

Stabilizing open quantum systems by Markovian reservoir engineering

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We study open quantum systems whose evolution is governed by a master equation of Kossakowski-Gorini-Sudarshan-Lindblad type and give a characterization of the convex set of steady states of such systems based on the generalized Bloch representation. It is shown that an isolated steady state of the Bloch equation cannot be a center, i.e., that the existence of a unique steady state implies attractivity and global asymptotic stability. Necessary and sufficient conditions for the existence of a unique steady state are derived and applied to different physical models, including two- and four-level atoms, (truncated) harmonic oscillators, and composite and decomposable systems. It is shown how these criteria could be exploited in principle for quantum reservoir engineering via coherent control and direct feedback to stabilize the system to a desired steady state. We also discuss the question of limit points of the dynamics. Despite the nonexistence of isolated centers, open quantum systems can have nontrivial invariant sets. These invariant sets are center manifolds that arise when the Bloch superoperator has purely imaginary eigenvalues and are closely related to decoherence-free subspaces.

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I. INTRODUCTION

The dynamics of open quantum systems and especially the possibility of controlling it have attracted significant interest recently. One of the fundamental tasks of interest is the stabilization of quantum states in the presence of dissipation. In recent years a large number of articles have been published on control of closed quantum systems or, more precisely, on systems that only interact coherently with a controller, with applications from quantum chemistry to quantum computing [1]. The essential idea in most of these articles is open-loop Hamiltonian engineering by applying control theory and optimization techniques. Although open-loop control design is a very important tool for controlling quantum dynamics, it has limitations. For instance, while open-loop Hamiltonian engineering can be used to mitigate the effects of decoherence, e.g., using dynamic decoupling schemes [2], or to implement quantum operations on logical qubits, protected against errors due to environmental interactions by a redundant encoding [3], Hamiltonian engineering has intrinsic limitations. One task that is difficult to achieve using Hamiltonian engineering alone is stabilization of quantum states.

Alternatively, we can try to engineer open quantum dynamics described by a Lindblad master equation [4,5] by changing not only the Hamiltonian terms but also the dissipative terms. Various ideas along these lines have been proposed in several articles [6–11]. There are two major sources of dissipative terms in the Lindblad equation: the interaction of the system with its environment and measurements we choose to perform on the system. Accordingly, we can engineer the open dynamics by either modifying the system's reservoir or by applying a carefully-designed quantum measurement. In this sense, the quantum Zeno effect is a simple model for reservoir engineering [12]. In addition, the open dynamics can be modified by feeding the measurement outcome (e.g., the

photocurrent from homodyne detection) back to the controller. This idea was first proposed in Ref. [11], where a feedback-modified master equation was derived and it was shown in Ref. [6] that such direct feedback could be used to stabilize arbitrary single-qubit states with respect to a rotating frame. More recently, there have been several attempts to extend this work to stabilize maximally entangled states using direct feedback [6–10]. The idea of reservoir engineering can also be used to stabilize the system in the decoherence-free subspace (DFS) [13]. In Ref. [14], it is illustrated that N atoms in a cavity can be entangled and driven into a DFS. In Ref. [15], several interesting physical examples are presented showing how to design the open dynamics such that the system can be stabilized in the desired dark state.

Such stabilization problems are a motivation for thorough investigation of the properties of a Lindblad master equation. Important questions include, for instance, which states can be stabilized given a certain general evolution of the system and certain resources. There are a number of classical articles discussing the stationary states and their (asymptotic) stability, as well as sufficient conditions for the existence of a unique stationary state [16–21]. More recently, a detailed analysis of the structure of the Hilbert space with respect to the Lindblad dynamics was carried out in Refs. [22,23], implying that all stationary states are contained in a subspace of the Hilbert space that is attractive. Necessary and sufficient conditions for the attractivity of a subspace or a subsystem have been further considered in Refs. [24,25]. Nonetheless there are still important issues that deserve further study. One is the issue of asymptotic stability of stationary states. It is often assumed that uniqueness implies attractivity of a steady state. Although this turns out to be true for the Lindblad equation, it does not follow trivially from the linearity of the master equation, and a rigorous derivation of this result is therefore desirable, as is a summary of various sufficient conditions for ensuring uniqueness of a stationary state. Similarly, linear dynamical systems can have invariant sets or center manifolds surrounding the set of steady states. The existence of such invariant sets usually precludes convergence of the system to a

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steady state, but criteria for the existence of nontrivial invariant sets are also of interest as they are natural decoherence-free subspaces. Finally, many investigations of the steady states have been based on considering the dynamics on the Hilbert space of the system, e.g., giving criteria for the attractivity of a subspace of the Hilbert space. However, since the steady states are points in the convex set of positive operators on this Hilbert space, such criteria are not always useful. For instance, only systems with steady states at the boundary of the state space (e.g., pure states) have (nontrivial) attractive subspaces of the Hilbert space. While these states may be of special interest, since the states at the boundary form a set of measure zero, most systems will have steady states in the interior. We may not be able to engineer a steady state at the boundary, but perhaps we could stabilize a state arbitrarily close to it, which may be entirely sufficient for practical purposes. Thus, complete characterization of the steady states requires considering the set of positive operators on the Hilbert space rather than the Hilbert space itself.

The purpose of this article is twofold: (i) to further investigate the properties of the stationary states of the Lindblad dynamics and the invariant set of the dynamics generated by imaginary eigenvalues, including the relationship between uniqueness and asymptotic stability and (ii) to present several sufficient conditions for the existence of a unique steady state, apply them to different physical models, and show how these criteria could in principle be used to stabilize an arbitrary quantum state using Hamiltonian and reservoir engineering. In Sec. II, we introduce the Bloch representation of Lindblad dynamics, which will be used throughout the article. In this representation, the spectrum of the dynamics can be easily derived and stability analysis can be conveniently presented. In Sec. III, we characterize the set of all stationary states as a convex set generated by a finite number of extremal points, analyze the properties of the extremal points, and give several sufficient conditions for the uniqueness of the stationary state. We also state a theorem that uniqueness implies attractivity, which is proved in the appendix. In Sec. IV these conditions are applied to different systems, including two- and four-level atoms, the quantum harmonic oscillator, and composite and decomposable systems, and several useful results are derived, including (i) if the Lindblad terms include the annihilation operator, then the system has a unique stationary state regardless of the other Lindblad terms or the Hamiltonian; (ii) for a composite system, if the Lindblad equation contains dissipation terms corresponding to annihilation operators for each subsystem, then the stationary state is also unique; (iii) how any pure or mixed state can be stabilized in principle via Hamiltonian and reservoir engineering. Finally, in Sec. V, we discuss the invariant set generated by the eigenstates of the dynamics with purely imaginary eigenvalues, and its relation to decoherence-free subspaces (DFS), including examples how to find or design a DFS.

II. BLOCH REPRESENTATION OF OPEN QUANTUM SYSTEM DYNAMICS

Under certain conditions the evolution of a quantum system interacting with its environment can be described by a quantum dynamical semigroup and shown to satisfy a Lindblad master

equation

$$\dot{\rho}(t) = -i[H, \rho(t)] + \mathcal{L}_D \rho(t) \equiv \mathcal{L} \rho(t), \quad (1)$$

where $\rho(t)$ is positive unit-trace operator on the system's Hilbert space \mathcal{H} representing the state of the system, H is a Hermitian operator on \mathcal{H} representing the Hamiltonian, $[A, B] = AB - BA$ is the commutator, and $\mathcal{L}_D \rho(t) = \sum_d \mathcal{D}[V_d] \rho(t)$, where V_d are operators on \mathcal{H} and

$$\mathcal{D}[V_d] \rho(t) = V_d \rho(t) V_d^\dagger - \frac{1}{2} [V_d^\dagger V_d \rho(t) + \rho(t) V_d^\dagger V_d]. \quad (2)$$

In this work we will consider only open quantum systems governed by a Lindblad master equation, evolving on a finite-dimensional Hilbert space $\mathcal{H} \simeq \mathbb{C}^N$.

From a mathematical point of view Eq. (1) is a complex matrix differential equation (DE). To use dynamical systems tools to study its stationary solutions and the stability, it is desirable to find a real representation for (1) by choosing an orthonormal basis $\sigma = \{\sigma_k\}_{k=1}^{N^2}$ for all Hermitian matrices on \mathcal{H} . Although any orthonormal basis will do, we shall use the generalized Pauli matrices, suitably normalized, setting $\sigma_k = \lambda_{rs}$, $k = r + (s - 1)N$ and $1 \leq r < s \leq N$, where

$$\lambda_{rs} = \frac{1}{\sqrt{2}}(|r\rangle\langle s| + |s\rangle\langle r|), \quad (3a)$$

$$\lambda_{sr} = \frac{1}{\sqrt{2}}(-i|r\rangle\langle s| + i|s\rangle\langle r|), \quad (3b)$$

$$\lambda_{rr} = \frac{1}{\sqrt{r+r^2}} \left(\sum_{k=1}^r |k\rangle\langle k| - r|r+1\rangle\langle r+1| \right). \quad (3c)$$

The state of the system ρ can then be represented as a real vector $\mathbf{r} = (r_k) \in \mathbb{R}^{N^2}$ of coordinates with respect to this basis $\{\sigma_k\}$,

$$\rho = \sum_{k=1}^{N^2} r_k \sigma_k = \sum_{k=1}^{N^2} \text{Tr}(\rho \sigma_k) \sigma_k$$

and the Lindblad dynamics (1) rewritten as a real DE:

$$\dot{\mathbf{r}} = \left(\mathbf{L} + \sum_d \mathbf{D}^{(d)} \right) \mathbf{r}, \quad (4)$$

where \mathbf{L} , $\mathbf{D}^{(d)}$ are real $N^2 \times N^2$ matrices with entries

$$L_{mn} = \text{Tr}[iH[\sigma_m, \sigma_n]], \quad (5a)$$

$$D_{mn}^{(d)} = \text{Tr}(V_d^\dagger \sigma_m V_d \sigma_n) - \frac{1}{2} \text{Tr}(V_d^\dagger V_d \{\sigma_m, \sigma_n\}), \quad (5b)$$

$\{A, B\} = AB + BA$ being the usual anticommutator. As $\sigma_{N^2} = \frac{1}{\sqrt{N}} \mathbb{I}$, we have $\dot{r}_N = 0$, and (5a) can be reduced to the dynamics on an $(N^2 - 1)$ -dimensional subspace,

$$\dot{\mathbf{s}}(t) = \mathbf{A} \mathbf{s}(t) + \mathbf{c}. \quad (6)$$

This is an affine-linear matrix DE in the state vector $\mathbf{s} = (r_1, \dots, r_{N^2-1})^T$. \mathbf{A} is an $(N^2 - 1) \times (N^2 - 1)$ real matrix with $A_{mn} = L_{mn} + \sum_d D_{mn}^{(d)}$ and \mathbf{c} a real column vector with $c_m = L_{mN} + \sum_d D_{mN}^{(d)}$. Notice that this essentially is the N -dimensional generalization of the standard Bloch equation for a two-level system, and we will henceforth refer to \mathbf{A} as the Bloch operator. The advantage of this representation is that all information of H and V is contained in \mathbf{A} and \mathbf{c} and it is easy

to perform a stability analysis of the Lindblad dynamics in matrix-vector form [26]. Defining $\tilde{\mathbf{A}} = \mathbf{L} + \sum_d \mathbf{D}^{(d)}$, we have the following relation:

$$\tilde{\mathbf{A}} = \begin{pmatrix} \mathbf{A} & \sqrt{N}\mathbf{c} \\ \mathbf{0}^T & 0 \end{pmatrix}.$$

Since $\text{Tr}(\rho^2) \leq 1$ for any physical state ρ , the Bloch vector \mathbf{s} must satisfy $\|\mathbf{s}\| \leq \sqrt{(N-1)/N}$, i.e., all physical states lie in a ball of radius $R = \sqrt{(N-1)/N}$. Note that for $N = 2$ the embedding into of the physical states into this ball is surjective, i.e., the set of physical states is the entire Bloch ball, but this is no longer true for $N > 2$.

III. CHARACTERIZATION OF THE STATIONARY STATES

A state ρ is a steady or stationary state of a dynamical system if $\dot{\rho} = 0$. Steady states are interesting both from a dynamical systems point of view, as well as for applications such as stabilizing the system in a desired state. Let $\mathfrak{E}_{\text{ss}} = \{\rho | \dot{\rho} = \mathcal{L}(\rho) = 0\}$ be the set of steady states for the dynamics given by (1). As (1) is linear in ρ , \mathfrak{E}_{ss} inherits the property of convexity from the set of all quantum states. \mathfrak{E}_{ss} includes special cases such as the so-called dark states, which are pure states $\rho = |\psi\rangle\langle\psi|$ satisfying $[H, \rho] = \mathcal{L}_D(\rho) = 0$. For some systems it is easy to see that there are steady states and what these are. For a Hamiltonian system ($\mathcal{L}_D \equiv 0$) it is obvious from Eq. (1), for instance, that the steady states are those that commute with the Hamiltonian, i.e., $\mathfrak{E}_{\text{ss}} = \{\rho : [H, \rho] = 0\}$. Similarly, for a system with $H = 0$ subject to measurement of the Hermitian observable M , the master equation (1) can be rewritten as $\dot{\rho} = \mathcal{D}[M]\rho = -\frac{1}{2}[M, [M, \rho]]$, and we can show that $\mathfrak{E}_{\text{ss}} = \{\rho : [M, \rho] = 0\}$. In general, assuming \mathbf{s}_0 is the Bloch vector associated with a particular steady state, the set of steady states \mathfrak{E}_{ss} for a system governed by a Lindblad master equation (LME) (1) can be written as $\{\mathbf{s}_0 : \mathbf{A}\mathbf{s}_0 + \mathbf{c} = \mathbf{0}\}$ in the Bloch representation. This is a convex subset of the affine hyperplane $\mathfrak{E}_{\text{ss}}^{\text{lin}} = \{\mathbf{s}_0 + \mathbf{v}\}$ in \mathbb{R}^{N^2-1} , where \mathbf{v} satisfies $\mathbf{A}\mathbf{v} = \mathbf{0}$. Moreover, using Brouwer's fixed point theorem, we can show that the set of steady states \mathfrak{E}_{ss} is always nonempty (see Appendix A) and we have:

Proposition 1. The Lindblad master equation (1) always has a steady state, i.e., the Bloch equation $\mathbf{A}\mathbf{s}_0 + \mathbf{c} = \mathbf{0}$ always has a solution and $\text{rank}(\mathbf{A}) = \text{rank}(\tilde{\mathbf{A}})$, where $\tilde{\mathbf{A}}$ is the matrix \mathbf{A} horizontally concatenated by the column vector \mathbf{c} .

As any convex set is the convex hull of its extremal points, we would like to characterize the extremal points of \mathfrak{E}_{ss} . A point in a convex set is called extremal if it cannot be written as a convex combination of any other points. See Fig. 1 for illustration of convex sets and extremal points. To this end, let $\text{supp}(\rho)$ be the smallest subspace \mathcal{S} of \mathcal{H} such that $\Pi^\perp \rho \Pi^\perp = 0$, where Π is the projector onto the subspace \mathcal{S} and Π^\perp is the projector onto the orthogonal complement of \mathcal{S} in \mathcal{H} .

Proposition 2. The steady state of \mathfrak{E}_{ss} is extremal if and only if it is the unique steady state in its support.

Proof. Since any convex set is the convex hull of its extremal points, the rank of the extremal point is the smallest among its neighboring points, and the rank of boundary points is smaller than that of points in the interior. Suppose that besides the extremal steady state ρ_0 , there is another steady state ρ_1 in

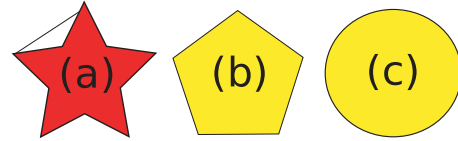


FIG. 1. (Color online) (a) Nonconvex set as a line segment connecting two points in the set is not contained in the set. (b) Convex set spanned by five unique extremal points given by the vertices of the polygon. (c) Convex set with infinitely many extremal points comprising the entire boundary.

the subspace $\text{supp}(\rho_0)$. Then any state ρ_2 which is a convex combination of ρ_0 and ρ_1 must also be in $\text{supp}(\rho_0)$. However, since ρ_0 is an extremal point, the rank of ρ_0 , which is equal to the dimension of $\text{supp}(\rho_0)$, must be lower than the rank of ρ_2 , which is impossible. Conversely, let ρ_s be the unique steady state in its support. Suppose it is not an extremal point, which means that there exist ρ_1 and ρ_2 with $\rho_s = a\rho_1 + (1-a)\rho_2$, $a > 0$. From Lemma 1 in Appendix B, ρ_1 and ρ_2 also lie in $\text{supp}(\rho_s)$, a contradiction to uniqueness of steady states in $\text{supp}(\rho_s)$. ■

We call a subspace \mathcal{S} invariant if any dynamical flow with initial state in \mathcal{S} remains in \mathcal{S} . It has been shown that if ρ_{ss} is a steady state then $\text{supp}(\rho_{\text{ss}})$ is invariant [23,25]. Furthermore, Proposition 1 shows that any invariant subspace contains at least one steady state. Thus, if ρ_{ss} is an extremal point of \mathfrak{E}_{ss} then $\text{supp}(\rho_{\text{ss}})$ is a *minimal* invariant subspace of the Hilbert space \mathcal{H} , i.e., there does not exist a proper subspace of $\text{supp}(\rho_k)$ that is invariant under the dynamics. It can also be shown that $\text{supp}(\rho_{\text{ss}})$ is attractive as a subspace of \mathcal{H} , and $\text{supp}(\rho_{\text{ss}})$ has been called a minimal collecting subspace in Ref. [23].

Different extremal steady states generally do not have orthogonal supports. For example, for a two level-system governed by the trivial Hamiltonian dynamics $H = 0$, \mathfrak{E}_{ss} is equal to the convex set of all states on \mathcal{H} , all pure states are extremal points, and it is easy to see that two arbitrary pure states generally do not have orthogonal supports. Just consider the pure states $\rho_1 = |0\rangle\langle 0|$ and $\rho_2 = \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)$, which are extremal states but $\text{supp}(\rho_1) \not\perp \text{supp}(\rho_2)$. However, in this case there is another extremal steady state $\rho_3 = |1\rangle\langle 1|$ with $\text{supp}(\rho_3) \subset \text{supp}(\rho_1) + \text{supp}(\rho_2)$ and $\text{supp}(\rho_3) \perp \text{supp}(\rho_1)$. In general, given two extremal steady states ρ_1 and ρ_2 , we have either $\text{supp}(\rho_1) \perp \text{supp}(\rho_2)$, or there exists another extremal steady state ρ_3 with $\text{supp}(\rho_3) \subset \text{supp}(\rho_1) + \text{supp}(\rho_2)$ such that $\text{supp}(\rho_1) \perp \text{supp}(\rho_3)$. That is to say, given an extremal steady state ρ_1 , if there exist other steady states, then we can always find another extremal steady state ρ_3 whose support is orthogonal to that of ρ_1 , $\text{supp}(\rho_1) \perp \text{supp}(\rho_3)$. Finally, let \mathcal{H}_{ss} be the union of the supports of all steady states ρ_{ss} . It can be shown (see, e.g., Ref. [23]) that we can choose a finite number of extremal steady states ρ_k with orthogonal supports, such that $\mathcal{H}_{\text{ss}} = \bigoplus_{k=1}^K \text{supp}(\rho_k)$. This decomposition is generally not unique, however. In the above example, any two orthonormal vectors of \mathcal{H} provide a valid decomposition of $\mathcal{H}_{\text{ss}} = \mathcal{H}$, and no basis is preferable. Therefore, such a decomposition of \mathcal{H}_{ss} is not necessarily physically meaningful, but it does give the following useful result:

Proposition 3. If a system governed by a LME (1) has two steady states, then there exist two proper orthogonal subspaces of \mathcal{H} that are both invariant.

In addition to the characterization of \mathfrak{E}_{ss} from the supports of its extremal points, it is also useful to characterize the steady states from the structure of the dynamical operators H and V_d in the LME (1).

Proposition 4. If ρ is a steady state at the boundary, then its support $\mathcal{S} = \text{supp}(\rho)$ is an invariant subspace for each of the Lindblad operators V_d . ■

Proof. A density operator ρ belongs to the boundary of $\mathfrak{D}(\mathcal{H})$ if it has zero eigenvalues, i.e., if $\text{rank}(\rho) = N_1 < N$. In this case, there exists a unitary operator U such that

$$\tilde{\rho} = U\rho U^\dagger = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^\dagger & R_{22} \end{bmatrix}, \quad (7)$$

where R_{11} is an $N_1 \times N_1$ matrix with full rank and R_{12} and R_{22} are $N_2 \times N_1$ and $N_2 \times N_2$ matrices with zero entries and $N_2 = N - N_1 = \dim \ker(\rho)$ and

$$\dot{\tilde{\rho}}(t) = -i[\tilde{H}, \tilde{\rho}(t)] + \sum_d \mathcal{D}[\tilde{V}_d] \tilde{\rho}(t) \quad (8)$$

with $\tilde{H} = UHU^\dagger$ and $\tilde{V}_d = UV_dU^\dagger$. Partitioning

$$\tilde{H} = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^\dagger & H_{22} \end{bmatrix}, \quad \tilde{V}_d = \begin{bmatrix} V_{11}^{(d)} & V_{12}^{(d)} \\ V_{21}^{(d)} & V_{22}^{(d)} \end{bmatrix}, \quad (9)$$

accordingly, it can be verified that a necessary and sufficient condition for ρ to be a steady state of the system is that $\dot{R}_{11} = \dot{R}_{12} = \dot{R}_{22} = 0$, where

$$\dot{R}_{11} = -i[H_{11}, R_{11}] + \sum_d \mathcal{D}[V_{11}^{(d)}] R_{11}, \quad (10a)$$

$$\dot{R}_{12} = -\frac{1}{2} R_{11} \sum_d (V_{11}^{(d)})^\dagger V_{12}^{(d)} + i R_{11} H_{12}, \quad (10b)$$

$$\dot{R}_{22} = \sum_d V_{21}^{(d)} R_{11} (V_{21}^{(d)})^\dagger. \quad (10c)$$

Since R_{11} is a positive operator with full rank and hence strictly positive, the third equation requires $V_{21}^{(d)} = 0$ for all d . The second equation is $R_{11}X = 0$ for $X = -\frac{1}{2} \sum_d (V_{11}^{(d)})^\dagger V_{12}^{(d)} + iH_{12}$, which shows that it will be satisfied if and only if the $N_1 \times N_2$ matrix X vanishes identically, which gives the equivalent conditions

$$0 = -i[H_{11}, R_{11}] + \sum_d \mathcal{D}[V_{11}^{(d)}] R_{11}, \quad (11a)$$

$$0 = -\frac{1}{2} \sum_d (V_{11}^{(d)})^\dagger V_{12}^{(d)} + iH_{12}, \quad (11b)$$

$$0 = V_{21}^{(d)} \quad \forall d. \quad (11c)$$

The last equation implies that if ρ is a steady state at the boundary, then all V_d have a block-tridiagonal structure and map operators defined on $\mathcal{S} = \text{supp}(\rho)$ to operators on \mathcal{S} , i.e., \mathcal{S} is an invariant subspace for all V_d .

The following theorem (proved in Appendix C) shows furthermore that uniqueness implies asymptotic stability:

Theorem 1. A steady state of the LME (1) is attractive, i.e., all other solutions converge to it, if and only if it is unique.

The fact that only isolated steady states can be attractive restricts the systems that admit attractive steady states. In particular, if there are two (or more) orthogonal subspaces \mathcal{H}_k of the Hilbert space \mathcal{H} , which are invariant under the dynamics, i.e., $\text{supp}\mathcal{L}(\mathfrak{D}(\mathcal{H}_k)) \subset \mathcal{H}_k$ for $k = 1, 2, \dots$, then the dynamics restricted to either invariant subspace must have at least one steady state on the subspace, and the set of steady states must contain the convex hull of the steady states on the \mathcal{H}_k subspaces. Thus we have:

Corollary 1. A system governed by LME (1) does not have a globally asymptotically stable equilibrium if there are two (or more) orthogonal subspaces of the Hilbert space that are invariant under the dynamics.

The previous results we give several equivalent useful sufficient conditions to ensure uniqueness of a steady state.

Condition 1. Given a system governed by a LME (1) with an extremal steady state ρ_{ss} , if there is no subspace orthogonal to $\text{supp}(\rho_{\text{ss}})$ that is invariant under all V_d , then ρ_{ss} is the unique steady state.

We compare Condition 1 with Theorem 2 in Ref. [15], which asserts that if there exists no other subspace that is invariant under all V_d orthogonal to the set of dark states, then the only steady states are the dark states. To prove that a given dark state is the unique stationary state, Theorem 2 in Ref. [15] requires that we show (i) uniqueness of the dark state and (ii) that there exists no other orthogonal invariant subspace. Since the dark states defined in Ref. [15] are extremal steady states, Condition 1 shows that (ii) is actually sufficient in that it implies uniqueness and hence attractivity of the steady state.

Condition 2. If there is no proper subspace of $\mathcal{S} \subsetneq \mathcal{H}$ that is invariant under all Lindblad generators V_d , then the system has a unique steady state in the interior.¹

Equation (11) also shows that if there are two orthogonal proper subspaces $\mathcal{H}_1 \perp \mathcal{H}_2$ of the Hilbert space that are invariant under the dynamics, then $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, and there exists a basis such that

$$H = \begin{bmatrix} H_{11} & 0 & H_{13} \\ 0 & H_{22} & 0 \\ H_{13}^\dagger & 0 & H_{33} \end{bmatrix}, \quad V_d = \begin{bmatrix} V_{11}^{(d)} & 0 & V_{13}^{(d)} \\ 0 & V_{22}^{(d)} & V_{23}^{(d)} \\ 0 & 0 & V_{33}^{(d)} \end{bmatrix},$$

for all d , and $iH_{13} - \frac{1}{2} \sum_d (V_{11}^{(d)})^\dagger V_{13}^{(d)} = 0$, i.e., in particular both subspaces are V_d -invariant for all V_d . Hence, if there are no two orthogonal proper subspaces of \mathcal{H} that are simultaneously V_d invariant for all V_d , then the system does not admit orthogonal proper subspaces that are invariant under the dynamics. Thus we have:

Condition 3. If there do not exist two orthogonal proper subspaces of \mathcal{H} that are simultaneously V_d invariant for all V_d , then the system has a unique fixed point, either at the boundary or in the interior.

The following applications show that these conditions are very useful to show attractivity of a steady state.

¹Proper subspace means we are excluding the trivial cases $\mathcal{S} = \{0\}$ and $\mathcal{S} = \mathcal{H}$.

IV. APPLICATIONS

A. Two and four-level atoms

Let us start with the simplest example, a two-level atom governed by the Lindblad master equation

$$\dot{\rho} = -i\Omega[\sigma_x, \rho] + \mathcal{D}[\sigma]\rho$$

with $\sigma = |0\rangle\langle 1|$. This model describes a two-level atom subject to spontaneous emission or a two-level atom interacting with a heavily damped cavity field after adiabatically eliminating the cavity mode. Noting that the Lindblad operator σ corresponds to a Jordan matrix $J_0(2)$, the previous results guarantee that this system has a unique (attractive) steady state. More interestingly, the previous results still guarantee the existence of a unique steady state if the atom is damped by a bath of harmonic oscillators

$$\dot{\rho} = [-iH, \rho] - \frac{\Gamma}{2}\bar{n}\mathcal{D}[\sigma^\dagger]\rho - \frac{\Gamma}{2}(\bar{n} + 1)\mathcal{D}[\sigma]\rho,$$

where $\bar{n} = (e^{\hbar\omega/k_B T} - 1)^{-1}$ is the average photon number. It suffices that one of the Lindblad term $\mathcal{D}[\sigma]\rho$ corresponds to an indecomposable Jordan matrix. In this simple case we can also infer the uniqueness of the steady state directly from the Bloch representation. We can decompose the Bloch matrix $\mathbf{A} = \mathbf{A}_H + \mathbf{A}_D$ into an antisymmetric matrix \mathbf{A}_H corresponding to the Hamiltonian part of the evolution and a diagonal and negative-definite matrix \mathbf{A}_D . Since $\mathbf{s}^T \mathbf{A} \mathbf{s} = \mathbf{s}^T \mathbf{A}_D \mathbf{s} < 0$ for any $\mathbf{s} \neq 0$, it follows that \mathbf{A}_D is invertible and the Bloch equation $\dot{\mathbf{s}} = \mathbf{A} \mathbf{s} + \mathbf{c}$ has a unique attractive stationary state.

On the other hand, if the atom is subjected to a continuous weak measurement such as $\dot{\rho} = \mathcal{D}[\sigma_z]\rho$, then we can easily verify that the pure states $|0\rangle$ and $|1\rangle$ are steady states. Hence, there are infinitely many steady states given by the convex hull of these extremal points, $\rho_{ss} = \alpha|0\rangle\langle 0| + (1 - \alpha)|1\rangle\langle 1|$ with $0 \leq \alpha \leq 1$. Of course, this is the well-known case of a depolarizing channel, which contracts the entire Bloch ball to the z axis, which is the measurement axis.

In the previous examples uniqueness of the steady state followed from similarity of at least one Lindblad operator V to an (indecomposable) Jordan matrix. When V is decomposable then the last example shows that the system can have infinitely many steady states, but similarity of a Lindblad operator to an indecomposable Jordan matrix is only a sufficient condition, i.e., it is not necessary for the existence of a unique steady state. If V has two or more Jordan blocks, for example, then each Jordan block defines an invariant subspace, but provided these subspaces are not orthogonal to each other, Condition 3 still applies, ensuring the uniqueness of the steady state.

For instance, a system governed by a LME $\dot{\rho} = \mathcal{D}[V]\rho$ with $V = S^{-1}JS$, $J = J_0(2) \oplus J_1(2)$ and

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

has a unique steady state because, although V has two eigenvalues 0 and 1 and two proper eigenvectors, the respective eigenspaces are *not* orthogonal and there are no two orthogonal subspaces that are invariant under V . Perhaps more interest-

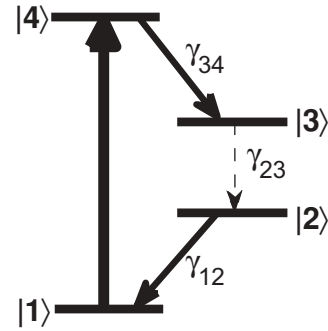


FIG. 2. Schematic plot of the four energy levels of one atom in a prototype system for a laser. The atom is pumped by an external field. The spontaneous decay rates satisfy $\gamma_{34}, \gamma_{12} \gg \gamma_{23}$. On the time scale when decay from level 3 to 2 can be ignored $\rho_{ss} = |3\rangle\langle 3|$ is the unique steady state of the system, realizing the population inversion.

ingly, for a system with a nontrivial Hamiltonian, e.g., $\dot{\rho} = -i[H, \rho] + \mathcal{D}[V]\rho$, uniqueness of the steady state can often be guaranteed even if V has two (or more) orthogonal invariant subspaces, if H suitably mixes the invariant subspaces.

Consider a four-level system with energy levels as illustrated in Fig. 2 and spontaneous emission rates γ_{34}, γ_{23} , and γ_{12} satisfying $\gamma_{34}, \gamma_{12} \gg \gamma_{23}$. This is a simple model for a laser. To derive stimulated emission we require population inversion, a cavity, and a gain medium composed of many atoms. For simplicity, we only consider one atom and try to describe the dynamics in the time scale such that the spontaneous decay $3 \rightarrow 2$ can be neglected. On this scale the Hamiltonian optical-pumping term H and the spontaneous decay term are

$$\begin{aligned} H &= \alpha(|1\rangle\langle 4| + |4\rangle\langle 1|), \\ V_1 &= \gamma_{34}|3\rangle\langle 4|, \\ V_2 &= \gamma_{12}|1\rangle\langle 2|. \end{aligned}$$

There are two invariant subspaces under V_1 and V_2 : $\mathcal{H}_1 = \text{span}\{|1\rangle, |2\rangle\}$ and $\mathcal{H}_2 = \text{span}\{|3\rangle, |4\rangle\}$. Hence, when $\alpha = 0$, we have two metastable states $|2\rangle$ and $|3\rangle$ in addition to the ground state $|1\rangle$, which is a steady state. However, for $\alpha \neq 0$ the pumping Hamiltonian H mixes up those two invariant subspaces, and through calculation we can easily find the unique steady state: $\rho_{ss} = |3\rangle\langle 3|$. Thus, on the time scales considered, population inversion between states $|3\rangle$ and $|2\rangle$ can be realized, but eventually spontaneous emission from $|3\rangle$ to $|2\rangle$ will kick in, resulting in the stimulated emission characteristic of a laser.

This is just one example of optical pumping, a technique widely used for state preparation in quantum optics. Although the principle of optical pumping is easy to understand intuitively for simple systems in that population cannot accumulate in energy levels being pumped, forcing the population to accumulate in states state not being pumped and not decaying to other states, it can be difficult to intuitively understand the dynamics in less straightforward cases. For example, what would happen if we applied an additional laser field coupling $|3\rangle$ and $|4\rangle$. Would the system still have a unique steady state? If so, what is the steady state? These questions are not easy to answer based on intuition, but we can very easily answer

them using the mathematical formalism developed, especially the Bloch equation. In fact, we easily verify that the system

$$\dot{\rho}(t) = -i[H, \rho] + \gamma_{34}\mathcal{D}[|3\rangle\langle 4|]\rho + \gamma_{12}\mathcal{D}[|1\rangle\langle 2|]\rho$$

with $H = \alpha(|1\rangle\langle 4| + |4\rangle\langle 1|) + \beta(|3\rangle\langle 4| + |4\rangle\langle 3|)$ has a unique steady state

$$\rho_{\text{ss}} = \frac{1}{\alpha^2 + \beta^2} \begin{bmatrix} \beta^2 & 0 & -\alpha\beta & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha\beta & 0 & \alpha^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

independent of γ_{12} and γ_{34} , provided $\gamma_{12}, \gamma_{34} \neq 0$. For $\beta = 0$ this state becomes $|3\rangle\langle 3|$, as intuition suggests.

B. Quantum harmonic oscillator

The harmonic oscillator plays an important role as a model for a wide range of physical systems from photon fields in cavities to nanomechanical oscillators to bosons in the Bose-Hubbard model for cold atoms in optical lattices. Although strictly speaking the harmonic oscillator is defined on an infinite-dimensional Hilbert space, the dynamics can often be restricted to a finite-dimensional subspace. For many interesting quantum processes the average energy of the system is finite and we can truncate the number of Fock states N_{max} from ∞ to a large but finite number. In many quantum optics experiments, for example, the intracavity field contains only a few photons, or has a number of photons in some finite range if it is driven by a field with limited intensity. In such cases the truncated harmonic oscillator is a good model for the underlying physical system provided N_{max} is large enough, and we can apply the previous results about stationary solutions and asymptotic stability.

Consider a harmonic oscillator with $H_0 = \hbar\omega c^\dagger c$, where c is the annihilation operator of the system, which on the truncated Hilbert space with $N_{\text{max}} = N$, takes the form

$$c \propto \sum_{n=0}^{N-1} \sqrt{n+1} |n\rangle\langle n+1|. \quad (12)$$

If there is a Lindblad term of the form $\mathcal{D}[c]\rho$, then we can infer from the previous analysis that the system has a unique and hence asymptotically stable steady state, regardless of whatever Hamiltonian control or interaction terms or other Lindblad terms are present. To see this note that the matrix representation of c is mathematically similar to the Jordan matrix

$$J_0(N) = \sum_{n=0}^{N-1} |n\rangle\langle n+1|. \quad (13)$$

It is easy to verify that J has a sole proper eigenvector whose generalized eigenspace is all of \mathcal{H} and thus does not admit two orthogonal proper invariant subspaces. Hence we can conclude from Condition 3 that for *any* dynamics governed by a LME (1) with a dissipation term $\mathcal{D}[c]\rho$, there is always a unique stationary solution to which any initial state will converge. In general, if (1) contains a Lindblad term $\mathcal{D}[V]\rho$ with V similar to a Jordan matrix $J_\alpha(N) = \alpha\mathbb{I}_N + J_0(N)$, then (1) always has a unique stationary state, no matter what the other terms are.

For example, the Lindblad equation for a damped cavity driven by a classical coherent field α is

$$\dot{\rho} = -\frac{1}{2}[\alpha^*c - \alpha c^\dagger, \rho] + \mathcal{D}[c]\rho = \mathcal{D}[\alpha\mathbb{I}_N + c]\rho,$$

showing that the system has a unique steady state. For $N = 4$ the steady state is

$$\rho_{\text{ss}} = \frac{1}{C} \begin{pmatrix} 1 + \alpha^2 A & -\alpha A & \alpha^2 B & -\alpha^3 \\ -\alpha A & \alpha^2 A & -\alpha^3 B & \alpha^4 \\ \alpha^2 B & -\alpha^3 B & \alpha^4 B & -\alpha^5 \\ -\alpha^3 & \alpha^4 & -\alpha^5 & \alpha^6 \end{pmatrix}$$

with α real, $A = \alpha^2 B + 1$, $B = \alpha^2 + 1$, and $C = 4\alpha^6 + 3\alpha^4 + 2\alpha^2 + 1$. When $\alpha = 0$, i.e., there is no driving field, we get ρ_{ss} is the ground (vacuum) state, as one would expect for a damped cavity, while for a nonzero driving field we stabilize a mixed state in the interior.

C. Composite systems

Many physical systems are composed of subsystems, each interacting with its environment, inducing dissipation. For example, consider N two-level atoms in a damped cavity driven by a coherent external field. Assuming the atom-atom and atom-cavity interactions are not too strong, and the main sources of dissipation are independent decay of atoms and the cavity mode, respectively, we obtain the Lindblad terms $\mathcal{D}[\sigma_n]$, $n = 1, \dots, N$, and $\mathcal{D}[c]$ in the LME (1), where σ_n is the decay operator $\sigma = |0\rangle\langle 1|$ for the n th atom and c is the annihilation operator of the cavity. Simulations suggest systems of this type always have a unique steady state, and this can be rigorously shown using the sufficient conditions derived.

A composite quantum system whose evolution is governed by a LME containing terms involving annihilation operators for each subsystem has a unique steady state, regardless of the Hamiltonian and any other Lindblad terms that may be present. This property can be inferred from Condition 3. Assume the full system is composed of K subsystems with Lindblad terms $\mathcal{D}[\sigma_k]\rho$, $k = 1, \dots, K$ and let \mathcal{H}_I be an invariant subspace for all σ_k . Then \mathcal{H}_I must contain the ground state $|\mathbf{0}\rangle = |0\rangle^{\otimes K}$ of the composite system as $\sigma_k|0\rangle = 0$ for all k . Hence, any simultaneously σ_k -invariant subspace must contain the state $|\mathbf{0}\rangle$ and there cannot exist two orthogonal proper subspaces of \mathcal{H} that are invariant under all σ_k . By Condition 3, the system has a unique steady state.

Thus, a system of N atoms in a damped cavity subject to a Lindblad master equation

$$\dot{\rho}(t) = -i[H, \rho(t)] + \gamma\mathcal{D}[c]\rho + \sum_{n=1}^N \gamma_n \mathcal{D}[\sigma_n]\rho$$

has a unique steady state, regardless of the Hamiltonian H . The steady state need *not* be $|\mathbf{0}\rangle$, however. In general, this will only be the case if $|\mathbf{0}\rangle$ is an eigenstate of H . Similarly, the presence of the two dissipation terms $\mathcal{D}[\sigma_k]$ in the LME for the two-atom model in [7]

$$\dot{\rho}(t) = -i[J + J^\dagger, \rho(t)] + \gamma\mathcal{D}[J]\rho + \sum_{k=1,2} \gamma_k \mathcal{D}[\sigma_k]\rho$$

with $J = \sigma_1 + \sigma_2$ ensures that there is a unique steady state provided $\gamma_k > 0$. This is no longer the case for $\gamma_k = 0$. In particular, in the regime where $\gamma_1, \gamma_2 \ll \gamma$ and the last two terms can be neglected as in Ref. [7], the reduced dynamics no longer has a unique steady state.

D. Decomposable systems

A system is decomposable if there exists a decomposition of the Hilbert space $\mathcal{H} = \bigoplus_{m=1}^M \mathcal{H}_m$ such that $\dot{\rho} = \bigoplus_{m=1}^M \dot{\rho}^{(m)}$ for any $\rho(0) = \bigoplus_{m=1}^M \rho^{(m)}(0)$, where $\rho^{(m)}(0)$ is an (unnormalized) density operator on \mathcal{H}_m . Decomposable systems cannot have asymptotically stable (attractive) steady states by Corollary 1.

One class of systems that are always decomposable, and hence never admit attractive steady states, are systems governed by a LME (1) with a single Lindblad operator V that is normal, i.e., $[V, V^\dagger] = 0$, and commutes with the Hamiltonian. This is easy to see. Normal operators are diagonalizable, i.e., there exists a unitary operator U such that $UVU^\dagger = D$ with D diagonal, and since $[H, V] = 0$, we can choose U such that it also diagonalizes H . Thus the system is fully decomposable, and it is easy to see in this case that every joint eigenstate of H and V is a steady state, and therefore there exists a steady-state manifold spanned by the convex hull of the projectors onto the joint eigenstates of H and V . In the absence of degenerate eigenvalues this manifold is exactly the $N - 1$ dimensional subspace of $\mathfrak{D}(\mathcal{H})$ consisting of operators diagonal in the joint eigenbasis of H and V .

A more interesting example of a physical system that is decomposable, and thus does not admit an attractive steady state, is a system of n indistinguishable two-level atoms in a cavity subject to collective decay, and possibly collective control of the atoms as well as collective homodyne detection of photons emitted from the cavity, as illustrated in Fig. 3. Let $\sigma = |0\rangle\langle 1|$ be the single-qubit annihilation operator and define the single-qubit Pauli operators $\sigma_x = \sigma + \sigma^\dagger$, $\sigma_y = i(\sigma - \sigma^\dagger)$ and $\sigma_z = -2[\sigma_x, \sigma_y]$. Choosing the collective measurement operator

$$M = \sum_{\ell=1}^n \sigma^{(\ell)}$$

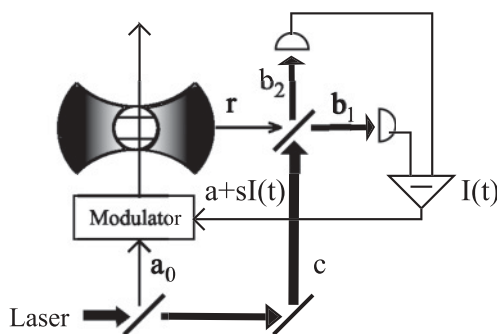


FIG. 3. Schematic diagram of an atomic system under direct quantum feedback control. A laser beam is split to generate the local oscillator c for homodyne detection as well as the driving field a_0 which is then modulated by the feedback photocurrent $I(t)$ to derive the final field $a + sI(t)$.

$\sigma^{(\ell)}$ being the n -fold tensor product whose ℓ th factor is σ , all others being the identity \mathbb{I}_2 , and the collective local control and feedback Hamiltonians

$$H_c = u_x J_x + u_y J_y + u_z J_z, \quad F = \lambda H_c,$$

where $J_a = \sum_{\ell=1}^n \sigma_a^{(\ell)}$ for $a \in \{x, y, z\}$, the evolution of the system is governed by the feedback-modified Lindblad master equation [11]

$$\dot{\rho}(t) = -i[H_0 + H_c + M^\dagger F + FM, \rho] + \mathcal{D}[M - iF]\rho,$$

assuming local decay of the atoms is negligible. It is easy to see from the master equation above that the system decomposes into eigenspaces of the (angular momentum) operator

$$J = J_x^2 + J_y^2 + J_z^2,$$

i.e., both the measurement operator M and the control and feedback Hamiltonians H_c and F (and hence $M^\dagger F + FM$) can be written in block-diagonal form with blocks determined by the eigenspaces of J . Therefore, the system is decomposable and we cannot stabilize *any* state, no matter how we choose $\mathbf{u} = (u_x, u_y, u_z, \lambda)$. For $n = 2$ this system was studied in Ref. [27] in the context of maximizing entanglement of a steady state on the $J = 1$ subspace using feedback, although the question of stability of the steady states was not considered. Although the system does not admit an attractive steady state in the whole space, we can verify that \mathfrak{E}_{ss} contains a line segment of steady states that intersects both the $J = 0$ and $J = 1$ subspaces in a unique state. Thus $J = 1$ subspace has a unique steady state determined by \mathbf{u} , to which all solutions with initial states in this subspace converge.

E. Feedback stabilization

An interesting possible application of the criteria for the existence of unique, attractive steady states is the possibility of engineering the dynamics such that the system has a desired attractive steady state by means of coherent control, measurements, and feedback. An special case of interest here is direct feedback. Systems subject to direct feedback as in the previous example, can be described by a simple *feedback-modified* master equation [11]:

$$\dot{\rho}(t) = -i[H, \rho(t)] + \mathcal{D}[M - iF]\rho(t), \quad (14)$$

where $H = H_0 + H_c + \frac{1}{2}(M^\dagger F + FM)$ is composed of a fixed internal Hamiltonian H_0 , a control Hamiltonian H_c , and a feedback correction term $\frac{1}{2}(M^\dagger F + FM)$. This master equation is of Lindblad form, and hence all of the previous results are directly applicable. Setting

$$V = M - iF, \quad (15a)$$

$$M = V + V^\dagger, \quad (15b)$$

$$F = i(V - V^\dagger), \quad (15c)$$

$$H_c = H - H_0 - \frac{1}{2}(M^\dagger F + FM), \quad (15d)$$

we see immediately that if the control and feedback Hamiltonian, H_c and F , and the measurement operator M are allowed to be arbitrary Hermitian operators, then we can generate *any* Lindblad dynamics. This is also true for a non-Hermitian measurement operator M as arises, e.g., for

homodyne detection, since the anti-Hermitian part of M can always be canceled by the effect of the feedback Hamiltonian F in $\mathcal{D}[M - iF]$. Given this level of control, it is not difficult to show that we can in principle render *any* given target state ρ_{ss} , pure or mixed, globally asymptotically stable by choosing appropriate H and V or, equivalently, by choosing appropriate H_c , F , and M .

To see how to do accomplish this in principle, let us first consider the generic case of a target state ρ_{ss} is in the interior of the convex set of the states with $\text{rank}(\rho_{ss}) = N$. A necessary and sufficient condition for ρ_{ss} to be an attractive steady state is (i) $-i[H, \rho_{ss}] + \mathcal{D}[V]\rho_{ss} = 0$ and (ii) no (proper) subspace of \mathcal{H} is invariant under (1).

The first condition ensures that ρ_{ss} is a steady state, and the latter ensures that it is the only steady state in the interior by Corollary 2. It is easy to see that choosing V and H such that

$$V = U\rho_{ss}^{-1/2}, \quad [H, \rho_{ss}] = 0, \quad (16)$$

where U is unitary, ensures that (i) is satisfied as

$$\begin{aligned} \mathcal{D}[V]\rho_{ss} &= U\rho_{ss}^{-1/2}\rho_{ss}\rho_{ss}^{-1/2}U^\dagger - \frac{1}{2}\{\rho_{ss}^{-1}, |n\rangle\langle n|\} \\ &= UU^\dagger - \frac{1}{2}\{\rho_{ss}^{-1}, \rho_{ss}\} = \mathbb{I} - \mathbb{I} = 0. \end{aligned}$$

To satisfy (ii) we must choose U such that V has no orthogonal invariant subspaces or, equivalently, H mixes up any two orthogonal invariant subspaces V may have. If $\rho_{ss} = \sum_k w_k \Pi_k$, where Π_k is the projector onto the k th eigenspace, then the invariance condition implies that U must not commute with any of the projection operators Π_k , or any partial sum of Π_k such as $\Pi_1 + \Pi_2$. To see this, suppose U commutes with $\Pi_n = |n\rangle\langle n|$, a projector onto an eigenspace of ρ_{ss} . Then $|n\rangle$ is a simultaneous eigenstate of U and ρ_{ss} with $U|n\rangle = e^{i\phi}|n\rangle$ and $\rho|n\rangle = \alpha|n\rangle$, where α must be real and positive as ρ_{ss} is a positive operator, and we have $[H, |n\rangle\langle n|] = 0$, $\{\rho_{ss}^{-1}, |n\rangle\langle n|\} = 2\alpha^{-1}|n\rangle\langle n|$, $|n\rangle$ is an eigenstate of V

$$V|n\rangle = U\rho^{-1/2}|n\rangle = U\alpha^{-1/2}|n\rangle = \alpha^{-1/2}e^{i\phi}|n\rangle,$$

and thus $V|n\rangle\langle n|V^\dagger = \alpha^{-1}|n\rangle\langle n|$ and $\mathcal{D}[V]|n\rangle\langle n| = 0$, i.e., $|n\rangle\langle n|$ is a steady state of the system at the boundary. Hence, the steady state is not unique, and ρ_{ss} cannot be attractive. In practice almost any randomly chosen unitary matrix U such as $U = \exp(i(X + X^\dagger))$, where X is a random matrix, will satisfy the above condition, and given a candidate U it is easy to check if it is suitable by calculating the eigenvalues of the superoperator \mathbf{A} in (6). Of course, choosing H and V of the form (16) is just one of many possible choices for condition (i) to hold. It is possible to find other suitable sets of operators (H, V) in terms of (H_c, F, M) when the class of practically realizable control and feedback operators or measurements is restricted. For example, we can easily verify that ρ_{ss} is the unique attractive steady state of a two-level system governed by the LME (1) with

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & (\sqrt{3} - 1)i \\ (3 - \sqrt{3})i & 1 \end{bmatrix},$$

$$\rho_{ss} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{bmatrix},$$

even though H and V do not satisfy (16). Thus, there are generally many possible choices for the control, measurement, and feedback operators that render a particular state in the interior asymptotically stable.

If the target state ρ_{ss} is in the boundary of the convex set of the states, i.e., $\text{rank}(\rho_{ss}) < N$, then the proof of Proposition 4 shows that we must have

$$0 = -i[H_{11}, R_{11}] + \mathcal{D}[V_{11}]R_{11}, \quad (17a)$$

$$0 = -\frac{1}{2}V_{11}^\dagger V_{12} + iH_{12}, \quad (17b)$$

$$0 = V_{21}. \quad (17c)$$

with $V_{k\ell}$ and $H_{k\ell}$ defined as in Eq. (9) to ensure that ρ is a steady state. To ensure uniqueness we must further ensure that there are no other steady states. This means, by Corollary 3, that (a) we must choose H_{11} and V_{11} such that R_{11} is the *unique* solution of (17a), and thus no subspace of $\mathcal{S} = \text{supp}(\rho_{ss})$ is invariant, and (b) we must choose the remaining operators H_{12} and V_{12} and V_{22} such that (17b) is satisfied and no subspace of \mathcal{S}^\perp is invariant, because if such a subspace \mathcal{S}_2 exists, then \mathcal{S}_1 and \mathcal{S}_2 will be two proper orthogonal invariant subspaces and ρ_{ss} will not be attractive.

One way to construct such a solution is by choosing H_{11} such that $[H_{11}, R_{11}] = 0$ and setting $V_{11} = U_{11}R_{11}^{-1/2}$, where U_{11} is a suitable unitary operator defined on \mathcal{S} as discussed in the previous section. Then we choose V_{22} such that no proper subspace of \mathcal{S}^\perp is invariant. Finally, we must choose V_{12} and H_{12} such that (17b) is satisfied and \mathcal{S}^\perp is itself not invariant. Although these constraints appear quite strict, in practice there are usually many solutions.

For example, suppose we want to stabilize the rank-3 mixed state $\rho_{ss} = \frac{1}{8}\text{diag}(1, 3, 4, 0)$ at the boundary. Then we partition ρ , V , and H as above, setting $V_{11} = UR_{11}$ with $R_{11} = \frac{1}{8}\text{diag}(1, 3, 4)$ and U a suitable unitary matrix such as

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then we choose H_{11} such that $[H_{11}, R_{11}] = 0$, e.g., we could set $H_{11} = V_{11}^\dagger V_{11}$, a choice, which ensures that ρ_{ss} is the unique steady state on the subspace $\mathcal{S} = \text{supp}(\rho_{ss})$. Next we choose V_{12} such that \mathcal{S}^\perp is not an invariant subspace. Any choice other than $V_{12} = (0, 0, 0)$ will do in this case, e.g., set $V_{12} = (1, 0, 0)$. Finally, we set $H_{12} = -\frac{i}{2}V_{11}^\dagger V_{12}$, $V_{21} = (0, 0, 0)^T$ and $V_{22} \neq 0$ to ensure that ρ_{ss} is the unique globally asymptotically stable state.

Note that the Hamiltonian, which was *not* crucial for stabilizing a state in the interior and could have been set to $H = 0$, does affect our ability to stabilize states in the boundary. We can stabilize a mixed state in the boundary only if $H_{12} \neq 0$. If $H_{12} = 0$, then Eq. (17b) implies $V_{11}^\dagger V_{12} = 0$, and there are two possibilities. If $V_{12} \neq 0$ but V_{11} has a zero eigenvalue, then the system restricted to the subspace \mathcal{S} has a pure state at the boundary and thus R_{11} cannot be the unique attractive steady state on \mathcal{S} . Alternatively, if $V_{12} = 0$, then V is decomposable with two orthogonal invariant subspaces \mathcal{S} and \mathcal{S}^\perp , and ρ_{ss} cannot be attractive either, consistent with what was observed in Ref. [28].

Target states at the boundary include pure states. \mathcal{S} in this case is a one-dimensional subspace of \mathcal{H} , and Eq. (17a) is trivially satisfied as H_{11} and V_{11} have rank 1, and the crucial task is to find a solution to Eq. (17b) such that no subspace of \mathcal{S}^\perp is invariant. If $H_{12} = 0$, then this is possible only if $V_{11} = 0$ and thus if V has a zero eigenvalue, as was observed in Ref. [28], but again, if $H \neq 0$, then there are many choices for H and V_{12} , V_{22} that stabilize a desired pure state. For example, we can easily check that the pure state $\rho_{ss} = |1\rangle\langle 1|$ is a steady state of the system $\dot{\rho} = -i[H, \rho] + \mathcal{D}[V]\rho$ if V is the irreducible Jordan matrix $J_a(N)$ with eigenvalue a and $H_{12} = -\frac{i}{2}a^*|1\rangle\langle 2|$.

V. INVARIANT SET OF DYNAMICS AND DECOHERENCE-FREE SUBSPACES

Having characterized the set of steady states, the question is whether the system always converges to one of these equilibria. The previous sections show that this is the case if the system has a unique steady state, as uniqueness implies asymptotic stability. In general, however, this is clearly not the case for a linear dynamical system. Rather, all solutions converge to a center manifold \mathfrak{E}_{inv} , which is an invariant set of the dynamics, consisting of both steady states and limit cycles [29]. Although we have seen that the Lindblad master equation (1) does not admit *isolated* centers, limit cycles often do exist for systems governed by a LME. This is easily seen when we consider the special case of Hamiltonian systems. In this case any eigenstate of the Hamiltonian is a steady state but no other dynamical flows converge to these steady states. For the Bloch equation (6) \mathfrak{E}_{inv} can be characterized explicitly. Consider the Jordan decomposition of the Bloch superoperator, $\mathbf{A} = \mathbf{S}\mathbf{J}\mathbf{S}^{-1}$, where \mathbf{J} is the canonical Jordan form. Let $\gamma_\ell = \alpha_\ell + i\beta_\ell$ be the eigenvalues of \mathbf{A} and Π_γ be the projector onto the (generalized) eigenspace of the eigenvalue γ , and let \mathcal{I} be the set of indices of the eigenvalues of \mathbf{A} with $\alpha_\ell = 0$.

Definition 1. Let \mathfrak{E}_{inv}^{lin} be the affine subspace of \mathbb{R}^{N^2-1} consisting of vectors of the form $\{\mathbf{s}_0 + \mathbf{w}\}$, where \mathbf{s}_0 is a solution of $\mathbf{A}\mathbf{s}_0 + \mathbf{c} = \mathbf{0}$, and $\mathbf{w} \in \mathfrak{E}_{cc}$, where $\mathfrak{E}_{cc} = \sum_{\ell \in \mathcal{I}} \Pi_{\gamma_\ell}(\mathbb{R}^{N^2-1})$ is the direct sum of the eigenspaces of \mathbf{A} corresponding to eigenvalues with zero real part. Then the invariant set $\mathfrak{E}_{inv} = \mathfrak{E}_{inv}^{lin} \cap \mathcal{D}_{\mathbb{R}}(\mathcal{H})$.

It is important to distinguish the invariant set \mathfrak{E}_{inv} , which is a set of Bloch vectors (or density operators), from the notion of an invariant subspace of the Hilbert space \mathcal{H} . In particular, as \mathfrak{E}_{inv} contains the set of steady states \mathfrak{E}_{ss} , it is always nonempty. Although $\text{supp}(\mathfrak{E}_{inv})$, i.e., the union of the supports of all states in \mathfrak{E}_{inv} , is clearly an invariant subspace of \mathcal{H} , in most cases $\text{supp}(\mathfrak{E}_{inv})$ will be the entire Hilbert space. In particular, this is the case if \mathfrak{E}_{inv} contains a single state in the interior, and $\text{supp}(\mathfrak{E}_{inv})$ will be a proper subspace of the Hilbert space only if all steady states are contained in a face at the boundary. This shows that proper invariant subspaces of the Hilbert space exist only for systems that have steady states at the boundary, and the maximal invariant subspace of the Hilbert space can only be less than the entire Hilbert space if there are no steady states in the interior.

Theorem 2. Every trajectory $\mathbf{s}(t)$ of a system governed by a Lindblad equation asymptotically converges to \mathfrak{E}_{inv} .

Proof. Let \mathbf{s}_0 be a solution of the affine-linear equation $\mathbf{A}\mathbf{s}_0 + \mathbf{c} = \mathbf{0}$, which exists by Prop. 1. $\Delta(t) = \mathbf{s}(t) - \mathbf{s}_0$ satisfies the homogeneous linear equation $\dot{\Delta}(t) = \mathbf{A}\Delta(t) = \mathbf{S}\mathbf{J}\mathbf{S}^{-1}\Delta(t)$, where $\mathbf{J} = \text{diag}(J_\ell)$ is the Jordan normal form of \mathbf{A} consisting of irreducible Jordan blocks J_ℓ of dimension k_ℓ with eigenvalue γ_ℓ . Setting $\mathbf{x}(t) = \mathbf{S}^{-1}\Delta(t)$ gives $\dot{\mathbf{x}}(t) = \mathbf{J}\mathbf{x}(t)$ and $\mathbf{x}(t) = e^{t\mathbf{J}}\mathbf{x}(0)$, where $e^{t\mathbf{J}}$ is block-diagonal with blocks

$$E_\ell(t) = e^{t\alpha_\ell} \begin{bmatrix} R_\ell & tR_\ell & \frac{1}{2}t^2R_\ell & \frac{1}{6}t^3R_\ell & \dots \\ 0 & R_\ell & tR_\ell & \frac{1}{2}t^2R_\ell & \dots \\ 0 & 0 & R_\ell & tR_\ell & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}, \quad (18)$$

where $R_\ell = 1$ if $\beta_\ell = 0$, otherwise

$$R_\ell = \begin{bmatrix} \cos(t\beta_\ell) & -\sin(t\beta_\ell) \\ \sin(t\beta_\ell) & \cos(t\beta_\ell) \end{bmatrix}. \quad (19)$$

Since the dynamical evolution is restricted to a bounded set, the matrix \mathbf{A} cannot have eigenvalues with positive real parts, i.e., $\alpha_\ell \leq 0$, and taking the limit for $t \rightarrow \infty$ shows that the Jordan blocks with $\alpha_\ell < 0$ are annihilated, and thus $\Delta(t) \rightarrow \mathbf{S}\mathbf{x}_\infty = \mathbf{w} \in \mathfrak{E}_{cc}$ and $\mathbf{s}(t) = \mathbf{s}_0 + \mathbf{S}\mathbf{x}(t) \rightarrow \mathbf{s}_0 + \mathbf{w}$. ■

The dimension of the invariant set, or more precisely, the affine hyperplane of \mathbb{R}^{N^2-1} it belongs to, is equal to the sum of the geometric multiplicities of the eigenvalues γ_ℓ with zero real part, while the dimension of the set of steady states is equal to the number of zero eigenvalues of \mathbf{A} . Thus, in general, the invariant set is much larger than the set of steady states of the system. Convergence to a steady state is guaranteed only if \mathbf{A} has no purely imaginary eigenvalues. In particular, if all eigenvalues of \mathbf{A} have negative real parts, i.e., $\mathcal{I} = \emptyset$, then all trajectories $\mathbf{s}(t)$ converge to the unique steady state $\mathbf{s}_{ss} = -\mathbf{A}^{-1}\mathbf{c}$. If \mathbf{A} has purely imaginary eigenvalues, then the steady states are centers and the invariant set contains center manifolds, which exponentially attract the dynamics [29]. In either case the trajectories of the system are

$$\mathbf{s}(t) = \mathbf{s}_0 + \mathbf{S}e^{t\mathbf{J}}\mathbf{S}^{-1}[\mathbf{s}(0) - \mathbf{s}_0], \quad (20)$$

and the distance of $\mathbf{s}(t)$ from the invariant subspace

$$d(\mathbf{s}(t), \mathfrak{E}_{inv}) = \|\mathbf{S}\mathbf{x}^\perp(t)\|, \quad (21)$$

where $\mathbf{x}^\perp(t) = \sum_{\ell \notin \mathcal{I}} E_\ell(t)\Pi_{\gamma_\ell}(\mathbf{x}(0))$. Equation (18) also shows that any eigenvalue with zero real part cannot have a nontrivial Jordan block as the dynamics would become unbounded otherwise. Thus the geometric and algebraic multiplicities of eigenvalues with zero real part must agree. Moreover, the eigenvalues of the (real) matrix \mathbf{A} occur in complex conjugate pairs $\gamma = \alpha \pm i\beta$. Thus, if \mathbf{A} has a pair of eigenvalues $\pm i\alpha$ with multiplicity k , then the center manifold (as a subset of \mathbb{R}^{N^2-1}) is at least $2k$ dimensional. Finally, as a unique steady state cannot be a center, it follows that if \mathbf{A} has purely imaginary eigenvalues, then it must also have at least one zero eigenvalue, and there will be a manifold of steady states, all of which are centers. The properties of the invariant set are nicely illustrated by the following example.

Consider a four-level system with $\dot{\rho}(t) = -i[H, \rho] + \mathcal{D}[V]\rho$, where

$$H = \frac{\sqrt{5}}{15} \begin{bmatrix} 6 & 2 & 1 & -2 \\ 2 & -6 & 2 & 1 \\ 1 & 2 & 2 & 6 \\ -2 & 1 & 6 & -2 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & -2 & -1 & 1 \\ 1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & -1 & -1 & 0 \end{bmatrix}.$$

V is indecomposable and has two proper and two generalized eigenvectors with eigenvalue 0. Let \mathcal{H}_0 be the subspace of \mathcal{H} spanned by the proper eigenvectors. H is block-diagonal with respect to a suitable orthonormal basis of $\mathcal{H}_0 \oplus \mathcal{H}_0^\perp$, and there is a 1D manifold of steady states

$$\rho_c(a) = \frac{1}{10} \begin{bmatrix} 3 - 20a & -5a + 1 & 2 - 15a & -5a + 1 \\ -5a + 1 & 10a + 2 & -15a - 1 & 10a + 2 \\ 2 - 15a & -15a - 1 & 3 & -15a - 1 \\ -5a + 1 & 10a + 2 & -15a - 1 & 10a + 2 \end{bmatrix},$$

where $a \in \frac{\sqrt{5}}{15}[-1, 1]$. We can verify that the Bloch matrix \mathbf{A} has a pair of purely imaginary eigenvalues $\pm 2i$ in addition to a 0 eigenvalue and that the invariant set $\mathfrak{E}_{\text{inv}}$ consists of all density matrices with support on the subspace \mathcal{H}_0 spanned by the proper eigenvectors of V defined above. In terms of the corresponding Bloch vectors the invariant set corresponds to the intersection of a three-dimensional invariant subspace of \mathbb{R}^{15} with $\mathfrak{D}_{\mathbb{R}}(\mathcal{H})$. This subspace is what we refer to as the ‘‘face’’ at the boundary, although note that this face is in fact homeomorphic to the 3D Bloch ball in this case. Figure 4(a) shows that all trajectories converge to $\mathfrak{E}_{\text{inv}}$, but Fig. 4(b) shows that the trajectories do not converge to steady states (except for a set of measure zero). Rather, states starting outside the invariant set converge to paths in $\mathfrak{E}_{\text{inv}}$, which in this example are circular closed loops. It is also important to note that most initial states, even initially pure states, converge to mixed states (with lower purity) with support on the invariant set [see Fig. 4(c)].

Although this example may seem rather artificial the properties of the invariant set and the convergence behavior illustrated here are relevant for real physical systems. One important class of physical systems with nontrivial invariant sets are those that possess (nontrivial) decoherence-free subspaces (DFS). By nontrivial we mean here that $\mathfrak{E}_{\text{inv}}$ or the DFS consists of more than one point. A DFS \mathcal{H}_{DFS} is generally defined to be a subspace of the Hilbert space \mathcal{H} that is invariant under the dynamics and on which we have unitary evolution. In general this means that $\mathcal{L}_D(\rho) = 0$ if $\text{supp}(\rho) \subset \mathcal{H}_{\text{DFS}}$. Thus there should exist a Hamiltonian H and Lindblad operators V_k such that $H|\psi\rangle \in \mathcal{H}_{\text{DFS}}$ for any $|\psi\rangle \in \mathcal{H}_{\text{DFS}}$ and $\sum_k \mathcal{D}[V_k]\rho = 0$ for any ρ with $\text{supp}(\rho) \subset \mathcal{H}_{\text{DFS}}$. We must be careful, however, because the decomposition of \mathcal{L}_D is not unique and the Lindblad terms can contribute to the Hamiltonian as we have already seen above, so the effective Hamiltonian on the subspace may not be the same as the system Hamiltonian without the bath.

Proposition 5. If a system governed by a LME has a DFS, then $\mathcal{H}_{\text{DFS}} \subset \text{supp}(\mathfrak{E}_{\text{inv}})$ and any state ρ with $\text{supp}(\rho) \subset \mathcal{H}_{\text{DFS}}$ belongs to $\mathfrak{E}_{\text{inv}}$.

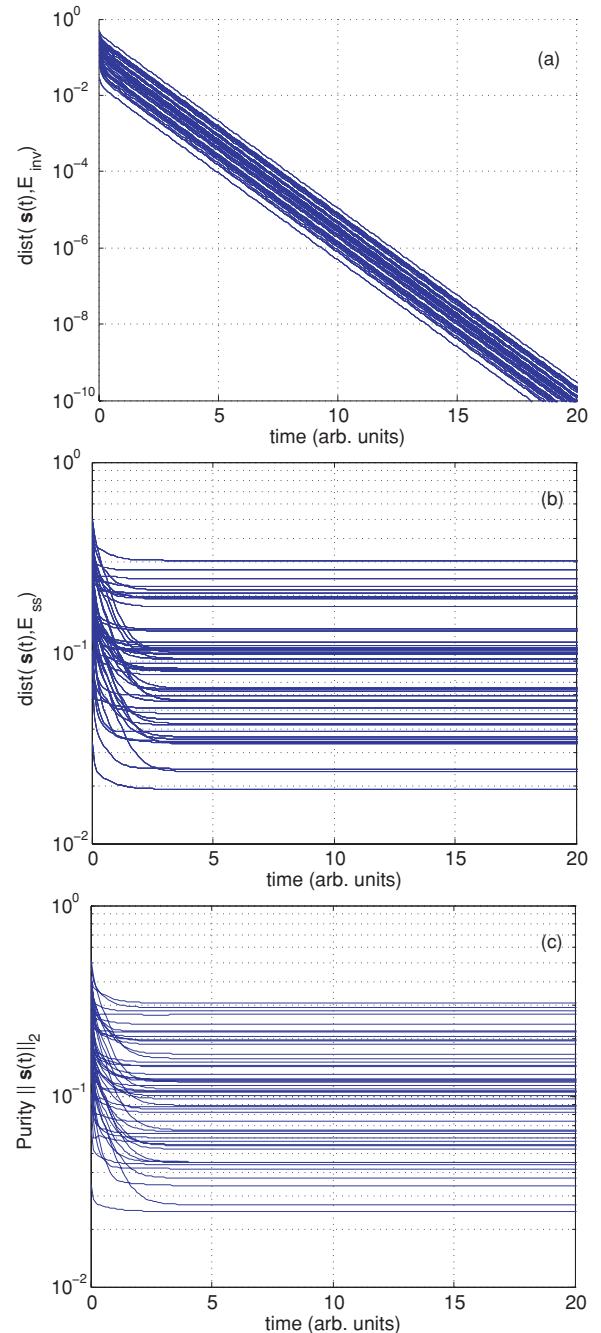


FIG. 4. (Color online) Semilogarithmic plots of (a) the distance of $\mathbf{s}(t)$ from the invariant set $\mathfrak{E}_{\text{inv}}$, (b) the distance from the set of steady states, and (c) the purity $\|\mathbf{s}(t)\|_2$ as a function of time for 50 trajectories starting with 50 random initial states \mathbf{s}_0 . (a) All of the trajectories converge to the invariant set at a constant rate, indicating exponential decay to the invariant set, but the distances of the trajectories from the smaller set of steady states $\mathfrak{E}_{\text{ss}} \subset \mathfrak{E}_{\text{inv}}$ in (b) do not decrease to zero; rather they converge to different limiting values, consistent with convergence of each trajectory to a different limit cycle inside the invariant set. As expected considering that the set of steady states \mathfrak{E}_{ss} is a measure-zero subset of $\mathfrak{E}_{\text{inv}}$, the limiting values of the distances are strictly positive, i.e., none of the 50 trajectories converges to a steady state. (c) The trajectories converge to various mixed states. All limiting values are far below $\frac{1}{2}\sqrt{3}$, the limiting value for a pure state, i.e., none of the 50 trajectories converges to a pure state, again as expected, as the set of pure states in $\mathfrak{E}_{\text{inv}}$ is a measure zero subset.

Proof. If \mathcal{H}_{DFS} is a proper subspace of \mathcal{H} , then the states with support on it correspond to a face F at the boundary of the state space of Bloch vectors or positive unit-trace operators ρ . As \mathcal{H}_{DFS} is an invariant subspace of \mathcal{H} , the face F must be invariant under the dynamics, i.e., $\mathbf{A}\mathbf{s} + \mathbf{c} \in F$ for any $\mathbf{s} \in F$, and thus the face F must contain a steady state \mathbf{s}_{ss} with $\mathbf{A}\mathbf{s}_{\text{ss}} + \mathbf{c} = \mathbf{0}$. Moreover, there exists a subspace S of \mathbb{R}^{N^2-1} such that for any $\mathbf{s} \in F$ we have $\mathbf{s} = \mathbf{s}_{\text{ss}} + \mathbf{v}$ with $\mathbf{v} \in S$. Let \mathbf{A}_H and \mathbf{A}_D be the Bloch operators associated with the Hamiltonian H and dissipative dynamics. We can take \mathbf{A}_H and \mathbf{A}_D to be the antisymmetric and symmetric parts of \mathbf{A} , respectively. If ρ is a state with support on \mathcal{H}_{DFS} , then its Bloch vector \mathbf{s} must satisfy

$$\begin{aligned} \mathbf{A}\mathbf{s} + \mathbf{c} &= \mathbf{A}\mathbf{s}_{\text{ss}} + \mathbf{A}\mathbf{v} + \mathbf{c} = \mathbf{A}\mathbf{v} \\ &= (\mathbf{A}_H + \mathbf{A}_D)\mathbf{v} = \mathbf{A}_H\mathbf{v} \end{aligned}$$

for all $\mathbf{v} \in S$. Due to the invariance property we have $\mathbf{A}_H\mathbf{v} \in S$ and as \mathbf{A}_H is a real antisymmetric matrix, it has purely imaginary eigenvalues. This shows that \mathbf{v} must be a linear combination of eigenvectors of \mathbf{A} with purely imaginary eigenvalues, i.e., $\mathbf{v} \in \mathfrak{E}_{\text{cc}}$ and $\mathbf{s} \in \mathfrak{E}_{\text{inv}}$. ■

There are many examples of systems that have decoherence-free subspaces. For instance, in the example above we can verify that \mathcal{H}_0 is a DFS as \mathcal{H}_0 is invariant under the Hamiltonian dynamics and for any ρ with support on \mathcal{H}_0 we have trivially $\mathcal{D}[V]\rho = 0$ as \mathcal{H}_0 is the subspace of \mathcal{H} spanned by the two (nonorthogonal) eigenvectors of V with eigenvalue 0. Hence, $\rho = \sum_{k=1,2} w_k |\psi_k\rangle\langle\psi_k|$ and $V|\psi_k\rangle = 0$ for $k = 1, 2$ implies

$$V|\psi_k\rangle\langle\psi_k|V^\dagger = V^\dagger V|\psi_k\rangle\langle\psi_k| = |\psi_k\rangle\langle\psi_k|V^\dagger V = 0$$

and thus $\mathcal{D}(V)\rho = 0$.

A simpler, more physical example is a three-level Λ system with decay of the excited state $|2\rangle$ given by the LME with $H = \text{diag}(0, 1, 0)$ and $V_1 = |1\rangle\langle 2|$, $V_2 = |3\rangle\langle 2|$. The system has a DFS spanned by the stable ground states $\mathcal{H}_{\text{DFS}} = \text{span}\{|1\rangle, |3\rangle\}$ as we clearly have $V_1|1\rangle = V_1|3\rangle = 0$ and $V_2|1\rangle = V_2|3\rangle = 0$ and thus $V_1|\psi\rangle = V_2|\psi\rangle = 0$ for all $\psi = \alpha|1\rangle + \beta|3\rangle$ and thus $\mathcal{D}[V_1]\rho = \mathcal{D}[V_2]\rho = 0$ for any

$$\rho = w_1|\psi_1\rangle\langle\psi_1| + w_2|\psi_2\rangle\langle\psi_2|, \quad |\psi_k\rangle \in \mathcal{H}_{\text{DFS}}, \quad (22)$$

for $k = 1, 2$, and \mathcal{H}_{DFS} is invariant under the Hamiltonian H . In this case it is easy to check that the corresponding invariant set $\mathfrak{E}_{\text{inv}}$ is precisely the face F at the boundary corresponding to density operators of the form (22). In fact, as the Hamiltonian is trivial on \mathcal{H}_{DFS} , all of the states with support on \mathcal{H}_{DFS} are actually steady states, i.e., $\mathfrak{E}_{\text{inv}} = \mathfrak{E}_{\text{ss}}$. This would no longer be the case if we changed the Hamiltonian to $H' = |1\rangle\langle 3| + |3\rangle\langle 1|$, for instance, but $\mathfrak{E}_{\text{inv}}$ would still be an invariant set. The requirement that \mathcal{H}_{DFS} be invariant under the Hamiltonian dynamics is very important. If we change the Hamiltonian above to $H'' = |1\rangle\langle 2| + |2\rangle\langle 1|$, for example, then the system no longer has a DFS. In fact, it is easy to check that the invariant set collapses to a single point, here $\mathfrak{E}_{\text{inv}} = \mathfrak{E}_{\text{ss}} = \{|3\rangle\langle 3|\}$. Other choices of the Hamiltonian will result in different steady states. As the states with support on a DFS must be contained in the invariant set $\mathfrak{E}_{\text{inv}}$, only systems with nontrivial $\mathfrak{E}_{\text{inv}}$ admit DFS's.

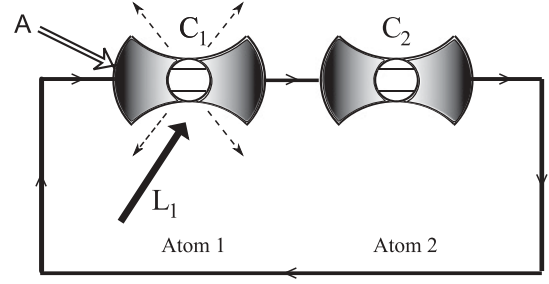


FIG. 5. Two atoms in separated cavities connected into a closed loop through optical fibers. The off-resonant driving field A generates an effective Hamiltonian $H_{\text{eff}} = Z_1 Z_2$. Atom 1 is also driven by a resonant laser field generating a local Hamiltonian X_1 . In the time scale we are interested in, only atom 1 experiences spontaneous decay.

Another example are two spins subject to the LME

$$\dot{\rho} = -i\alpha[Z_1 Z_2, \rho] + \gamma_1 \mathcal{D}[\sigma_1]\rho,$$

where σ_k is the decay operator for spin k and $Z_k = \sigma_k \sigma_k^\dagger - \sigma_k^\dagger \sigma_k$. Here we have an effective Ising interaction term and a decay term for the first spin. This model might describe an electron spin weakly coupled to stable a nuclear spin. The same model was derived for two atoms in separate cavities connected by optical fibers (Fig. 5) in the large-detuning regime [30,31]. In the latter case we could achieve $\gamma_2 \ll \gamma_1$ by choosing different Q factors for the two cavities so that on certain time scale that one atom experiences spontaneous decay while the other does not. For a system of this type the Hilbert space has a natural tensor product structure $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and we immediately expect the invariant set to be $\{\rho = |0\rangle_1\langle 0|_1 \otimes \rho_2\}$ as subsystem 2 is clearly unaffected by dissipation, $\mathcal{D}[\sigma_1](\rho_1 \otimes \rho_2) = (\mathcal{D}[\sigma_1]\rho_1) \otimes \rho_2$. It is easy to see that H is invariant on $\mathfrak{E}_{\text{inv}}$, and the states $|0\rangle \otimes |\psi\rangle$ with $|\psi\rangle \in \mathcal{H}_2$ also form a DFS. However, if atom 1 is driven by a resonant laser field Ω , then the Lindblad dynamics become

$$\dot{\rho} = -i\alpha[Z_1 Z_2, \rho] - i\Omega[X_1, \rho] + \gamma_1 \mathcal{D}[\sigma_1]\rho$$

and the DFS disappears. The system still has a 1D manifold of steady states but the Bloch superoperator \mathbf{A} no longer has purely imaginary eigenvalues $\pm i\gamma$ with $\gamma > 0$, i.e., the invariant set collapses to the 1D manifold of steady states.

VI. CONCLUSION

We have theoretically investigated the convex set of the steady states and the invariant set of the Lindblad master equation, derived several sufficient conditions for the existence of a unique steady state, and applied these to different physical systems. One interesting result is that if one Lindblad term corresponds to an annihilation operator of the system, then the stationary state is unique. Another useful result is that a composite system has a unique steady state if the Lindblad equation contains dissipation terms corresponding to annihilation operators for each subsystem. In both cases the result still holds if other dissipation terms are present, and regardless of the Hamiltonian. We also show that uniqueness implies asymptotic stability of the steady state and hence

global attractivity. On the other hand, if there are at least two steady states, then there is a convex set of steady states, none of which are asymptotically stable. Furthermore, in this case even convergence to a steady state is *not* guaranteed as there can be a larger invariant set surrounding the steady states, corresponding to the center manifold generated by the eigenspaces of the Bloch superoperator \mathbf{A} with purely imaginary eigenvalues. The invariant set is closely related to decoherence-free subspaces; in particular, any state ρ with support on a DFS belongs to the invariant set.

This characterization of the set of steady states and the invariant set can be used to stabilize desired states using Hamiltonian and reservoir engineering, and we illustrate how in principle any state, pure or mixed, can be stabilized this way. This can be extended to engineering decoherence-free subspaces. The latter are naturally attractive but attractivity of a subspace is a weak property in that almost all initial states will generally converge to mixed state trajectories with support on the subspace, not stationary pure states. One possibility of implementing such reservoir engineering is via direct feedback, e.g., by homodyne detection, which yields a feedback-modified master equation [11] with Lindblad terms depending on the measurement and feedback Hamiltonians. This dependence shows that feedback can change the reservoir operators, and we have shown that in the absence of restrictions on the control, measurement, and feedback operators, any state can be rendered asymptotically stable by means of direct feedback. It will be interesting to consider what states can be stabilized, for example, given a restricted set of available measurements and control and feedback Hamiltonians for specific physical models.

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APPENDIX A: EXISTENCE OF STEADY STATES

We can use Brouwer's fixed point theorem and Cantor's intersection theorem to prove that any dynamical system whose flow is a continuous map ϕ_t from a disk D^n to itself must have a fixed point, and the assumption that the domain is the disk D^n can be relaxed to any simple-connected compact set. Specifically, we have:

Theorem 3. Let $\dot{x} = f(x)$ be a dynamical system with a flow ϕ_t from a simply connected compact set D to itself. If ϕ_t is continuous, then there exists a fixed point.

Proof. For any given $T > 0$, $\phi_T : D \rightarrow D$ is a continuous map from D to itself. Applying Brouwer's fixed point theorem, there exists at least one fixed point. Denote the set of fixed points as S_T and observe that as a closed subset of a compact set S_T is compact. Similarly, we can find the set of fixed points $S_{T/2}$ for $\phi_{T/2}$, which is also compact and satisfies

$S_{T/2} \subset S_T$ as a fixed point of $\phi_{T/2}$ is also a fixed point of ϕ_T . By iterating this procedure we can construct a sequence of nonempty compact netting sets $\{S_{T/2^k}\} : \dots \subset S_{T/2^k} \subset S_{T/2^{k-1}} \subset \dots \subset S_{T/2} \subset S_T$. By Cantor's intersection theorem, the intersection of $\{S_{T/2^k}\}$ is nonempty. Let x_0 be one of the points in the intersection. Then for any $T' = nT/2^k$, we have $\phi_{T'}(x_0) = x_0$. Since such T' is dense for $[0, +\infty)$ and ϕ_t a continuous flow, we know that for any time t , $\phi_t(x_0) = x_0$, i.e., x_0 is a fixed of the dynamical system. ■

Since the set of physical states is a compact simply connected set and the master equation clearly continuous, we can conclude that any system governed by a Lindblad master equation has a physical stationary state.

APPENDIX B: EXTREMAL POINTS OF CONVEX SET OF STEADY STATES

Lemma 1. If $\rho_s = s\rho_0 + (1-s)\rho_1$ is a convex combination of the positive operators ρ_0, ρ_1 with $0 < s < 1$, then $\text{rank} \rho_s$ is constant and the support of ρ_0 and ρ_1 is contained in the support of ρ_s .

Proof. If $\text{rank}(\rho_s) = k$, then there exists a basis such that $\rho_s = \text{diag}(r_1, \dots, r_k, 0, \dots)$ with $r_\ell \geq 0$ and $\sum_{\ell=1}^k r_\ell = 1$, i.e., the last $N-k$ rows and columns of ρ_s are 0. Since ρ_0 and ρ_1 are positive operators and $s > 0$, this is possible only if the last $N-k$ rows and columns of ρ_0 and ρ_1 are zero, and thus the support of ρ_0 and ρ_1 is contained in the support of ρ_s . Furthermore, the rank of all ρ_s on the open line segment $0 < s < 1$ must be the same. If there were two intermediate points with $\text{rank}(\rho_s) < \text{rank}(\rho_t)$ and $0 < s < t < 1$, then the support of ρ_0 and ρ_t would have to be contained in the support of ρ_s by the previous argument, which is impossible as $\text{rank}(\rho_s) < \text{rank}(\rho_t)$. Similarly, for $0 < t < s < 1$. ■

Theorem 4. Let \mathcal{H}_s be the smallest subspace of \mathcal{H} that contains the support of all steady states. There exist a finite number of extremal steady states ρ_k such that \mathcal{E}_{ss} is the convex hull of $\{\rho_k\}$ and $\mathcal{H}_s = \oplus_k \text{supp}(\rho_k)$.

Proof. We know that a convex set is the convex hull of its extremal points but there may be many extremal points with nonorthogonal supports. Thus, what we need to show is that we can always choose a subset of the extremal points with mutually orthogonal supports that generates the entire convex set of steady states. Given two extremal steady states ρ_1, ρ_2 , either $\text{supp}(\rho_1) \perp \text{supp}(\rho_2)$, or we can find another steady state ρ_3 with $\text{supp}(\rho_2) \subset \text{supp}(\rho_1) + \text{supp}(\rho_2)$ such that $\text{supp}(\rho_1) \perp \text{supp}(\rho_3)$. Assuming we have already constructed $\mathcal{H}_0 = \oplus_{\ell=1}^{k-1} \text{supp}(\rho_\ell)$ with $\text{supp}(\rho_\ell)$ mutually orthogonal, let ρ_k be another extremal point with $\text{supp}(\rho_k)$ not included in \mathcal{H}_0 and define $\mathcal{H}_1 = \mathcal{H}_0 + \text{supp}(\rho_k)$. By connecting ρ_k and a fixed point with full rank in \mathcal{H}_0 , we can find another steady state with full rank in \mathcal{H}_1 . So \mathcal{H}_1 is an invariant subspace under the dynamics on \mathcal{H} . Therefore, in the following, we will restrict the dynamics H and V_k on the subspace \mathcal{H}_1 . Define P to be the projection operator of \mathcal{H}_0 and P^\perp to be the orthogonal projection operator of P with respect to \mathcal{H}_1 . In the block-diagonal diagonal form with respect to P and P^\perp ,

$$\rho_k = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}, \quad H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \quad V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix},$$

where without loss of generality we only consider one Lindblad term. Since \mathcal{H}_0 is an invariant subspace under the dynamics, we have

$$0 = V_{21} \quad (\text{B1a})$$

$$0 = -\frac{1}{2} \sum_j V_{11}^\dagger V_{12} + iH_{12}. \quad (\text{B1b})$$

For a steady state ρ_k , we have $0 = \dot{\rho} = -i[H, \rho_k] + \mathcal{D}[V]\rho$. Since ρ_k is an extremal point, ρ_{22} has full rank. Moreover, as ρ_k is stationary in \mathcal{H}_1 , it is also stationary restricted to a subspace $P^\perp \mathcal{H}_1$, which means

$$0 = -i[H_{22}, \rho_{22}] + \mathcal{D}[V_{22}]\rho_{22}.$$

Substituting this as well as (B1) into $\dot{\rho}_{22} = 0$, we find

$$\rho_{22} V_{12}^\dagger V_{12} + V_{12}^\dagger V_{12} \rho_{22} = 0,$$

which means $V_{12} = 0$ since ρ_{22} has full rank. Together with (B1) we have $[H, P] = [V, P] = 0$. Hence $P^\perp \mathcal{H}_1$ is also an invariant space under the dynamics restricted on \mathcal{H}_1 . Combining the condition that \mathcal{H}_1 is an invariant subspace under the dynamics on \mathcal{H} , we conclude that $P^\perp \mathcal{H}_1$ is also an invariant subspace under the dynamics on \mathcal{H} . There must exist an extremal fixed point $\bar{\rho}_k$ in $P^\perp \mathcal{H}_1$ with support orthogonal to \mathcal{H}_0 . We have $\mathcal{H}_1 = H_0 \oplus \text{supp}(\bar{\rho}_k)$. Continuing this process until all fixed points are included in $\bigoplus_k \text{supp}(\rho_k)$, we finally obtain $\mathcal{H}_s = \bigoplus_k \text{supp}(\rho_k)$. This construction can be completed in a finite number of steps as the dimension of \mathcal{H}_s is finite. ■

APPENDIX C: PROOF OF “NO ISOLATED CENTERS” THEOREM

Suppose \mathbf{A} has a pair of purely imaginary eigenvalues $\pm i\alpha$. Let \mathbf{e} be an eigenvector of \mathbf{A} corresponding to the eigenvalue $+i\alpha$ with $\alpha > 0$, i.e., $\mathcal{L}_{tot}(\mathbf{e}) = \mathbf{A}\mathbf{e} = i\alpha\mathbf{e}$. In the Schrödinger picture, we have $e^{t\mathbf{A}}\mathbf{e} = e^{i\alpha t}\mathbf{e}$. Let E be the operator, corresponding to \mathbf{e} , in the adjoint operator space, with $\mathcal{L}_{tot}^\dagger(E) = i\alpha E$. In the Heisenberg picture, the adjoint dynamics gives $E(t) = e^{i\alpha t}E$, and $E(t)^\dagger E(t) = E^\dagger E$, with $E^\dagger E$ always positive. We can scale E such that $\|E^\dagger E\|_\infty = 1$. Thus, $E^\dagger E$ is a positive matrix with maximum eigenvalue $\lambda_{\max} = 1$ and $|\phi_0\rangle$ as the associated eigenvector. Hence, we have $E^\dagger E|\phi_0\rangle = |\phi_0\rangle$ and $\text{Tr}(E^\dagger E\rho_0) = \text{Tr}(\lambda_{\max}\rho_0) = 1$, where $\rho_0 = |\phi_0\rangle\langle\phi_0|$. Let us consider in the Schrödinger picture, the evolution of $\rho(t)$ with initial state $\rho(0) = \rho_0$. We define the average state

$$\bar{\rho}(T) = \frac{1}{T} \int_0^T \rho(t) dt. \quad (\text{C1})$$

Setting $D = E^\dagger E$, switching between the Schrödinger and Heisenberg picture, and using the Kadison inequality $D(t) \geq E(t)^\dagger E(t)$, we obtain

$$\begin{aligned} \text{Tr}(\bar{\rho}D) &= \frac{1}{T} \int_0^T \text{Tr}[\rho(t)D] dt \\ &= \frac{1}{T} \int_0^T \text{Tr}[\rho D(t)] dt \end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{T} \int_0^T \text{Tr}[\rho_0 E(t)^\dagger E(t)] dt \\ &= \frac{1}{T} \int_0^T \text{Tr}[\rho_0 E^\dagger E] dt \\ &= \text{Tr}(\rho_0 E^\dagger E) = \|E^\dagger E\|_\infty = 1. \end{aligned}$$

On the other hand, we have $\text{Tr}(\bar{\rho}E^\dagger E) \leq 1$ and thus $\text{Tr}(\bar{\rho}E^\dagger E) = 1$.

If the unique steady state ρ_{ss} is in the interior of the convex set of physical states, i.e., ρ_{ss} has full rank, then $\bar{\rho}$ must have full rank for sufficiently large T as well, and this is possible only if $E^\dagger E = \mathbb{I}$, i.e., E is unitary. Next we calculate the term $E^\dagger \mathcal{L}_{tot}^\dagger(E)$. From the evolution:

$$\mathcal{L}_{tot}^\dagger(E) = [iH, E] + \sum_j \left[V_j^\dagger E V_j - \frac{1}{2}(E V_j^\dagger V_j + V_j^\dagger V_j E) \right],$$

we have

$$\begin{aligned} E^\dagger \mathcal{L}_{tot}^\dagger(E) &= E^\dagger [iH, E] \\ &\quad + \sum_j \left(E^\dagger V_j^\dagger E V_j - \frac{1}{2} V_j^\dagger V_j - \frac{1}{2} E^\dagger V_j^\dagger V_j E \right). \end{aligned} \quad (\text{C2})$$

Since E is unitary, we have $E^\dagger \mathcal{L}_{tot}^\dagger(E) = E^\dagger i\alpha E = i\alpha I$. Taking the trace at both sides in (C2),

$$\begin{aligned} \sum_j \text{Tr}(E^\dagger V_j^\dagger E V_j) &= \frac{1}{2} \sum_j [\text{Tr}(E^\dagger E V_j^\dagger V_j) + \text{Tr}(E^\dagger V_j^\dagger V_j E)] \\ &\quad + i\alpha N \end{aligned}$$

On the other hand, by Cauchy Schwartz inequality,

$$\begin{aligned} \left| \sum_j \text{Tr}(E^\dagger V_j^\dagger E V_j) \right| &\leq \sum_j |\text{Tr}((E^\dagger V_j^\dagger E)(V_j))| \\ &\leq \sum_j \sqrt{\text{Tr}(E^\dagger V_j^\dagger E E^\dagger V_j E)} \sqrt{\text{Tr}(V_j^\dagger V_j)} \\ &\leq \sum_j \frac{1}{2} [\text{Tr}(E^\dagger V_j^\dagger V_j E) + \text{Tr}(V_j^\dagger V_j)]. \end{aligned}$$

Therefore, we must have $\alpha = 0$, which contradicts the initial assumption that $\alpha > 0$.

When the unique steady state is a mixed state at the boundary with $1 < \text{rank}(\rho_{ss}) < N$, then we can partition the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that ρ_{ss} vanishes on \mathcal{H}_2 . It has been shown that in this case all solutions are attracted to states with support on \mathcal{H}_1 . Thus, we can restrict the dynamics to \mathcal{H}_1 , i.e., the support of ρ_{ss} , and the same arguments as above imply that $E^\dagger E$ must equal the identity on the \mathcal{H}_1 subspace, $E^\dagger E|_{\mathcal{H}_1} = \mathbb{I}_{\mathcal{H}_1}$, which leads to a contradiction.

If the unique fixed point ρ_{ss} happens to be a pure state at the boundary, then it is easy to see that there cannot be any loop paths, because the state $\bar{\rho}$ averaged over one period would have to equal the stationary state ρ_{ss} , which is not possible because a rank 1 projector cannot be written as a linear combination of other states.

Moreover, if s_{ss} is a center that belongs to a face F in the boundary, then the entire center manifold it belongs to must be contained in F as otherwise there

would be loop planes intersecting the boundary and physical states evolving into nonphysical states, which is forbidden.

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