Exponential quadratic operators and evolution of bosonic systems coupled to a heat bath

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Using exponential quadratic operators, we present a general framework for studying the exact dynamics of system-bath interaction in which the Hamiltonian is described by the quadratic form of bosonic operators. To demonstrate the versatility of the approach, we study how the environment affects the squeezing of quadrature components of the system. We further propose that the squeezing can be enhanced when parity kicks are applied to the system.

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I. INTRODUCTION

Coupling between system and environment is ubiquitous in all quantum processes (e.g., in quantum information processing). Such coupling usually results in (i) the energy decay of the quantum system and (ii) the destruction of the relative phases of several superposed quantum states, and thus the linear superposition of several quantum states turn into a classical mixture. However, the environment can also help us; for example, entanglement between two systems can be generated via a common environment [1].

Although it seems impossible to model the environment exactly in many cases and thus difficult to obtain the exact dynamics of a system-environment interaction, quantitative analysis based on approximate description of the environment is needed in many cases: for instance, the analysis of decoherence-suppressing methods [2–10], the discussion of the entanglement of two systems coupled to the environment, and the study of the quantum dissipation of systems [11,12]. An extensively adopted approach to model the environment, which is also called a reservoir or bath, is to introduce a set of harmonic oscillators with different frequencies. In this case, the interaction between the system and the environment is modeled by coupling the system to these harmonic oscillators through an appropriate interaction Hamiltonian. Several methods have been proposed to study the coupling between the system and a set of harmonic oscillators. In quantum optics (e.g., Refs. [13–15]), a frequently used method to analyze Markovian process is either a master equation or a Langevin equation. Another method is the path integral approach [16], which was extensively developed in Refs. [11,17], but this method is very complicated. Furthermore, different approximations are used in all of these methods to make the problem tractable for either analytical or numerical calculations.

In this article, we introduce a method to calculate the evolution of the bosonic system coupled to the environment. The total Hamiltonian is described by a quadratic form of the bosonic operators. Our method is based on some properties of exponential quadratic operators. As shown, this method provides a feasible way to calculate the effect of the environment on the system. As an example, we apply our method to study the environmental effect on the generation of squeezed states. Moreover, we also use our method to study the system-environment interaction when the parity kicks are applied to the system. We find that the parity kicks can help us to obtain a better squeezing.

II. EXPONENTIAL QUADRATIC OPERATORS

For a set of annihilation operator $a_i (1 \leqslant i \leqslant n)$, exponential quadratic operators (EQOs; see [18–20]) are expressions of the form

$$Q = e^{\sum_{i,j}(c_{ij}a_i a_j + d_{ij}a_i a_j^\dagger + e_{ij}a_i^\dagger a_j^\dagger)}. \tag{1}$$

The orders of a_i and $a_i^\dagger$ are not important since $[a_i,a_i^\dagger] = 1$, and the exponent of a number is easy to handle. This equation can also be written in the following way: $Q = e^{\frac{1}{2}\Lambda^T R \Lambda}$ in which $\Lambda^T = (a_1^\dagger, a_2^\dagger, \ldots, a_n^\dagger, a_1, a_2, \ldots, a_n)$ and R is a symmetric matrix. If we define

$$S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

then we have

$$Q\Lambda^T Q^{-1} = \Lambda^T e^{-RS}, \tag{2}$$

where the multiplication in $Q\Lambda^T Q^{-1}$ is understood to act on each term of Λ.

III. COUPLING BETWEEN OSCILLATOR AND RESERVOIR

Consider a system comprising a harmonic oscillator with annihilation operator a and a reservoir consisting of a set of oscillators with annihilation operator b_k for each mode. The Hamiltonian of system-reservoir is described by

$$H = \hbar\omega a^\dagger a + \sum_k \hbar\omega_k b_k^\dagger b_k + \hbar \sum_k \gamma_k (a b_k^\dagger + b_k a^\dagger), \tag{3}$$

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where the first, second, and third terms are the system, reservoir, and system-reservoir interaction Hamiltonians, respectively. Here, γ_k are the coefficients representing the coupling strength between the system and the mode k of reservoir; these coupling constants are typically much smaller than the other frequencies in the Hamiltonian. For simplicity, but without loss of generality, we regard these couplings as reals.

We calculate the evolution of $a(a^\dagger)$ in the Heisenberg picture by using Eq. (2). For $U = e^{-iHt/\hbar}$, we put $\Lambda^T = (a^\dagger, b_1^\dagger, b_2^\dagger, \ldots, b_n^\dagger, a, b_1, b_2, \ldots, b_n)$ and

$$R = \begin{pmatrix} & P \\ P & \end{pmatrix},$$

where

$$P = \begin{pmatrix} i\omega t & i\gamma_1 t & i\gamma_2 t & \cdots & i\gamma_n t \\ i\gamma_1 t & i\omega_1 t & & & \\ i\gamma_2 t & & i\omega_2 t & & \\ \vdots & & & \ddots & \\ i\gamma_n t & & & & i\omega_n t \end{pmatrix}. \tag{4}$$

Thus, to calculate the evolution of a $(a^\dagger)$, we need only to calculate the matrix e^{-RS}.

IV. COUPLING BETWEEN SYSTEM AND RESERVOIR DURING A SQUEEZING PROCESS

To see the power of the technique, let us consider a Hamiltonian for degenerate parametric amplification with a classical pump under the influence of a reservoir in a squeezing process. The Hamiltonian can be expressed as

$$H = \hbar\omega a^\dagger a + \frac{1}{2} i\hbar\epsilon[e^{2i\omega t}a^2 - e^{-2i\omega t}(a^\dagger)^2] + \sum_{j=1}^{n} \hbar\omega_j b_j^\dagger b_j + \hbar\left(a^\dagger \sum_{j=1}^{n} \gamma_j b_j + \text{h.c.}\right). \tag{5}$$

In order to remove the time dependence in the Hamiltonian, we transfer the Hamiltonian into a rotating reference frame with $U = \exp(iH_0 t/\hbar)$ with $H_0 = \hbar\omega(a^\dagger a + \sum_{j=1}^{n} b_j^\dagger b_j)$. Thus, in the rotating reference frame, the Hamiltonian in Eq. (5) becomes

$$H_I = -\frac{1}{2} i\hbar\epsilon[(a^\dagger)^2 - a^2] + \sum_{j=1}^{n} \hbar(\omega_i - \omega) b_j^\dagger b_j + \hbar\left(a^\dagger \sum_{j=1}^{n} \gamma_j b_j + \text{h.c.}\right). \tag{6}$$

Note that the first term is the usual squeezing Hamiltonian. We can easily find the matrix R corresponding to $-iH_I t/\hbar$. By analyzing e^{-RS}, numerically if necessary, we obtain the evolution of $a^\dagger(t)$ and $a(t)$ and thus the solution of all quantities associated with a squeezing process. The most important one among them is $\langle[\Delta\boldsymbol{(}a(t) + a(t)^\dagger\boldsymbol{)}]^2\rangle = \langle(\Delta X)^2\rangle$.

V. PARITY KICKS IN THE SQUEEZING PROCESS

Using appropriate time-varying control fields, it is well known that one could alleviate decoherence effects through a sequence of frequent parity kicks. However, earlier papers discuss only how to protect an initial state of the system. Here, with the calculation tools presented previously, we can do something. We can study how to protect the state engineering, in particular, how to *produce* an enhanced squeezed state. Without the environment noise, H_I finally will drive the system into a highly squeezed state. However, with the environmental noise, the noise and the decoherence *compete*. We study such competing process with parity kicks on and find the time when the system is maximally squeezed. As in Ref. [4], we introduce an extra Hamiltonian (in the rotating reference frame).

$$H_I' = H_I + H_{\text{kick}}(t),$$

where $H_{\text{kick}}(t) = H_{\text{kick}}$ for $t_i \leqslant t \leqslant t_i + \tau$ and $H_{\text{kick}} = 0$ otherwise. We require $t_{i+1} - t_i = \tau_0$ for all i. Moreover, we assume $\tau \ll \tau_0$ and $H_{\text{kick}}(t)$ is strong enough during the kick periods that we can neglect the effect of H_I, which is $e^{-iH_I\tau/\hbar} \approx e^{-iH_{\text{kick}}\tau/\hbar}$. Under these conditions, we model parity kicks as unitary operators $P = e^{-iH_{\text{kick}}\tau/\hbar}$ acting on system at a set of time t_i. Since we want to eliminate the influence of coupling between system and reservoir, we require P to have following properties: $PH_{\text{system}}P = H_{\text{system}}$, $PH_{\text{bath}}P = H_{\text{bath}}$, and $PH_{\text{int}}P = -H_{\text{int}}$. The three Hamiltonians are defined in Eq. (6). It is easy to verify that $P = e^{-i\pi a^\dagger a}$ satisfies these equations. Thus, the unitary operator corresponding to two such periods would be $Pe^{-iH_I\tau_0/\hbar}Pe^{-iH_I\tau_0/\hbar} = e^{-(i\tau_0/\hbar)(H_{\text{system}}+H_{\text{bath}}-H_{\text{int}})}e^{-(i\tau_0/\hbar)(H_{\text{system}}+H_{\text{bath}}+H_{\text{int}})} \doteq Y$. Intuitively, it shows that the interaction between system and reservoir of different periods cancel each other out. In fact, it has been proved that when $\tau_0 \to 0$, the system and the reservoir are totally decoupled.

We use numerical computation to verify this effect in the squeezing process. To this end, we calculate the evolution of $a^\dagger(t)$ and $a(t)$. We have

$$a^\dagger(2n\tau_0) = Y^{\dagger n} a^\dagger Y^n.$$

To use the EQO method shown in Eq. (2) to solve this expression, we note that if

$$e^{Y_1}\Lambda^T e^{-Y_1} = \Lambda^T P_1,$$
$$e^{Y_2}\Lambda^T e^{-Y_2} = \Lambda^T P_2,$$

then

$$e^{Y_2}e^{Y_1}\Lambda^T e^{-Y_1}e^{-Y_2} = \Lambda^T P_2 P_1.$$

Thus, we know that we need only to calculate the M in

$$Y^\dagger \Lambda^T Y = \Lambda^T M,$$

and M^n would be the desired transforming matrix. Again, we use this property and see that we only need to calculate

$$e^{(i\tau_0/\hbar)(H_{\text{system}}+H_{\text{bath}}\pm H_{\text{int}})}\Lambda^T e^{-(i\tau_0/\hbar)(H_{\text{system}}+H_{\text{bath}}\pm H_{\text{int}})}. \tag{7}$$

For simplicity, we consider the ground-state situation. The procedure is entirely general and applies for the case of $T > 0$K. We compute the variance $\langle(\Delta X)^2\rangle$ with two types of coupling: namely, the Lorentzian spectrum and

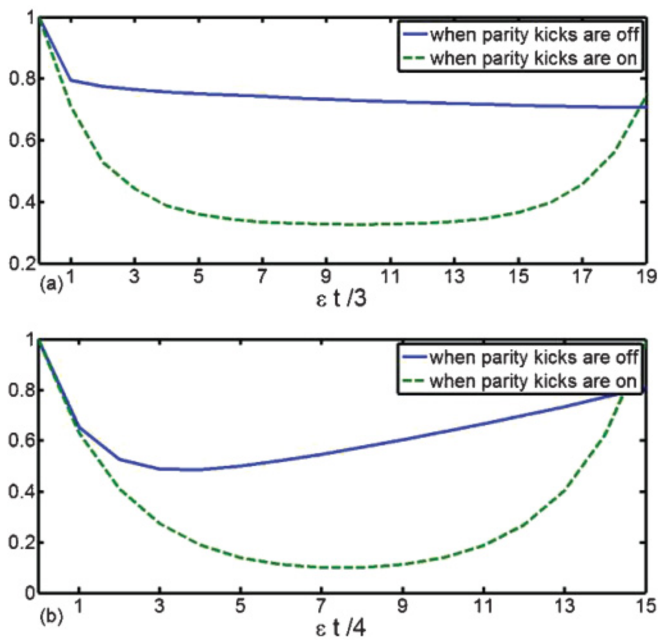

FIG. 1. (Color online) The variance $\langle(\Delta X)^2\rangle$ vs rescaled time ϵt when parity kicks are on and off respectively. Panel (a) refers to a Lorentzian spectrum and panel (b) refers to an Ohmic spectrum. We find that a better squeezing can be obtained when parity kicks are applied to the system.

the Ohmic spectrum. For the Lorentzian spectrum $\gamma_j = g(\omega_j) = \eta\Gamma/\sqrt{(\omega_j-\omega)^2+\Gamma^2}$, as an example of numerical calculations, we assume $\Gamma = 2\times 10^9$ Hz, $\eta = 5\times 10^7$ Hz, the squeezing parameter $\epsilon = 10^8$ Hz, and the kick period $\tau_0 = 1.67\times 10^{-9}$ s. For the Ohmic spectrum $\gamma_j = g(\omega_j) = \sqrt{\xi\omega_j}e^{-\omega_j/\omega_c}$, we assume $\xi = 10^6$ Hz, $\omega_c = 10^9$ Hz, the squeezing parameter $\epsilon = 5\times 10^7$ Hz, and the kick period $\tau_0 = 2.5\times 10^{-9}$ s. For both spectra, we assume the frequencies associated with the system and reservoir to be $\omega = 10^9$ Hz and $\omega_j = j\times 10^7$ Hz $(j = 1,2,\ldots,200)$ respectively. With these parameters, the variance $\langle(\Delta X)^2\rangle$ versus rescaled time ϵt is plotted in Fig. 1, which shows that a better squeezing can be obtained if parity kicks are applied to the system. To check the validity of our numerical calculation, we examine whether the commutation relations between creation and annihilation operators are satisfied, which is equivalent to whether the transforming matrix M of $a^\dagger$ and a is symplectic. We set

$$\Omega = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$$

and then calculate $L = M^T\Omega M - \Omega$. We find that the maximum element in L is less than 10^{-7} through all the time, which indicates the transforming matrix is very close to symplectic matrix.

VI. DISCUSSION

It is interesting to do a comparison between several methods, including the widely used Markovian master equation [13]. Under the Hamiltonian (3) with a Lorentzian spectrum as shown in Ref. [21], we can obtain an exact solution

$$\begin{aligned} a(t) &= \left[u(t)e^{-\Gamma t/2}a + \sum u_j(t)b_j\right]e^{-i\omega t} \\ &= \left\{\left[\cos(\Theta t/2) + \frac{\Gamma}{\Theta}\sin(\Theta t/2)\right] + \sum u_j(t)b_j\right\}e^{-i\omega t}. \end{aligned} \tag{8}$$

The constant Θ is given by $\Theta = \sqrt{4\pi\eta^2 D - \Gamma^2}$, where D is the density of reservoir modes and $u_j(t)$ are some complicated functions [21]. However, the master equation, in the Markovian approximation, can be written as

$$\begin{aligned} \frac{\partial\rho}{\partial t} &= -i\omega[a^\dagger a,\rho] + \frac{\lambda}{2}[2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a] \\ &\quad + \lambda\bar{n}[a^\dagger\rho a + a\rho a^\dagger - a^\dagger a\rho - \rho a a^\dagger], \end{aligned} \tag{9}$$

where $\lambda = 2\pi Dg(\omega)^2$ is a constant, which represents the decay rate of the harmonic oscillator. For simplicity, we assume the temperature of reservoir to be zero and the initial state of system to be $|1\rangle$. We then calculate the the probability $P(t) = \langle 1|\text{tr}_R[\rho(t)]|1\rangle$, which can be used to observe the decay of system. We can also compute $P(t)$ when the coupling strengths γ_k are constants. For example, we assume that parameters of the Lorentzian spectrum in Fig. 2(a) to be $\gamma_j = g(\omega_j) = 2.8209\times 10^{12}/\sqrt{(\omega_j-\omega)^2+10^{12}}$ Hz and the flat spectrum in Fig. 2(b) to be $\gamma_j = 5.6419\times 10^6$ Hz with $j = 1,2,\ldots,200$. For the flat spectrum, we assume the frequencies associated with the system and reservoir to be $\omega = 10^9$ Hz and $\omega_j = j\times$

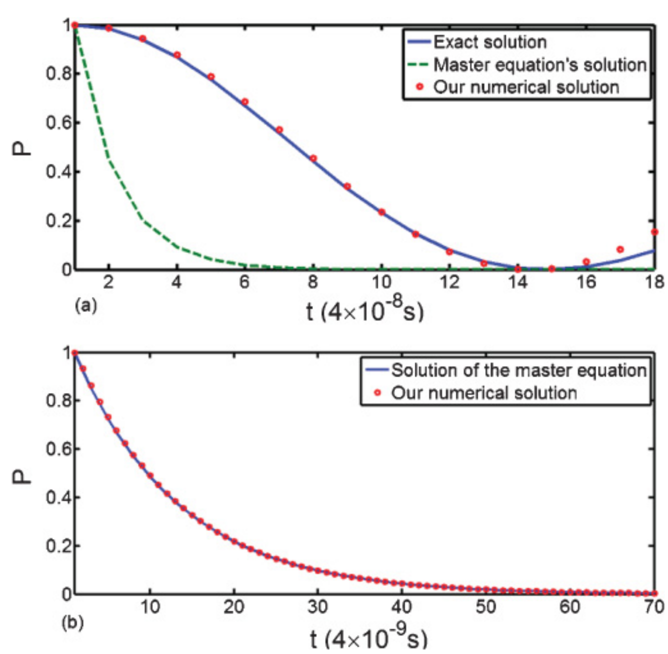

FIG. 2. (Color online) (a) The probability of the system being in state $|1\rangle$ with a Lorentzian spectrum. We can see that our numerical solution is close to the exact solution. On the other hand, we can see master equation is not valid for this situation. (We set the reservoir to have 200 equally distributed oscillators while getting the numerical solution of our method.) (b) The probability of the system being in state $|1\rangle$ when the coupling strengths γ_k are constant. We can see that the lines of our numerical solution and master equation's solution coincide with each other.

10^7 Hz ($j = 1,2,\ldots,200$), respectively. For the Lorentzian spectrum, however, we change the frequencies of the reservoir to be $\omega_j = (50 + j/2) \times 10^7$ Hz ($j = 1,2,\ldots,200$) due to the shape of the Lorentzian spectrum, which varies dramatically at the center and is negligible at two sides. By making this change, we can sample the spectrum better. Then we plot Fig. 2. We can find that while the master equation leads to a good approximate solution in some cases, it fails sometimes. Thus, our method is more reliable, and the accuracy can be further improved by using better numerical methods.

We also note that the parity kicks can be done by increasing the frequency of the harmonic oscillator for a short time interval. For example, this can be achieved in the ion trap by changing the electric field. (Also see Refs. [22,23] for schemes of generating squeezed states in ion traps.)

VII. CONCLUSION

We have shown that for a general Hamiltonian with bononic quadratic forms, we can compute dynamics of a system using exponential quadratic operators. Our method provides substantial improvement over computation involving master equations as we do not need to solve any differential equations, and it provides numerical solutions for Hamiltonians that can be written in quadratic forms of creation and annihilation operators. Thus, this technique compares well with the dynamics of the system under a master equation, but it is in some sense more appealing as it could provide analytical expressions for some cases. In particular, we analyze the effect of reservoir in a squeezing process, and we propose a possible scheme to improve the degree of squeezing. Our method can be applied to study the problem on the quantization of nanomechanical systems; further work will be presented elsewhere.

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[1] D. Braun, Phys. Rev. Lett. **89**, 277901 (2002); J. P. Paz and A. J. Roncaglia, *ibid.* **100**, 220401 (2008).
[2] N. Erez, G. Gordon, M. Nest, and G. Kurizki, Nature **452**, 724 (2008).
[3] B. Misra and E. C. G. Sudarshan, J. Math. Phys. **18**, 756 (1977).
[4] D. Vitali and P. Tombesi, Phys. Rev. A **59**, 4178 (1999).
[5] L. Viola and E. Knill, Phys. Rev. Lett. **94**, 060502 (2005).
[6] K. Khodjasteh and D. A. Lidar, Phys. Rev. A **75**, 062310 (2007).
[7] M. A. de Ponte, M. C. de Oliveira, and M. H. Y. Moussa, Ann. Phys. **317**, 72 (2005).
[8] A. G. Kofman and G. Kurizki, Phys. Rev. Lett. **93**, 130406 (2004).
[9] U. Herzog, Opt. Commun. **179**, 381 (2000)
[10] L.-A. Wu, M. S. Byrd, and D. A. Lidar, Phys. Rev. Lett. **89**, 127901 (2002).
[11] A. J. Leggett, S. Chakravarty, A. T. Dorsey, P. A. Fisher, A. Garg, and W. Zwerger, Rev. Mod. Phys. **59**, 1 (1987).
[12] U. Weiss, *Quantum Dissipative Systems* (World Scientific, Singapore, 2008).
[13] W. H. Louisell, *Quantum Statistical Properties of Radiation* (John Wiley and Sons, New York, 1973).
[14] C. W. Gardiner, *Quantum Noise* (Springer, Berlin, 1991).
[15] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, 1997).
[16] R. P. Feynman and F. L. Vernon, Ann. Phys. **24**, 181 (1963).
[17] A. O. Caldeira and A. J. Leggett, Physica A **121**, 587 (1983).
[18] R. Balian and E. Brezin, Nuovo Cimento **64**, 37 (1969).
[19] X. B. Wang, S. X. Yu, and Y. D. Zhang, J. Phys. A **27**, 6563 (1994).
[20] X. B. Wang, C. H. Oh, and L. C. Kwek, J. Phys. A **31**, 4329 (1998).
[21] Y. X. Liu, C. P. Sun, and S. X. Yu, Phys. Rev. A **63**, 033816 (2001).
[22] J. I. Cirac, A. S. Parkins, R. Blatt, and P. Zoller, Phys. Rev. Lett. **70**, 556 (1993).
[23] H. P. Zeng and F. C. Lin, Phys. Rev. A **52**, 809 (1995).