Non-Markovianity of the damped Jaynes-Cummings model with detuning

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(Received 25 February 2010; published 23 June 2010)

The degree of non-Markovian behavior of a damped Jaynes-Cummings model with detuning is investigated. Our attention is focused on the effects of the detuning and the width of the Lorentzian spectral density on the degree of non-Markovian behavior. It is found that an increase of the detuning can make the information exchange between the qubit and the reservoir more rapid, and this leads to an increase in the degree of non-Markovianity for some cases, while an increase of the spectral width always leads to a decrease in the degree of non-Markovianity.

DOI: 10.1103/PhysRevA.81.062124 PACS number(s): 03.65.Yz, 42.50.Lc, 03.65.Ta

I. INTRODUCTION

All realistic quantum systems are open to essentially uncontrollable environments that act as sources of decoherence and dissipation. Therefore the study of open quantum systems taking into account the effect of the environment on the dynamical evolution of the system of interest has become more and more interesting [1,2]. According to the character of the environment, the quantum dynamical processes can be simply classified into Markovian processes with no memory effect and non-Markovian ones with pronounced memory effect.

For memoryless Markovian open systems, the environment acts as a sink for the system information. Because of the system-reservoir interaction, the system of interest loses information into the environment, and this lost information plays no further role in the system dynamics. However, in the non-Markovian case, owing to the memory effect, the information lost by the system during the interaction with the environment will return to the system at a later time. This shows much more complicated behaviors than in the Markovian case.

Most of the results on open system dynamics are based on the Markovian approximation [1,2]. Recent studies have shown that the limits of Markovian and non-Markovian quantum processes play an increasingly important role in many fields of physics, such as quantum optics [2–4], solid state physics [5], and quantum information science [6,7]. Recently, non-Markovian dynamics has been investigated in many works [8–12].

In order to study non-Markovian dynamics quantitatively, following the first computable measure of Markovianity for quantum channels introduced in Ref. [13], some measures for the degree of non-Markovianity have been introduced [14–17]. Reference [14] shows that the difference between Markovian and non-Markovian processes can be measured through the continuous increment of the state distinguishability. The increment can be interpreted as the revival of information flow from the reservoir to the system.

When we consider particular models such as the damped Jaynes-Cummings model with detuning, the following main questions arise: What are the major determinants for the non-Markovianity and how does the non-Markovianity depend on them? The object of this paper is to study the degree of non-Markovian behavior of a damped Jaynes-Cummings model with detuning. We mainly focus on the effects of the detuning Δ and the width of the Lorentzian spectral density λ on the degree of non-Markovian behavior. We find that there exists a transition point of λ at which the process is divided into Markovian and non-Markovian regimes; the value of the transition point is determined by the value of the detuning Δ. We also give the physical origin of the nonmonotonic behavior of the non-Markovianity in the case of short reservoir correlation time.

The paper is organized as follows. In Sec. II, we present the model and its analytical solution. In Sec. III, we review briefly the measure for the degree of non-Markovian behavior and give the non-Markovianity of the model. Section IV is devoted to studying the effects of the detuning Δ and λ on the non-Markovianity. Finally, we give our conclusions in Sec. V.

II. THE MODEL

In this paper we consider a qubit with excited state $|e\rangle$ and ground state $|g\rangle$ that interacts with a reservoir formed by the quantized modes of a high-Q cavity. The total Hamiltonian of this typical model reads

$$H = \frac{1}{2}\omega_0\sigma_+\sigma_- + \sum_k \omega_k b_k^\dagger b_k + (\sigma_+ B + \sigma_- B^\dagger) \qquad (1)$$

with $B = \sum_k g_k b_k$, where ω_0 is the transition frequency of the qubit, and σ_+ and σ_- are the system raising and lowering operators, respectively; the index k labels the field modes of the reservoir with frequencies ω_k; $b_k^\dagger$ and b_k are respectively the modes' creation and annihilation operators; and g_k denote the coupling constants. At zero temperature, this Hamiltonian represents one of the few open quantum systems amenable to an exact solution [1,18,19].

If we restrict ourselves to the case of a single excitation in the whole system, we can expand the state of the total system at any time t as

$$|\phi(t)\rangle = c_1(t)|e\rangle_S|0\rangle_R + \sum_k c_k(t)|g\rangle_S|1_k\rangle_R, \qquad (2)$$

where $|0\rangle_R$ denotes the vacuum state of the reservoir and $|1_k\rangle_R$ is the state of the reservoir with only one excitation in the kth mode. Letting $t = 0$, we obtain an initial state of the form

$$|\phi(0)\rangle = c_1(0)|e\rangle_S|0\rangle_R + \sum_k c_k(0)|g\rangle_S|1_k\rangle_R. \qquad (3)$$

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The time evolution of these probability amplitudes is governed by a series of differential equations that is easily derived from the Schrödinger equation:

$$\dot{c}_1(t) = -i\sum_k g_k \exp[i(\omega_0-\omega_k)t]c_k(t), \quad (4)$$

$$\dot{c}_k(t) = -ig_k^* \exp[-i(\omega_0-\omega_k)t]c_1(t). \quad (5)$$

Assuming that there are no photons in the initial state, that is, $c_k(0)=0$, we solve Eq. (5) and substitute the solution into Eq. (4) to obtain a closed equation for $c_1(t)$:

$$\dot{c}_1(t) = -\int_0^t dt_1 f(t-t_1)c_1(t). \quad (6)$$

The correlation function $f(t-t_1)$ is related to the spectral density $J(\omega)$ of the reservoir by

$$f(t-t_1) = \int d\omega J(\omega)\exp[i(\omega_0-\omega)(t-t_1)]. \quad (7)$$

The exact form of $c_1(t)$ thus depends on the particular choice of the spectral density of the reservoir. Here, we investigate the detuning case of a Lorentzian spectral density [1]

$$J(\omega) = \frac{1}{2\pi}\frac{\gamma_0\lambda^2}{(\omega_0-\omega-\Delta)^2+\lambda^2}, \quad (8)$$

where $\Delta=\omega_0-\omega_c$ is the detuning of ω_c and ω_0, and ω_c is the center frequency of the cavity. It is worth noting that the effective coupling between the qubit and its environment decreases when the value of the detuning Δ increases. The parameter λ defines the spectral width of the reservoir and is connected to the reservoir correlation time $\tau_R=\lambda^{-1}$. On the other hand, the parameter γ_0 can be shown to be related to the decay of the excited state of the qubit in the Markovian limit of a flat spectrum. The relaxation time scale τ_S over which the state of the system changes is then related to γ_0 by $\tau_S=\gamma_0^{-1}$. In order to compute the exact probability amplitude $c_1(t)$, we evaluate the reservoir correlation function $f(t-t_1)$ using the spectral density $J(\omega)$:

$$f(t-t_1) = \frac{1}{2}\gamma_0\lambda\exp[-(\lambda-i\Delta)(t-t_1)]. \quad (9)$$

For this $f(t-t_1)$, the differential equation (6) for the probability amplitude $c_1(t)$ can be easily solved to give the exact solution by Laplace transform,

$$c_1(t) = c_1(0)h(t), \quad (10)$$

where

$$h(t) = e^{-(\lambda-i\Delta)t/2}\left[\cosh\left(\frac{dt}{2}\right)+\frac{\lambda-i\Delta}{d}\sinh\left(\frac{dt}{2}\right)\right] \quad (11)$$

with $d=\sqrt{(\lambda-i\Delta)^2-2\gamma_0\lambda}$.

The time-dependent decay rate is

$$\gamma(t) = \mathrm{Re}\left\{\frac{2\gamma_0\lambda\sinh(dt/2)}{d\cosh(dt/2)+(-i\Delta+\lambda)\sinh(dt/2)}\right\}, \quad (12)$$

where Re(z) denotes the real part of z. For the resonance case, namely, $\Delta=0$, Eq. (12) reduces to

$$\gamma(t) = \frac{2\gamma_0\lambda\sinh(d't/2)}{d'\cosh(d't/2)+\lambda\sinh(d't/2)} \quad (13)$$

with $d'=\sqrt{\lambda^2-2\gamma_0\lambda}$. This is the very time-dependent decay rate Eq. (10.47) of Ref. [1]. We should note that Eq. (13) holds for all cases except $\lambda=2\gamma_0$. When $\lambda=2\gamma_0$, the time-dependent rate reduces to $\gamma(t)=2\gamma_0$; when $\lambda>2\gamma_0$, $\gamma(t)>0$ for all $t>0$; when $\lambda>2\gamma_0$, $\gamma(t)<0$ for some time intervals.

III. THE DEGREE OF NON-MARKOVIANITY

The measure for non-Markovianity defined in Ref. [14] is

$$\mathcal{N} = \max_{\rho_{1,2}(0)}\int_{\sigma>0} dt\sigma(t,\rho_{1,2}(0)), \quad (14)$$

where $\sigma(t,\rho_{1,2}(0))$ is the rate of change of the trace distance, which can be defined as

$$\sigma(t,\rho_{1,2}(0)) = \frac{d}{dt}D(\rho_1(t),\rho_2(t)). \quad (15)$$

$D(\rho_1,\rho_2)=\frac{1}{2}|\rho_1-\rho_2|$ is the trace distance of the quantum states ρ_1 and ρ_2, describing the distinguishability between the two states and satisfying $0\leqslant D\leqslant 1$. For convenience, we abbreviate $\sigma(t,\rho_{1,2}(0))$ to $\sigma(t)$. We should note that $\sigma(t)\leqslant 0$ for all dynamical semigroups and all time-dependent Markovian processes, while, if there exists a pair of initial states and a certain time t such that $\sigma(t)>0$, the process is non-Markovian. Physically, this means that for non-Markovian dynamics the distinguishability of the pair of states increases at certain times. This can be interpreted as a flow of information from the environment back to the system, which enhances the possibility of distinguishing the two states.

From Eq. (14) we find that to obtain the degree of non-Markovianity we should take the maximum over all initial states $\rho_{1,2}(0)$, which is difficult to obtain because of optimization. However, by drawing a sufficiently large sample of random pairs of initial states, Breuer *et al.* [14] have shown by strong numerical evidence that for the off-resonant case the maximum is attained for the initial states $\rho_1(0)=|0\rangle\langle 0|$ and $\rho_2(0)=|1\rangle\langle 1|$. For other cases, any observed growth of the trace distance is a clear signature of non-Markovian behavior and leads to a lower bound for $\mathcal{N}$.

For these two initial states, the trace distance has a simple expression,

$$D(\rho_1(t),\rho_2(t)) = |h(t)|^2. \quad (16)$$

From Eq. (16) we can find a direct link between the time behavior of the single-qubit excited-state population $|h(t)|^2$ and the trace distance $D(\rho_1(t),\rho_2(t))$. This relation shows that for the off-resonant case a return of information from the reservoir back into the system is always accompanied by a feedback of energy.

After some calculations, we finally obtain the rate of change of the trace distance,

$$\sigma(t) = e^{-\lambda t}\{\mu[\cosh(at)-\cos(bt)]+\nu\sinh(at)-\xi\sin(bt)\}, \quad (17)$$

where a and b denote the real and the imaginary parts of d, respectively; $\mu=(1/2|d|^2)(\lambda a^2-\lambda b^2-\lambda\Delta^2-\lambda^3-2ab\Delta)$; $\nu=(1/2|d|^2)(a^3+ab^2-\lambda^2 a+a\Delta^2+2b\Delta\lambda)$; $\xi=(1/2|d|^2)(b^3+ba^2+\lambda^2 b-b\Delta^2+2a\Delta\lambda)$; and $|d|$ stands for the absolute value of d. Having Eqs. (14) and (17) in mind,

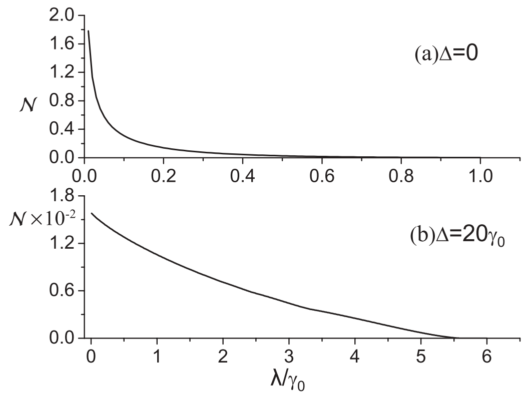

FIG. 1. Non-Markovianity $\mathcal{N}$ as a function of λ for two different values of Δ.

we can study the non-Markovianity of the model for different parameters.

IV. THE EFFECTS OF Δ AND λ ON THE NON-MARKOVIANITY

In this section, we consider the effects of the detuning and the width of the Lorentzian spectral density. Figure 1 shows the non-Markovianity $\mathcal{N}$ as a function of λ for $\Delta = 0$ (the resonance case) and $\Delta = 20\gamma_0$ (the detuning case). From Fig. 1 we see that for both cases the non-Markovianty decreases with increasing λ. This can be understood as follows. The parameter λ is connected to the reservoir correlation time $\tau_R = \lambda^{-1}$. The increase of λ indicates a decrease of the reservoir correlation time, and hence the non-Markoviantity becomes weaker.

Another intriguing aspect is how the non-Markovianity depends on the detuning Δ for certain λ. Figure 2(a) shows the non-Markovianity $\mathcal{N}$ as a function of Δ for $\lambda = 0.01\gamma_0$, namely, the case of long reservoir correlation time. From Fig. 2(a) we see that in this case the non-Markovianty decreases with increasing Δ. This can be illustrated as follows. The parameter Δ is the detuning between the center frequency of the cavity and the qubit transition frequency. When the value of the detuning Δ increases, the effective coupling between the qubit and the reservoir decreases, and thus the amount of information exchanged between the qubit and the reservoir is reduced. This leads to a decrease of the reversed

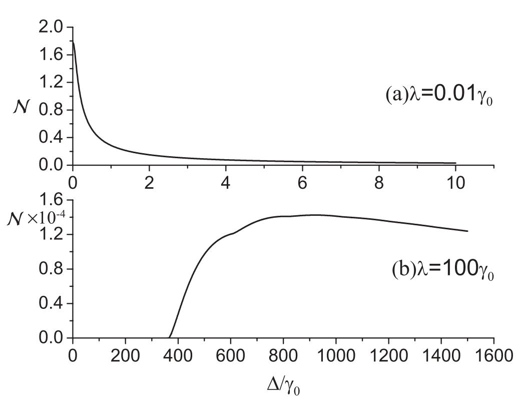

FIG. 2. Non-Markovianity $\mathcal{N}$ as a function of Δ for two different values of λ.

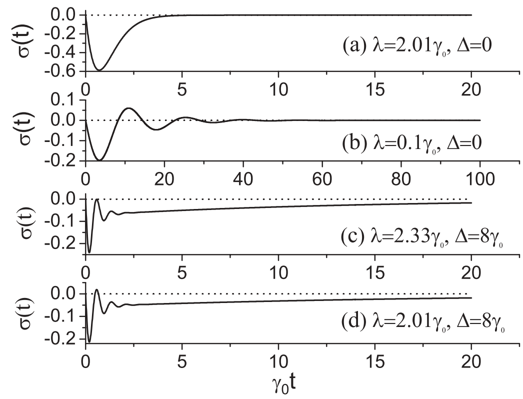

FIG. 3. Dynamics of the change rate of the trace distance for different values of λ and Δ.

information flow from environment to system. and hence the non-Markovianity becomes weaker.

Is this a general property for all values of λ? The answer is no. Figure 2(b) shows the non-Markovianity $\mathcal{N}$ as a function of Δ for $\lambda = 100\gamma_0$, namely, short reservoir correlation times. Figure 2(b) exhibits a nonmonotonic behavior: when $\Delta < 380\gamma_0$, the non-Markovianity is nearly zero, and then increases with increasing Δ; after it reaches a maximum value, it decreases with further increase of Δ. We note that similar behavior can be found in Ref. [14] but its physical origin is not exactly known.

In order to understand the physical origin of the non-monotonic behavior of the non-Markovianity, we investigate the effects of Δ on the change rate of the trace distance to gain insight into the physical processes characterizing the dynamics. From a numerical calculation, we find that λ and Δ determine the presence or the absence of oscillations around the stationary value 0 of the change rate of the trace distance. Figure 3 shows the dynamics of the change rate of the trace distance for different values of λ and Δ. In the resonance case ($\Delta = 0$), when $\lambda > 2\gamma_0$, there is no oscillation and $\sigma(t)$ converges to zero for long times [Fig. 3(a)]. This indicates that the behavior of the system is Markovian and irreversible decay occurs. When $\lambda < 2\gamma_0$, oscillations are present and for some time intervals $\sigma(t)$ is larger than zero. This indicates that in these time intervals the flow of energy and information from the system to the environment is reversed and then non-Markovian dynamics occurs [Fig. 3(b)]. In other words, the transition between the Markovian and the non-Marlovian regimes occurs at the transition point $\lambda_t = 2\gamma_0$.

For the detuning case (namely $\Delta \neq 0$), numerical calculation shows that, for some values of λ that are larger than $2\gamma_0$, $\sigma(t)$ exhibits oscillations, and for some time intervals $\sigma(t)$ is larger than zero. This indicates that the process is non-Markovian and the transition point should be different from that in the resonance case. Take the case $\Delta = 8\gamma_0$ as an example; we find that the transition point is $\lambda_t = 2.32\gamma_0$, that is, when $\lambda > 2.32\gamma_0$ there is no positive value of $\sigma(t)$ [see Fig. 3(c)], while, when $\lambda < 2.32\gamma_0$, $\sigma(t)$ is positive in some time intervals [see Fig. 3(d)].

The numerical calculation also shows that the value of the transition point depends on the detuning Δ. In Fig. 4 we give the values of the transition point for different values of Δ.

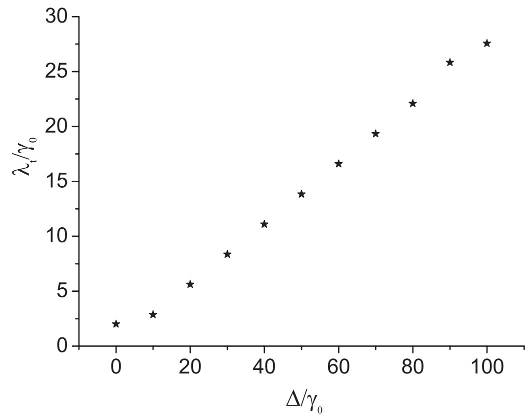

FIG. 4. The transition point for different values of Δ.

Figure 4 shows that the values of the transition point increase with increasing Δ. This can be understood as follows. It is well known [3] that for the Jaynes-Cummings model, the atomic inversion exhibits Rabi oscillation as time evolves and the Rabi frequency is determined by the detuning, that is, the larger the detuning, the larger the Rabi frequency. We know that the oscillations of the atomic inversion represent the exchanging of energy between the atom and the field. When the Rabi frequency becomes larger, the energy exchange will become more rapid. This means that an increase of the detuning makes the exchange between the atom and the field more rapid. From Eq. (16), we know that there is a direct link between the information exchange and the energy. Thus the result shown above can be generalized to our case, that is, an increase of the detuning makes the energy (and at the same time the information) exchange between the qubit and reservoir more rapid too. This will lead to a quick reversal of information from reservoir to qubit. As a result, the non-Markovian phenomenon occurs for a correlation time that is not long.

Then the physical origin of the nonmonotonic behavior shown in Fig. 2(b) can be explained as follows. Remember that the increase of the detuning has two kinds of effect on the dynamics of the system, that is, (i) it makes the information exchange between the qubit and the reservoir more rapid and (ii) it makes the effective coupling between the qubit and the reservoir decrease. The non-Markovianity is determined by the competition between the two effects, that is, when the former plays the dominating role, the non-Markovianity will increase with increasing Δ; otherwise, the non-Markovianity will decrease with increasing Δ.

V. CONCLUSION

In this paper, we have studied the effects of the detuning Δ and the width of the Lorentzian spectral density λ on the degree of non-Markovian behavior of a damped Jaynes-Cummings model with detuning. The non-Markovianity is measured by the the non-Markovian behavior measure given in Ref. [14]. We find that, for the off-resonant case, a return of information from the reservoir back into the system is always accompanied by a feedback of energy. We also find that there exists a transition point that divides the dynamics into Markovian and non-Markovian regimes; the value of the transition point is determined by the detuning Δ: the larger the detuning Δ, the larger the value of the transition point. In addition, we show that the increasing of the detuning has two kinds of effect on the dynamics of the system, that is, (i) it makes the information exchange between the qubit and the reservoir more rapid and (ii) it makes the effective coupling between the qubit and the reservoir decrease. The competition between the two effects can lead to nonmonotonic behavior of the non-Markovianity in the case of short reservoir correlation time.

ACKNOWLEDGMENTS

This work is financially supported by the Fundamental Research Fund of Beijing Institute of Technology (Grant No. 20081742010) and the National Science Foundation of China (Grant No. 10974016). J.G.L. wishes to thank Xiu-San Xing for valuable discussions.

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