mathcal{T}_q(\rho^j_{A_1A_n})\right)\\ &= \sum_j p_j\mathcal{T}_q(\rho^j_{A_1A_2})+\cdots+\sum_j p_j\mathcal{T}_q(\rho^j_{A_1A_n})\\ &\geqslant \mathcal{T}_q(\rho_{A_1A_2})+\cdots+\mathcal{T}_q(\rho_{A_1A_n}),\end{aligned}\tag{49}$$

where $\rho^j_{A_1A_i}$ is the reduced density matrix of $|\psi_j\rangle_{A_1(A_2\ldots A_n)}$ onto subsystem A_1A_i for each $i=2,\ldots,n$ and the last inequality is by definition of Tsallis-q entanglement for each $\rho_{A_1A_i}$. □

Now, let us consider the polygamy of multiqubit entanglement using Tsallis-q entropy. We first note that the function $g_q(x)$ in Eq. (20) can also relate CoA and TEoA of a two-qubit state ρ_{AB}: By letting $\rho_{AB}=\sum_i p_i|\psi_i\rangle_{AB}\langle\psi_i|$ be an optimal decomposition for its CoA, that is,

$$\mathcal{C}^a(\rho_{AB})=\sum_i p_i\mathcal{C}(|\psi_i\rangle_{AB}),\tag{50}$$

we have

$$\begin{aligned}g_q(\mathcal{C}^a(\rho_{AB})) &= g_q\left(\sum_i p_i\mathcal{C}(|\psi_i\rangle_{AB})\right)\\ &\leqslant \sum_i p_i g_q(\mathcal{C}(|\psi_i\rangle_{AB}))\\ &= \sum_i p_i\mathcal{T}_q(|\psi_i\rangle_{AB})\\ &\leqslant \mathcal{T}^a_q(\rho_{AB}),\end{aligned}\tag{51}$$

where the first inequality can be assured by the convexity of $g_q(x)$ and the last inequality is by the definition of TEoA. Because $g_q(x)$ is convex for $1\leqslant q\leqslant 4$, Eq. (51) is thus true for this region of q. Furthermore, $g_q(x)$ satisfies the property of Eq. (42) for $1\leqslant q\leqslant 2$ or $3\leqslant q\leqslant 4$. Thus, we have the following theorem of the polygamy inequality in multiqubit systems.

Theorem 3. For any multiqubit state $\rho_{A_1\cdots A_n}$ and $1\leqslant q\leqslant 2$ or $3\leqslant q\leqslant 4$, we have

$$\mathcal{T}_q(\rho_{A_1(A_2\cdots A_n)})\leqslant \mathcal{T}^a_q(\rho_{A_1A_2})+\cdots+\mathcal{T}^a_q(\rho_{A_1A_n}),\tag{52}$$

where $\mathcal{T}_q(\rho_{A_1(A_2\cdots A_n)})$ is the Tsallis-q entanglement of $|\psi\rangle_{A_1(A_2\cdots A_n)}$ with respect to the bipartite cut between A_1 and $A_2\cdots A_n$, and $\mathcal{T}^a_q(\rho_{A_1A_i})$ is the TEoA of the reduced density matrix $\rho_{A_1A_i}$ for $i=2,\ldots,n$.

Proof 2. We first prove the theorem for a n-qubit pure state and generalize it into mixed states.

For the case when q tends to 1, Tsallis-q entanglement converges to EoA in Eq. (10). It was shown that the polygamy inequality of multiqubit systems can be shown in terms of EoA [25]. For the case when $q=2$ or 3, it is also straightforward from Eqs. (29) and (34).

For an n-qubit pure state $|\psi\rangle_{A_1(A_2\cdots A_n)}$ and $1<q<2$ or $3<q<4$, let us first assume that $(\mathcal{C}^a_{A_1A_2})^2+\cdots+(\mathcal{C}^a_{A_1A_n})^2\leqslant 1$ in Eq. (34). Then we have

$$\begin{aligned}\mathcal{T}_q(|\psi\rangle_{A_1(A_2\cdots A_n)}) &= g_q(\mathcal{C}_{A_1(A_2\cdots A_n)})\\ &\leqslant g_q\left(\sqrt{(\mathcal{C}^a_{A_1A_2})^2+\cdots+(\mathcal{C}^a_{A_1A_n})^2}\right)\\ &\leqslant g_q(\mathcal{C}^a_{A_1A_2})\\ &\quad + g_q\left(\sqrt{(\mathcal{C}^a_{A_1A_3})^2+\cdots+(\mathcal{C}^a_{A_1A_n})^2}\right)\\ &\quad \vdots\\ &\leqslant g_q(\mathcal{C}^a_{A_1A_2})+\cdots+g_q(\mathcal{C}^a_{A_1A_n})\\ &\leqslant \mathcal{T}^a_q(\rho_{A_1A_2})+\cdots+\mathcal{T}^a_q(\rho_{A_1A_n}),\end{aligned}\tag{53}$$

where the first inequality is due to the monotonicity of the function $g_q(x)$, the second and third inequalities are obtained by iterative use of Eq. (42), and the last inequality is by Eq. (51).

Now, let us assume that $(\mathcal{C}^a_{A_1A_2})^2+\cdots+(\mathcal{C}^a_{A_1A_n})^2>1$. Due to the monotonicity of $g_q(x)$, we first note that

$$\begin{aligned}\mathcal{T}_q(|\psi\rangle_{A_1(A_2\cdots A_n)}) &= g_q(\mathcal{C}(|\psi\rangle_{A_1(A_2\cdots A_n)}))\\ &\leqslant g_q(1)\\ &= \frac{1}{q-1}\left(1-\frac{1}{2^{q-1}}\right),\end{aligned}\tag{54}$$

for any multiqubit pure state $|\psi\rangle_{A_1(A_2\cdots A_n)}$, and $q>1$. By letting $\gamma=\frac{1}{q-1}(1-\frac{1}{2^{q-1}})$, it is thus enough to show that $\mathcal{T}^a_q(\rho_{A_1A_2})+\cdots+\mathcal{T}^a_q(\rho_{A_1A_n})\geqslant\gamma$.

Here, we note that there exists $k\in\{2,\ldots,n-1\}$ such that

$$\begin{aligned}&(\mathcal{C}^a_{A_1A_2})^2+\cdots+(\mathcal{C}^a_{A_1A_k})^2\leqslant 1,\\ &(\mathcal{C}^a_{A_1A_2})^2+\cdots+(\mathcal{C}^a_{A_1A_{k+1}})^2>1.\end{aligned}\tag{55}$$

If we let

$$T:=(\mathcal{C}^a_{A_1A_2})^2+\cdots+(\mathcal{C}^a_{A_1A_{k+1}})^2-1,\tag{56}$$

we have

$$\begin{aligned}\gamma &= g_q(1)\\ &= g_q\left(\sqrt{(\mathcal{C}^a_{A_1A_2})^2+\cdots+(\mathcal{C}^a_{A_1A_{k+1}})^2-T}\right)\\ &\leqslant g_q\left(\sqrt{(\mathcal{C}^a_{A_1A_2})^2+\cdots+(\mathcal{C}^a_{A_1A_k})^2}\right)\\ &\quad + g_q\left(\sqrt{(\mathcal{C}^a_{A_1A_{k+1}})^2-T}\right)\\ &\leqslant g_q(\mathcal{C}^a_{A_1A_2})+\cdots+q_q(\mathcal{C}^a_{A_1A_k})+q_q(\mathcal{C}^a_{A_1A_{k+1}})\\ &\leqslant \mathcal{T}^a_q(\rho_{A_1A_2})+\cdots+\mathcal{T}^a_q(\rho_{A_1A_n}),\end{aligned}\tag{57}$$

where the first inequality is by using Eq. (42) with respect to $(\mathcal{C}^a_{A_1A_2})^2+\cdots+(\mathcal{C}^a_{A_1A_k})^2$ and $(\mathcal{C}^a_{A_1A_{k+1}})^2-T$, the second inequality is by iterative use of Eq. (42) on $(\mathcal{C}^a_{A_1A_2})^2+\cdots+(\mathcal{C}^a_{A_1A_k})^2$, and the last inequality is by Eq. (51).

For an n-qubit mixed state $\rho_{A_1(A_2\cdots A_n)}$, let $\rho_{A_1(A_2\cdots A_n)}=\sum_j p_j|\psi_j\rangle_{A_1(A_2\cdots A_n)}\langle\psi_j|$ be an optimal decomposition for TEoA such that $\mathcal{T}^a_q(\rho_{A_1(A_2\cdots A_n)})=\sum_j p_j\mathcal{T}_q(|\psi_j\rangle_{A_1(A_2\cdots A_n)})$.

Because each $|\psi_j\rangle_{A_1(A_2\cdots A_n)}$ in the decomposition is an n-qubit pure state, we have

$$\begin{aligned}\mathcal{T}_q^a(\rho_{A_1(A_2\cdots A_n)}) &= \sum_j p_j \mathcal{T}_q^a(|\psi_j\rangle_{A_1(A_2\cdots A_n)})\\ &\leqslant \sum_j p_j\left(\mathcal{T}_q^a(\rho^j_{A_1A_2})+\cdots+\mathcal{T}_q^a(\rho^j_{A_1A_n})\right)\\ &= \sum_j p_j \mathcal{T}_q^a(\rho^j_{A_1A_2})+\cdots+\sum_j p_j \mathcal{T}_q^a(\rho^j_{A_1A_n})\\ &\leqslant \mathcal{T}_q^a(\rho_{A_1A_2})+\cdots+\mathcal{T}_q^a(\rho_{A_1A_n}),\end{aligned} \tag{58}$$

where $\rho^j_{A_1A_i}$ is the reduced density matrix of $|\psi_j\rangle_{A_1(A_2\cdots A_n)}$ onto subsystem A_1A_i for each $i = 2,\ldots,n$ and the last inequality is by definition of TEoA for each $\rho_{A_1A_i}$. □

Although Theorem 3 provides the polygamy inequality of multiqubit entanglement in terms of TEoA for $1 \leqslant q \leqslant 2$ or $3 \leqslant q \leqslant 4$, it is also clear that Eq. (52) is also true for q slightly larger than 4 or less than 1 due to its continuity with respect to q.

IV. CONCLUSION

Using Tsallis-q entropy, we have established a class of bipartite entanglement measures, Tsallis-q entanglement, and provided its analytic formula in two-qubit systems for $1 \leqslant q \leqslant 4$. Based on the functional relation between concurrence and the Tsallis-q entanglement, we have shown that the monogamy of multiqubit entanglement can be mathematically characterized in terms of Tsallis-q entanglement for $2 \leqslant q \leqslant 3$. We have also provided a polygamy inequality of multiqubit entanglement in terms of TEoA for $1 \leqslant q \leqslant 2$ or $3 \leqslant q \leqslant 4$.

Besides the mathematical characterization of multipartite entanglement, the monogamous and polygamous properties of multipartite quantum entanglement also play an important role in many areas of quantum information theory. Monogamy and polygamy inequalities provide us with an efficient way to classify multipartite entanglement. For example, it is known that there are two inequivalent classes of genuine three-qubit entanglement: One of them is the Greenberger-Horne-Zeilinger (GHZ) class [26] and the other one is the W class [27]. This classification is under stochastic local operations and classical communication (SLOCC) [27], however, these two inequivalent classes also show distinct properties in monogamy and polygamy inequalities. The monogamy and polygamy inequalities are saturated by W-class states, whereas they are never saturated by GHZ-class states [28]. In other words, although the classification of genuine three-qubit entangled states is operational under SLOCC, monogamy and polygamy inequalities can indeed provide us with an analytic way for the classification. Thus it would also be an important task to investigate general MoE in higher-dimensional systems for possible classification of multiparty higher-dimensional quantum entanglement.

As we have mentioned before, MoE can restrict the possible correlation between authorized users and the eavesdropper, and this can be used for the security proof in quantum cryptography. Because higher-dimensional quantum systems rather than qubits are preferred in some physical systems for stronger security in quantum key distribution (QKD) [29], this also shows the importance of characterizing MoE in higher-dimensional systems for the security proof of high-dimensional QKD.

The class of monogamy and polygamy inequalities of multiqubit entanglement we provided here consists of infinitely many inequalities parameterized by q. We believe that our result will provide useful tools and strong candidates for general monogamy and polygamy relations of multipartite entanglement in higher-dimensional quantum systems, which is one of the most important and necessary topics in the study of multipartite quantum entanglement.

Although we have provided an analytic characterization of Tsallis-q entanglement, it would also be an interesting and important task to find out the operational meaning of Tsallis-q entanglement. We believe that the operational meaning of Tsallis-q entanglement with respect to the selective choice of q will lead us to the physical intuition about the monogamy and polygamy inequalities: For example, the similar polygamy property in disjoint intervals $1 < q < 2$ and $3 < q < 4$.

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[1] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[2] W. K. Wootters and W. H. Zurek, Nature (London) **299**, 802 (1982).
[3] B. M. Terhal, IBM J. Res. Dev. **48**, 71 (2004).
[4] L. Masanes, Phys. Rev. Lett. **102**, 140501 (2009).
[5] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[6] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000).
[7] T. J. Osborne and F. Verstraete, Phys. Rev. Lett. **96**, 220503 (2006).
[8] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
[9] A. Rényi, in *Proceedings of the Fourth Berkeley Symposium on Mathematics, Statistics and Probability* (University of California Press, Berkeley, 1960), Vol. 1, pp. 547–561.
[10] R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A **210**, 377 (1996).
[11] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
[12] P. T. Landsberg and V. Vedral, Phys. Lett. A **247**, 211 (1998).
[13] G. Vidal, J. Mod. Opt. **47**, 355 (2000).
[14] S. Abe, Phys. Rev. E **66**, 046134 (2002).
[15] S. Abe and A. K. Rajagopal, Physica A **289**, 157 (2001).
[16] C. Tsallis, S. Lloyd, and M. Baranger, Phys. Rev. A **63**, 042104 (2001).
[17] R. Rossignoli and N. Canosa, Phys. Rev. A **66**, 042306 (2002).
[18] A. K. Rajagopal and R. W. Rendell, Phys. Rev. A **72**, 022322 (2005).

[19] J. Batle, A. R. Plastino, M. Casas, and A. Plastino, J. Phys. A **35**, 10311 (2002).

[20] J. S. Kim and B. C. Sanders, e-print arXiv:0911.5180.

[21] O. Cohen, Phys. Rev. Lett. **80**, 2493 (1998).

[22] Due to the existence of the decomposition satisfying Eqs. (21) and (22), we have

$$
\begin{aligned}
g_q(\mathcal{C}(\rho_{AB})) &= g_q\left(\sum_i p_i \mathcal{C}(|\phi_i\rangle_{AB})\right) \\
&= \sum_i p_i g_q(\mathcal{C}(|\phi_i\rangle_{AB})) \\
&= \sum_i p_i \mathcal{T}_q(|\phi_i\rangle_{AB}) \\
&\geqslant \mathcal{T}_q(\rho_{AB}).
\end{aligned}
$$

Conversely, the existence of the optimal decomposition of $\rho_{AB} = \sum_j q_j |\mu_j\rangle_{AB}\langle\mu_j|$ for Tsallis-q entanglement leads us to

$$
\begin{aligned}
\mathcal{T}_q(\rho_{AB}) &= \sum_j q_j \mathcal{T}_q(|\mu_j\rangle_{AB}) \\
&= \sum_j q_j g_q(\mathcal{C}(|\mu_j\rangle_{AB})) \\
&\geqslant g_q\left(\sum_j q_j \mathcal{C}(|\mu_j\rangle_{AB})\right) \\
&\geqslant g_q(\mathcal{C}(\rho_{AB})),
\end{aligned}
$$

where the first and second inequalities are due to the convexity and monotonicity of $g_q(x)$.

[23] T. Laustsen, F. Verstraete, and S. J. van Enk, Quantum Inf. Comput. **3**, 64 (2003).

[24] G. Gour, S. Bandyopadhay, and B. C. Sanders, J. Math. Phys. **48**, 012108 (2007).

[25] F. Buscemi, G. Gour, and J. S. Kim, Phys. Rev. A **80**, 012324 (2009).

[26] D. M. Greenberger, M. A. Horne, and A. Zeilinger, *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer, Dordrecht, 1989), p. 69.

[27] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).

[28] J. S. Kim and B. C. Sanders, J. Phys. A **41**, 495301 (2008).

[29] S. Groblacher, T. Jennewein, A. Vaziri, G. Weihs, and A. Zeilinger, New J. Phys. **8**, 75 (2006).