Entanglement dynamics of nonidentical oscillators under decohering environments

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We study the evolution of entanglement for a pair of coupled nonidentical harmonic oscillators in contact with an environment. For both cases of a common bath and of two separate baths for each of the oscillators, a full master equation is provided without rotating-wave approximation. The entanglement dynamics is analyzed as a function of the diversity between the oscillators' frequencies and their positive or negative mutual coupling and also the correlation between the occupation numbers. The singular effect of the resonance condition (identical oscillators) and its relationship with the possibility of preserving asymptotic entanglement are discussed. The importance of the bath's memory properties is investigated by comparing Markovian and non-Markovian evolutions.

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I. INTRODUCTION

Coupled harmonic oscillators are the first approximation for a broad class of extended systems not only in different fields of physics, but also in chemistry and biology. Within the quantum formalism, they are the basis of the description of electromagnetic-field interactions in quantum optics and approximate lattice systems in different traps in atomic physics [1,2]. Moreover, in the last few years, there has been impressive progress toward the cooling and back-action evasion measurement of macroscopic—in terms of number of atoms—harmonic oscillators, by allowing the observation of their quantum behavior. Two main classes of systems experimentally realized are nanoelectromechanical structures (NEMS) [3,4] and different types of optomechanical systems [5] where nano- and micromechanical devices, cavities, or suspended mirrors, respectively, are coupled to single electrons or light. As an example, the observation of NEMS extremely near to the ground state of motion, with an occupation factor of just 3.8 [4], has recently been reported. These experiments would allow the observation of coherent quantum-superposition states, entanglement, and the study of decoherence processes in a controllable manner on massive solid-state objects.

Phenomena associated with the *coupling* of these quantum oscillators have been revisited within the context of quantum information in many theoretical studies during the last decade [6–12]. Entropy and entanglement in extended systems with many degrees of freedom (harmonic chains or lattices) [6] have been characterized in fundamental and thermal states by exploring scaling laws and connections with phase transitions [7]. An important advantage is that these systems admit Gaussian-state solutions having a well-defined [13] computable [14] measure of entanglement, the logarithmic negativity [15]. The question of the generation of entanglement has also been addressed by considering oscillators whose parameters are modulated in time [8,9]. In these studies, losses were generally neglected, while recently, the effects of decoherence on a pair of entangled oscillators have been considered in the presence of dissipation through baths of infinite oscillators [10–12]. Our aim, in this paper, is to analyze a rather unexplored aspect of this problem, that is, *the effect of the diversity* on the entanglement between coupled harmonic oscillators in different situations. Indeed, instead of considering identical oscillators, we look at the effects of detuning between their frequencies ω_1 and ω_2.

The interest about diversity effects on entanglement is both theoretical and related to experimental issues. We mention, for instance, the effect of diversity of frequencies for two photons entering a beam splitter. This leads to a completely different output with respect to the case of indistinguishable photons [16,17].[1] Coupling between different harmonic modes has also been extensively studied in quantum optics in the presence of nonlinear interactions and parametric coupling, by allowing, for instance, generation of entanglement between photons pairs and intense light beams [19,20] of different colors. In that context, however, the frequency diversity of each pair of oscillators is compensated by a third mode and is, in general, not relevant, while here we focus on pairs of mechanical oscillators off-resonance and with constant couplings. The main expected consequence is indeed an effective decoupling of the oscillators due to fast rotation of their interaction term, and we will show the effects on the robustness of entanglement.

It is clear that the identity of the oscillators, in general, is a very peculiar and strong assumption, which is not always justified and introduces a symmetry into the system with extreme consequences. It is indeed important to clarify the effects of relaxation of this symmetry by introducing some diversity. In an extension from two coupled oscillators to an array, this will imply a break in the translational symmetry. Apart from the fundamental interest, we point out that, in experimental realization of engineered arrays of massive quantum oscillators, some diversity between them might actually be unavoidable. Coupled oscillators with different frequencies have also been suggested for quantum-limited measurements [21]. Finally, in many experiments, coupled oscillators actually model different physical entities (for instance, radiation and a moving mirror in optomechanics), and a symmetric Hamiltonian would describe only a very special case [22].

Our analysis of entanglement evolution encompasses both diversity between harmonic oscillators and dissipation. Once

[1] Notice that the robustness of equal light modes or quantum interference has recently been tested against another kind of diversity—on states—, which is dissimilar to the source that is considered [18].

the oscillators are prepared in some entangled state (for instance, through a sudden switch of their coupling), we look at its robustness for increasing diversity and coupling strength. It is well known that an object whose degrees of freedom are all coupled to an environment will decohere into a thermal state with the same temperature as the heat bath [2]. The thermal state, unless temperatures are very low, is separable and highly entropic so that after thermalization, all entanglement shared between the coupled oscillators disappears [23]. As the dissipating oscillators reach a separable state in a finite time, and not asymptotically, the name of sudden death has been suggested [24]. Still, under certain conditions on the ways in which the oscillators dissipate, entanglement can survive asymptotically [25]. The transition from a quantum to a classical behavior in these systems does not only convey fundamental interest, but it is also important in view of applications of harmonic systems that operate at the quantum limit, which include quantum information processing and not shot-noise-limited measurements of displacement, forces, or charges [26]. These phenomena have generally been studied for identical oscillators and, in particular, previous works provided a master equation description in the case of a fully symmetric Hamiltonian [10,11]. Off-resonance oscillators, in the presence of a common bath, have recently been considered in Ref. [12] in the case of vanishing asymptotic coupling, by showing that lower temperatures are needed to asymptotically maintain entanglement. In this work, we systematically analyze the role of diversity on the entanglement dynamics and provide the full master equations for $\omega_1 \neq \omega_2$: (i) both for common and for separate baths for the two oscillators, (ii) without the approximation of rotating waves, which is generally assumed when modeling light fields, and (iii) by comparing results to the non-Markovian case.

Recently there have been some works focusing on memory effects and non-Markovianity. In the case of continuous variables, non-Markovian effects on entanglement evolution have been discussed [10,27,28]. In Ref. [27], the authors considered two identical oscillators coupled to separate baths in the very high-temperature regime and analyzed how the matching between the frequency of the oscillators and the spectral density of the bath affects both Markovian and non-Markovian dynamics. In Ref. [10], the non-Markovian evolution has been compared with the case where both the Markovian limit and the rotating-wave approximation in the system-bath coupling are taken. Since the effects of these two approximations are not easily separable, it is difficult to single out non-Markovian corrections from that analysis. For the sake of comparison, our study of entanglement dynamics in the presence of diversity is also extended to the non-Markovian case, by looking at both common and separate baths and by showing that deviations are often actually negligible.

In Sec. II, we introduce the model of a pair of oscillators with different frequencies and coupled through position both between them and with the baths of oscillators. Non-Markovian master equations in the weak-coupling limit with one common and two separate baths are presented, and all the details are provided in the Appendix. Temporal decay of entanglement between the oscillators and the correlations between the occupation numbers are shown in Sec. III by analyzing the role of frequency diversity when their coupling strength also is varied. Apart from the entanglement dynamics, we also discuss its robustness (asymptotic entanglement for a common bath) in the context of the symmetry of the system in Sec. III B. Non-Markovian deviations from these results are shown in Sec. IV, and further discussion and conclusions are left for Sec. V.

II. MODEL AND MASTER EQUATIONS

We consider two harmonic oscillators with the same mass and different frequencies coupled to a thermal bath. As discussed in Ref. [29], depending on the distance between the two oscillators, different modelizations of the system-bath interaction can be performed. We will discuss the case of two distant objects, which amounts to considering the coupling with two independent baths, and compare it with the zero-distance scenario (common bath). By analyzing the role of diversity on the quantum features of this system, we also generalize previous works on identical oscillators that dissipate in common and separate baths [10]. The nonresonant problem has been addressed by Paz and Roncaglia in Refs. [11,12], where they studied entanglement dynamics of two uncoupled oscillators that interact only through a common bath.

A. Separate baths

The model Hamiltonian, with each oscillator coupled to an infinite number of oscillators (separate baths), is $H^{\rm sep} = H_S + H_B^{\rm sep} + H_{\rm SB}^{\rm sep}$. The system Hamiltonian

$$H_S = \frac{p_1^2}{2} + \frac{1}{2}\omega_1^2 x_1^2 + \frac{p_2^2}{2} + \frac{1}{2}\omega_2^2 x_2^2 + \lambda x_1 x_2 \tag{1}$$

describes two oscillators with different frequencies $\omega_{1,2}$, coupled through their positions, while

$$H_B^{\rm sep} = \sum_k \sum_{i=1}^{2} \left(\frac{P_k^{(i)2}}{2} + \frac{1}{2}\Omega_k^{(i)2} X_k^{(i)2} \right) \tag{2}$$

is the free Hamiltonian of two (identical) bosonic baths, and

$$H_{\rm SB}^{\rm sep} = \sum_k \lambda_k^{(1)} X_k^{(1)} x_1 + \sum_k \lambda_k^{(2)} X_k^{(2)} x_2 \tag{3}$$

encompasses the system-bath interaction.

The master equation for the reduced density matrix of the two oscillators, up to the second order in $H_{\rm SB}^{\rm sep}$ (weak-coupling limit) is, by assuming $\hbar = 1$,

$$\frac{d\rho}{dt} = -i[H_S,\rho] - \frac{1}{2}\sum_{i,j=1}^{2} \{ i\epsilon_{ij}^2 [x_i x_j,\rho] + D_{ij} [x_i,[x_j,\rho]] + i\Gamma_{ij} [x_i,\{p_j,\rho\}] - F_{ij} [x_i,[p_j,\rho]] \}. \tag{4}$$

In the Appendix, we give an explicit derivation of the master equation together with the definition of the coefficients. Then, ρ is subject to energy renormalization (ϵ_{ij}), dissipation Γ_{ij}, and diffusion D_{ij}, F_{ij}. Coefficients in Eq. (4) depend on time. In spite of the non-Markovianity, the master equation is known to always be local in time within the weak coupling limit [30]. Moreover, the exact master equation for identical coupled oscillators with a common bath has also been shown to be local in time [31,32].

In Sec. III, we will consider the Markovian limit [performed sending $t \to \infty$ in Eqs. (A17)–(A20)]. Non-Markovian corrections will be discussed in Sec. IV. We anticipate that we will focus on the Markovian limit because corrections caused by such an approximation turn out to be negligible in most cases, both for equal or for different frequencies ω_1 and ω_2, and for common or for separate baths.

To obtain an explicit expression for $\epsilon_{ij}, D_{ij}, \Gamma_{ij}$, and F_{ij}, we need to know the density of states of the baths, defined for both of them as

$$J(\Omega) = \sum_k \frac{\lambda_k^2}{\Omega_k} \delta(\Omega - \Omega_k). \tag{5}$$

Here, we will explicitly consider the Ohmic environment with a Lorentz-Drude cutoff function, whose spectral density is

$$J(\Omega) = \frac{2\gamma}{\pi} \Omega \frac{\Lambda^2}{\Lambda^2 + \Omega^2}. \tag{6}$$

We learned from the master equation, Eq. (4), that the system Hamiltonian is renormalized because of the presence of ϵ_{ij}, and this renormalization turns out to depend on the frequency cutoff Λ. This undesirable nonphysical effect can be removed by adding to the initial Hamiltonian counterterms [33], which exactly compensate the asymptotic values of ϵ_{ij}. This is accomplished by replacing $\epsilon_{ij}(t)$ with $\epsilon_{ij}(t) - \epsilon_{ij}(\infty)$ in Eq. (4). Then, in the Markovian case, we will simply drop them from the master equation. This renormalization procedure amounts to redefining the natural frequency in terms of the observed frequency.

B. Common bath

Now, we introduce the case of a common bath modeled by

$$H_B^c = \sum_k \left(\frac{P_k^2}{2} + \frac{1}{2} \Omega_k^2 X_k^2 \right), \tag{7}$$

and we consider the same spectral density in Eq. (6). The interaction term reads

$$H_{\rm SB}^c = \sum_k \lambda_k X_k (x_1 + x_2), \tag{8}$$

which means that the system is coupled to the bath through the mode $x_+ = (x_1 + x_2)/\sqrt{2}$, while the mode $x_- = (x_1 - x_2)/\sqrt{2}$ is decoupled. The master equation in this case is

$$\begin{aligned}\frac{d\rho}{dt} = & -i[H_S,\rho] - \frac{i}{2} \sum_{i \neq j} \left(\bar{\epsilon}_{ii}^2 - \bar{\epsilon}_{jj}^2\right) x_j \rho x_i \\ & - \frac{1}{\sqrt{2}} \sum_{i=1}^{2} \{ i\bar{\epsilon}_{ii}^2 [x_i x_+, \rho] + \bar{D}_{ii} [x_+, [x_i, \rho]] \\ & + i\bar{\Gamma}_{ii} [x_+, \{p_i, \rho\}] - \bar{F}_{ii} [x_+, [p_i, \rho]] \}\end{aligned} \tag{9}$$

(see the Appendix for the definition of the coefficients). Coupled systems that dissipate in a common bath have recently been a subject of interest because of the possibility for asymptotically preserving entanglement at high temperatures, which makes this a major distinctive feature with respect to the case of separate baths [11]. In Sec. III B, we will study entanglement evolution with Eq. (9) by revisiting the possibility for asymptotically maintaining entanglement in the presence of diversity (frequency detuning). As a matter of fact, whereas, in the case of equal frequencies, the decoupled mode x_- is an eigenmode of the isolated system, this is no longer true once $\omega_1 \neq \omega_2$. We will see how asymptotic entanglement can be preserved in the presence of diversity, through a proper choice of the system-bath coupling constants.

C. Symmetry in λ sign

A closer look at the form of the total Hamiltonian in the presence of separate baths allows us to single out its symmetry properties. $H^{\rm sep}$ is clearly modified by the transformation $\lambda \to -\lambda$. However, once a canonical transformation U is introduced such that $Ux_2U^\dagger = -x_2$ and $Up_2U^\dagger = -p_2$, it is easy to show that the master equation is invariant under the combined action of U and the flip of λ. In fact, while H_S is left unmodified, the change in $H_{\rm SB}^{\rm sep}$ (that would correspond to flip $\lambda_k^{(2)}$ in $-\lambda_k^{(2)}$) is ineffective with respect to the evolution of the reduced density matrix, where ρ is determined only by contributions proportional to $(\lambda_k^{(i)})^2$. A simple consequence of this symmetry can be observed by studying the evolution of the two-mode squeezed state,

$$|\Psi_{\rm TMS}\rangle = \sqrt{1-\mu} \sum_{n=0}^{\infty} \mu^{n/2} |n\rangle|n\rangle, \tag{10}$$

where $\mu = \tanh^2 r$, and r is the squeezing amplitude. For this state, the canonical transformation U amounts to changing r to $-r$. Then, the same evolution must be obtained by considering λ in $H^{\rm sep}$ and by considering a given squeezing r in the initial condition, or $-\lambda$ and $-r$.

All these considerations are true only in the case of separate baths. Once the common bath is taken, because of the action of U, the mode coupled to the bath is x_- instead of x_+. Therefore, in the case of separate baths, results for opposite λ are equivalent to a change in the sign of the initial squeezing, while this is not the case for a common bath.

III. ENTANGLEMENT AND QUANTUM CORRELATIONS

The calculation of bipartite entanglement for mixed states is generally an unsolved task. Nevertheless, the criterion of positivity of the partial transposed density matrix of Gaussian two-mode states ρ^{T_B} is necessary and sufficient for their separability [13,34,35]. The amount of entanglement can be measured through the logarithmic negativity, defined as $E_{\mathcal{N}} = \log_2 ||\rho^{T_B}||$, where $||\rho^{T_B}||$ is the trace norm of ρ^{T_B} [14]. For any two-mode Gaussian state, the logarithmic negativity is $E_{\mathcal{N}} = \max[0, -\log_2 2\lambda_-]$, where λ_- is the smallest symplectic eigenvalue of ρ^{T_B}.

Because the system is connected to heat baths, the dissipating degree of freedom will ultimately attain the thermal state corresponding to the system Hamiltonian. The properties of entanglement in the thermal state are well studied, whence, it is known that only for very low temperatures, entanglement is present between the two oscillators [23]. Therefore, we focus our attention on the time evolution of the decoherence process itself, namely, how fast an initially entangled state decays into a separable one, for separate (Sec. III A) and common (Sec. III B) baths.

A. Entanglement decay for separate baths

We consider, as a nonseparable initial state, the two-mode squeezed vacuum Eq. (10), with squeezing parameter r. This state has an amount of entanglement proportional to r and consists of the orthogonal modes $x_\pm = (x_1 \pm x_2)/\sqrt{2}$, which are simultaneously squeezed and stretched, respectively, by an amount r. Such a state, for infinite squeezing, gives the maximally entangled state for continuous variables, which has been known for a long time because of the famous paper by Einstein *et al.* [36].

The entanglement evolution in the presence of separate baths is shown in Fig. 1(a) for different parameter choices, by showing the effects of increasing the coupling strength and the detuning. Entanglement, in general, decays with an oscillatory behavior with an oscillation amplitude that depends on the λ strength [9]. We study the last time at which entanglement vanishes, t_F. In Figs. 2 and 3, we scan how long it takes for such an entangled state (with $r = 2$) to become separable when the oscillators' detuning and coupling strength are varied. Indeed, the last time t_F is represented in Fig. 3 for different values of the ratio between frequencies (ω_2/ω_1) and the coupling between oscillators (λ/ω_1^2).

For this choice of parameters, Fig. 2 shows that entanglement is present for a few tens of periods. Due to the scaling of temperature, time, and couplings with the frequency of the first oscillator ω_1, it is clear that results shown in Figs. 2 and 3 are not symmetric by exchange of the oscillators' role. In other words, even if it is physically equivalent to have the first oscillator with half the frequency of the second one or the second oscillator with half the frequency of the first one, due to the scaling, these figures are not symmetrical with respect to the line $\omega_2/\omega_1 = 1$.

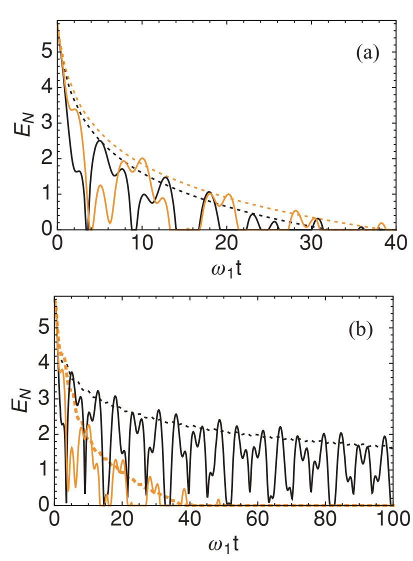

FIG. 1. (Color online) Time evolution of an entangled two-mode squeezed initial state Eq. (10) with $r = 2$, in the case of (a) separate baths and (b) a common bath. The parameters are $k_B T = 10\omega_1$, $\gamma = 0.001\omega_1$, and cutoff frequency $\Lambda = 50\omega_1$. The four curves correspond to the points in Figs. 2 and 4, namely, A (dotted black), B (solid black), C [dotted orange (gray)], and D [solid orange (gray)]. Two features are apparent: In the presence of a nonzero coupling λ (continuous lines), entanglement oscillates strongly; and second, in the case of separate baths, any initial entanglement vanishes fast, while, in the case of a common bath, it can be made to survive by having identical frequencies.

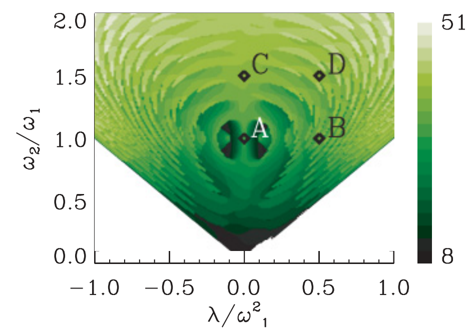

FIG. 2. (Color online) Parameter scan of t_F (see text) as a function of the oscillators' coupling and frequency detuning, in the case of separate baths. We have restricted the scan within a limited detuning ($\omega_2 \leqslant 2\omega_1$) as a result of the absence of any important features outside this region. In addition, the condition $\lambda < \omega_1\omega_2$ ensures reality of the eigenfrequencies in the problem. The behavior of t_F is basically monotonic in ω_2/ω_1, with a superimposed arenalike shape, which comes from the oscillatory nature of entanglement. See text for more details. The dots A, B, C, and D are given in Fig. 1 as examples.

Figure 2 shows two main features: (i) an increase of the survival time t_F when ω_2/ω_1 is increased; (ii) an increase of t_F with $|\lambda|$ in the absence of diversity ($\omega_2 = \omega_1$), while, for $\omega_2 \gtrsim 2\omega_1$, this dependence is weakened. This can also be appreciated in the perspective presented in Fig. 3. Both features can be explained in terms of the eigenmodes of the system $Q_\pm$ in Eq. (A10) and their eigenfrequencies $\Omega_\pm$ in Eq. (A9). Since the eigenmodes do not interact with each other, they can be regarded as independent channels for decoherence. The eigenmodes will only interact with near-resonant frequencies in the bath. In addition, all variances at the thermal states they approach are dependent upon the fraction $\Omega_\pm/2k_B T$.

As far as feature (i) is concerned, in the absence of coupling ($\lambda = 0$, implying $\Omega_{+,-} = \omega_{2,1}$) and for $\omega_2 \to \infty$, the effective temperature of the final thermal state reached by the eigenmode Q_+ will vanish, $T_{\text{eff},+} = k_B T/\omega_2 \to 0$.

The eigenmodes will, therefore, reach a thermal state ($T_{\text{eff},-} = k_B T/\omega_1$) and a ground state ($T_{\text{eff},+} = 0$),

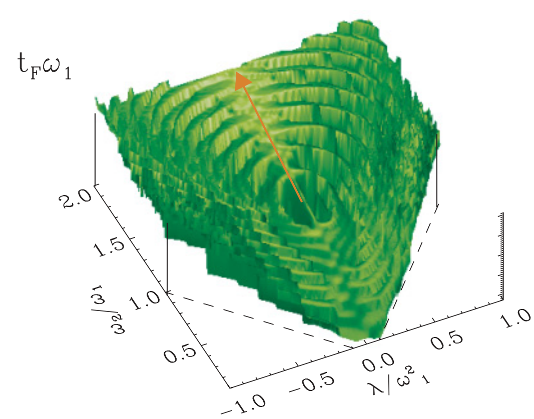

FIG. 3. (Color online) Three-dimensional representation of Fig. 2. Here, the arenalike shape can be better appreciated.

respectively. Since the presence of entanglement is a competition between the reduced purities of individual oscillators and the total purity [37], just by improving the final purity of oscillator Q_+, its time evolution has an overall higher purity, thus, making entanglement higher (and its survival time longer). This effect is equivalent to reducing the real temperatures of the baths (while keeping everything else constant). While the decoherence is given by the coupling γ and is, hence, not reduced, the reached final state is purer; and, thus, entanglement is seen to survive longer.

As far as feature (ii) is concerned, for $\omega_2 = \omega_1$, the eigenfrequencies of the system are $\Omega_\pm = \sqrt{\omega_1^2 \pm |\lambda|}$. By increasing $|\lambda|$ up to ω_1^2, the eigenfrequency Ω_- vanishes, and the amount of bath modes that have a similar frequency, given by the spectral density, also vanishes [$J(\Omega_- \to 0) \to 0$]. That is, the amount of bath modes with which this degree of freedom interacts tends to zero, and, therefore, the amount of decoherence suffered. This effect, however, is partially compensated by the fact that a vanishing Ω_- would imply that this mode will reach a thermal state of effective infinite temperature. Were it not so, the entanglement would survive asymptotically, as in the case of a common bath, where, for $\omega_2 = \omega_1$, the mode Ω_- does not decohere.

We then see that all major effects related to entanglement decay between coupled oscillators in the presence of heat baths can be explained in terms of: (1) eigenfrequencies (where they lie within the spectral density of the heat baths), and (2) effective temperatures reached by the eigenmodes after thermalization.

A minor feature is that the increase of t_F with λ and ω_2 is not completely smooth due to entanglement oscillations, and this gives rise to the arena-shaped dependence that is clearer in Fig. 3. In general, the dependence of the decoherence time on the oscillation frequency of the system is negligible for optics experiments at ambient temperatures ($T_{\mathrm{eff},\pm} = k_B T/\Omega_\pm \simeq 0$), but becomes extremely important in the presence of the lower frequencies of mechanical oscillators [38].

Notice that we have restricted the coupling within the boundary $|\lambda| < \omega_1\omega_2$ (triangle in the lower part of Fig. 2), otherwise, one of the eigenfrequencies (Ω_-) would become imaginary, which means that trajectories would be unbounded which, although not unphysical in principle, leads to leaks in any type of experiment.

B. Common bath

Now, let us compare the previous results with the oscillators' evolution with dissipation through a common bath, Eq. (9). This model can be considered a limit case while separate baths would model the opposite case, when baths are completely uncorrelated. The presence of asymptotic entanglement in this situation has been studied in Refs. [10,11,25], by showing that entanglement can survive for sufficiently low temperatures (or conversely, for high enough initial squeezing). In the case in which there are different frequencies, Paz and Roncaglia noticed in Ref. [12] that this asymptotic behavior disappears as the detuning is increased, which means that it is highly dependent on the frequency matching between oscillators and the fact that each oscillator's

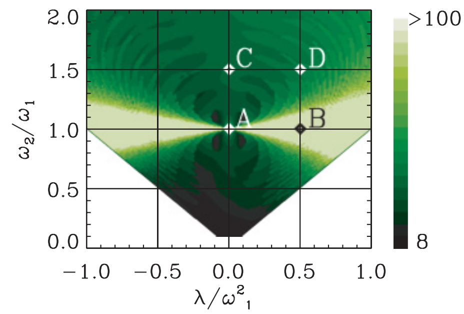

FIG. 4. (Color online) Parameter scan of t_F (see text) as a function of the oscillators' coupling and frequency detuning, in the case of a common bath. As seen here, t_F is huge whenever the detuning is small and on the order of tens of periods for high detuning. As expected, at resonance, one of the eigenmodes decouples from the baths, which leads to an infinite t_F. We have truncated the plot at heights of $\omega_1 t_F = 210$, otherwise, this mountain riff is infinitely high (but only strictly infinite when $\omega_1 = \omega_2$).

bath can be regarded as perfectly correlated to the other one. In physical realizations, these two conditions will hardly be met, and it will be interesting to discover the effect of deviations from it. We have then studied the effects of frequency diversity in the case of a common bath, by looking again at the robustness of entanglement in terms of the decay time t_F [Fig. 1(b)]. In Figs. 4 and 5, we show results equivalent to those represented in Figs. 2 and 3 but for a common bath. There, we see that, in the resonant case, entanglement never vanishes, and so the survival time in that case is infinite. In Fig. 5, diverging times t_F are recognized along the line $\omega_1 = \omega_2$.

When the oscillators' frequencies begin to differ, survival diminishes fast to values similar to the uncorrelated bath's case. This phenomenon can be determined from the master equation, Eq. (9). The fact that the coupling to the bath is $\gamma(x_1 + x_2)$ establishes that the mode x_- remains decoupled at all times, thus, by keeping its coherence and by contributing positively to a nonzero entanglement. If the frequencies are different, that mode gets coupled to x_+, which is itself coupled to the bath, and its decoherence is transferred to x_-. This way, both modes

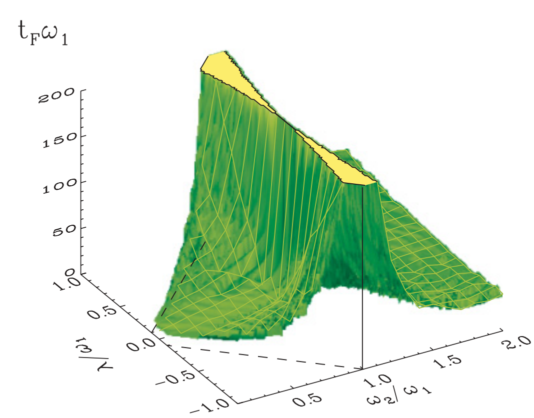

FIG. 5. (Color online) Three-dimensional representation of Fig. 4. Apart from the riff at equal frequencies, the shape is very similar to the case of separate baths, with an arenalike bowl shape. We have drawn a mesh to guide the reader's eye. Again, heights over $t_F = 210/\omega_1$ have been truncated (yellow region).

are decohered and end up in a thermal state, with no presence of entanglement. Hence, the survival time is finite. Because of this, it is necessary to stress the fact that even if the frequency difference is infinitesimal, decoherence will eventually appear, although it does so in a time inversely proportional to the infinitesimal. Thus, increasing the difference in frequencies just exacerbates the transfer velocity of decoherence from mode x_+ to mode x_-. This argument shows how artificial the equal frequencies assumption is.

We point out that asymptotic entanglement can also be found in the presence of frequency diversity and is not related to the symmetry that is present for $\omega_1 = \omega_2$. As a matter of fact, if the couplings to the common bath are different for each oscillator (i.e., $\gamma_1 \neq \gamma_2$), we can restore asymptotic entanglement even off-resonance, for $\omega_1 \neq \omega_2$. By noticing the relation between $x_{1,2}$ with the eigenmodes of the system $Q_\pm$, we see that the mode Q_+ can be uncoupled from the bath when the angle θ fulfills

$$\cos\theta = \frac{\gamma_1}{\sqrt{\gamma_1^2 + \gamma_2^2}}, \tag{11}$$

with $\theta = \theta(\omega_1,\omega_2,\lambda)$ given in the Appendix. Thus, half of the system is kept in a pure state, and we can show that entanglement survives asymptotically exactly in the same fashion as it did for equal frequencies before. In other words, with different couplings to the baths together with different frequencies, the system can retain asymptotic entanglement if Eq. (11) is fulfilled. In any case, this phenomenon is as unique as the equal frequencies phenomenon, and requires fine-tuned parameters (by tuning bath couplings or frequency difference) in order to be observable. It should then be regarded as exceptional as well. Our argument clarifies that asymptotic entanglement is *not* a consequence of a symmetry in the system, but *comes from a finely tuned decoherent-free degree of freedom of the system.*

C. Twin oscillators

We have considered entanglement as an indicator of the quantumness of our system, measurable for the family of (Gaussian) states considered here. The discrimination between the predictions of classical and quantum theories for coupled harmonic oscillators has also been mainly studied in optics. In that context, there have been many theoretical predictions experimentally confirmed, focusing on the violation of different classical inequalities or the positivity of variances [39]. An example is the variance of the difference of the occupation numbers:

$$d = \langle : (n_1 - n_2)^2 : \rangle, \tag{12}$$

where, as usual, n_i is the occupation number operator of each oscillator. d has been considered in a two-mode squeezed state generated by parametric oscillators in optics to characterize twin beams [1]. The quantum character of the correlations in the occupation numbers comes from the negativity of the variance d and can only follow from the negativity of the corresponding quasiprobability, the Glauber-Sudarshan representation in this case. Here, in analogy with the optical case, we consider *twin* oscillators by looking at the temporal dynamics of d. The correspondent fourth-order moments can be obtained from the covariance matrix because the states we are dealing with are Gaussian. In Fig. 6, we plot two examples that compare the evolution of this variance and entanglement for the common and separate baths cases. We have taken only the negative part of this correlation—identifying quantum behavior—and inverted it [by plotting max$(0,-d)$] for ease of comparison with entanglement. For the initial entangled state in Eq. (10), $d = -2\mu/(1-\mu)$ is always negative. In other words, for squeezed states, the occupation number of one oscillator determines the other occupation number. We observe that, by starting from this value, d decays with large oscillations and that the sudden deaths of entanglement and of this correlation coincide up to a fraction of a period. Still, entanglement evolution is smoother, and twin oscillators temporarily lose their quantum correlations even for entangled oscillators.

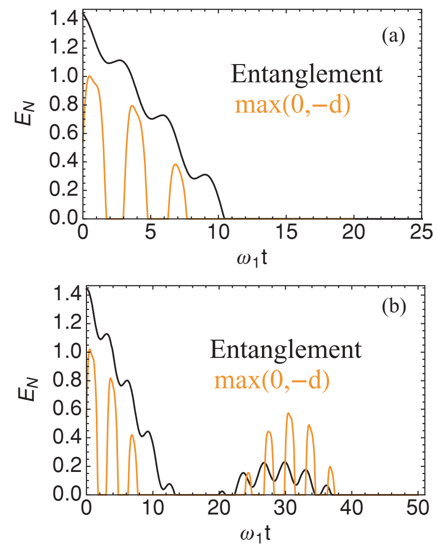

FIG. 6. (Color online) Comparison between the time evolutions of entanglement (black) and the correlation measure max$[0, -\langle : (n_1 - n_2)^2 : \rangle]$ [orange (gray)], for the case of (a) separate baths and (b) a common bath. We have used equal frequencies, coupling $\lambda = 0.1\omega_1^2$, damping $\gamma = 0.001\omega_1$, temperature $k_B T/\omega_1 = 10$, and initial squeezing parameter $r = 0.5$. The envelope of the correlation measure seems to be closely related to the entanglement, except for a delay of half a period.

IV. NON-MARKOVIAN EVOLUTION

Until now, we have considered the Markovian master equation, in the sense that the time-dependent coefficients related to the heat bath have been replaced by their asymptotic value, obtained by integrating up to infinite time. This implies a complete absence of memory in the bath, which does not retain instantaneous information on the dynamics of the two oscillators. However, this simplification can be dropped, and we are left with a complete non-Markovian description *within the weak-coupling approximation*. In this section, we analyze how these corrections affect the entanglement evolution we discussed so far.

In Ref. [27], a comparison has been made by considering temperatures 2 orders of magnitude greater than the frequency cutoff. The authors found that, when the frequency of the oscillators falls inside the bath-spectral density, entanglement persists for a longer time than in a Markovian channel. When there is no resonance between reservoir and oscillators, non-Markovian correlations accelerate decoherence and generate entanglement oscillations.

As is known, when all the important frequencies are much lower than the cutoff, the density of states of Eq. (6) is expected to generate almost memoryless friction [40]. Then, by taking $\omega_1 \ll \Lambda$, we compared Markovian and non-Markovian entanglement dynamics for different choices of the parameters of the system and the bath $(\omega_2/\omega_1,\lambda/\omega_1^2,\gamma/\omega_1,k_BT/\omega_1,r)$. The following considerations are valid for separate baths as well as for a common environment. In the case of very small coupling ($\gamma \sim 10^{-3}\omega_1$), non-Markovian corrections are practically unobservable, independent of the value of the system parameters up to relevant temperatures. To observe some appreciable deviations, we need to reach temperatures on the order of $10\omega_1$. This result is understood by considering that, for small values of the system-bath coupling, the role played by the bath itself is highly reduced. Since non-Markovian corrections on the values of the coefficients are relevant only during the first stage of the evolution, it can also be predicted that if the initial state is robust enough (as, for instance, in the case of squeezing $r = 2$ discussed in the previous sections), non-Markovianity will have marginal effects. It is, in fact, true that, in the case of $\gamma \sim 10^{-3}\omega_1$ and $T \sim 10\omega_1$, a correction to the entanglement decay time can only be observed by assuming low initial squeezing ($r \lesssim 0.1$). When γ starts to increase, it is possible to deal with cases where some observable corrections due to non-Markovianity also emerge in the low-temperature regime. In particular, the case of equal frequencies turns out to be the most sensitive one. To give an example, in Fig. 7, we compared the two evolutions for two different values of r. Starting from $r = 2$, non-Markovian corrections are almost negligible, by giving corrections of a few percent to the value

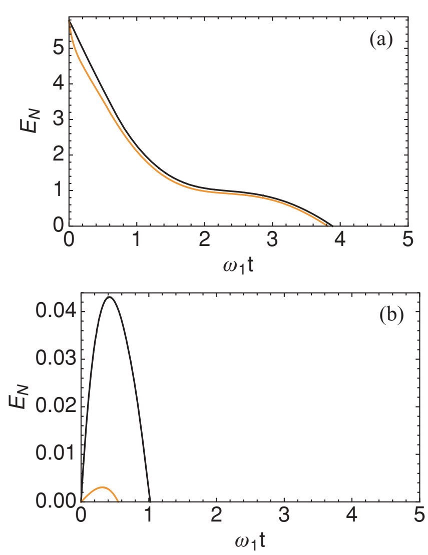

FIG. 7. (Color online) Comparison between Markovian [orange (gray)] and non-Markovian (black) entanglement dynamics. The system parameters are $\omega_1 = \omega_2$, $\lambda = 0.2\omega_1^2$, $\gamma = 0.01 \times 2/\pi\omega_1$, $\Lambda = 20\omega_1$, and $k_BT = 10\omega_1$. The initial squeezings are (a) $r = 2$ and (b) $r = 0$.

of t_F found within the Markovian approximation. In order to observe a noticeable difference, we must start with a factorized state ($r = 0$). As can be observed in Fig. 7, in this case, while memoryless baths are able to support a very small quantity of entanglement, the initial kick given by non-Markovian coefficients enhances entanglement generation, its maximum amount is amplified for about 1 order of magnitude, and its death time is increased. It is, however, worth noting that the absolute value reached is, in any case, very low.

V. DISCUSSION AND CONCLUSION

During the past few years, a series of papers has investigated entanglement dynamics of coupled harmonic oscillators in contact with thermal environments. Nevertheless, an analysis concerning the role played by the diversity of the two oscillators was missing, with a few exceptions [11,12,41]. Here, we have studied nonidentical oscillators from a broad perspective, by considering the effects of common and separate baths, oscillators' detuning and coupling strength, as well as non-Markovianity.

We started our discussion with the case of separate baths, which seems to be more physically meaningful. As one should expect, after a transient, unless the temperature is very low, the initial two-mode squeezed state $|\Psi_{\rm TMS}\rangle$ becomes separable. By studying the decay time, we observed a series of interesting features that we have qualitatively explained. First of all, by increasing ω_2/ω_1 and by keeping all other parameters unchanged, entanglement survival time increases as well. In the limit of $\omega_2/\omega_1 \gg 1$, Ω_+ and Ω_- tend to ω_2 and ω_1, respectively. Since the asymptotic thermal state that the system is reaching depends on the frequencies, the purity of the final state is enhanced together with the coherence time necessary to reach it. On the other hand, near $\omega_2 = \omega_1$, entanglement survival time can also be raised by increasing $|\lambda|$. When $|\lambda| \to \omega_1^2$, Ω_- is close to zero. Here, we argue that the behavior we observe is the result of two competing effects. If, on one hand, the thermal equilibrium state is less pure because of the presence of a vanishing frequency, on the other hand, the number of decohering channels goes to zero, since Ω_- falls in a poorly populated region of the spectral density, and increases, in this way, the time necessary to reach the thermal equilibrium.

If the two oscillators share a common bath, the observation of asymptotic entanglement at relevant temperatures becomes possible. One of the two eigenmodes of the system can result, in fact, decoupled from the environment. As a consequence of this decoupling, the asymptotic density matrix will result as the direct sum of the matrix of a pure state and a thermal state, corresponding to the mode that interacts with the bath. When the two oscillators have identical frequencies and bath couplings, the mode x_- gets decoupled, independent of the value of λ, giving rise to the plateau we observe in Figs. 4 and 5. It is important to stress that this regime can be achieved not only when the oscillators are resonant, but also in the presence of frequency detuning, through proper engineering of the system-bath interaction. Conversely, if their bath couplings are different $\gamma_1 \neq \gamma_2$, frequencies ω_1 and ω_2 can be found so that this special regime is achieved.

We also studied the dynamical evolution of the degree of quantumness of the system by means of the quantity d, which was introduced in Eq. (12). The negativity of this quantity is direct evidence of the negativity of the Glauber-Sudarshan quasiprobability distribution of the state. The agreement between the behavior of d and the entanglement is worth investigating, since a strong relationship between them is not known.

As for the Markovian versus the non-Markovian debate, we compared the two behaviors in a wide range of parameters. While, in Ref. [27], the authors investigated what changes when the frequency of the oscillators goes outside the spectral distribution of the bath in the high-temperature regime, we tried to conduct an extensive analysis for $\omega_1,\omega_2 \ll \Lambda$. Our results indicate that non-Markovian corrections can be observed, and are important, only for a reduced subset of initial conditions. To see them, we must put a relevant coupling between oscillators and bath, ω_1 should be close to ω_2, and the initial state should be (almost) disentangled. Since the master equation has been obtained in the weak-coupling limit, the first of these assumptions seems, at least, questionable. If we release only one of them, Markovian and non-Markovian evolutions are almost identical.

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APPENDIX: DERIVATION OF THE MASTER EQUATION

A system-bath model is usually described through the Hamiltonian $H = H_S + H_B + V$, where H_S refers to the system, H_B refers to the bath, and where V is the term containing the interaction. Generally, the coupling term can be written as $V = \sum_k S_k \otimes B_k$, where S_k denotes system operators and B_k denotes bath operators. The master equation for the reduced density matrix to the second order in the coupling strength (for a general discussion, see, for instance, Ref. [42]) is given by

$$\begin{aligned}\frac{d\rho}{dt} &= -i[H_S,\rho]\\ &\quad - \int_0^t d\tau \sum_{l,m}\{C_{l,m}(\tau)[S_l\tilde{S}_m(-\tau)\rho - \tilde{S}_m(-\tau)\rho S_l]\\ &\quad + C_{m,l}(-\tau)[\rho\tilde{S}_m(-\tau)S_l - S_l\rho\tilde{S}_m(-\tau)]\}. \qquad (A1)\end{aligned}$$

Here, $C_{l,m}(\tau)$ are the bath's correlation functions, defined by

$$C_{l,m}(\tau) = \mathrm{Tr}_B[\tilde{B}_l(\tau)\tilde{B}_m(0)R_0], \qquad (A2)$$

where $R_0 = (e^{-\beta H_B}/\mathrm{Tr}e^{-\beta H_B})$, the equilibrium density matrix of the bath. In Eqs. (A1) and (A2), the tilde indicates the interaction picture taken with respect to $H_0 = H_S + H_B$. For instance, $\tilde{B}_l(\tau) = e^{iH_0\tau}B_l e^{-iH_0\tau}$.

In the following, the correlation functions will appear combined as $C^A_{l,m}(\tau) = C_{l,m}(\tau) + C_{m,l}(-\tau)$ and $C^C_{l,m}(\tau) = C_{l,m}(\tau) - C_{m,l}(-\tau)$.

A. Separate baths

In our model, where the Hamiltonian is H^{sep}, we can identify $B_1 = \sum_k \lambda_k^{(1)} X_k^{(1)}$ and $B_2 = \sum_k \lambda_k^{(2)} X_k^{(2)}$ and, correspondingly, $S_1 = x_1, S_2 = x_2$. Their expressions in the interaction picture are

$$\tilde{B}_i = \sum_k \lambda_k^{(i)}\left[X_k^{(i)}\cos\Omega_k^{(i)}t + P_k^{(i)}\frac{\sin\Omega_k^{(i)}t}{m\Omega_k^{(i)}}\right], \qquad (A3)$$

and $\tilde{x}_i(\tau) = \alpha_{i1}x_1 + \alpha_{i2}x_2 + \beta_{i1}p_1 + \beta_{i2}p_2$, with

$$\alpha_{11} = \cos^2\theta\cos\Omega_-\tau + \sin^2\theta\cos\Omega_+\tau, \qquad (A4)$$

$$\alpha_{12} = \frac{\sin 2\theta}{2}(\cos\Omega_+\tau - \cos\Omega_-\tau), \qquad (A5)$$

$$\beta_{11} = \frac{\cos^2\theta}{\Omega_-}\sin\Omega_-\tau + \frac{\sin^2\theta}{\Omega_+}\sin\Omega_+\tau, \qquad (A6)$$

$$\beta_{12} = \frac{\sin 2\theta}{2}\left(\frac{\sin\Omega_+\tau}{\Omega_+} - \frac{\sin\Omega_-\tau}{\Omega_-}\right), \qquad (A7)$$

and $\alpha_{22}(\theta) = \alpha_{11}(\pi/2-\theta)$, $\beta_{22}(\theta) = \beta_{11}(\pi/2-\theta)$, $\alpha_{21} = \alpha_{12}$, and $\beta_{21} = \beta_{12}$. The coefficients appearing in Eqs. (A4)–(A7) are defined as

$$\theta = \frac{1}{2}\arctan\left(\frac{2\lambda}{\omega_2^2 - \omega_1^2}\right), \qquad (A8)$$

and

$$\Omega_\pm = \sqrt{\frac{\omega_1^2 + \omega_2^2}{2} \pm \frac{\sqrt{4\lambda^2 + (\omega_2^2 - \omega_1^2)^2}}{2}}. \qquad (A9)$$

The normal modes for the position are

$$Q_+ = \cos\theta x_2 + \sin\theta x_1, \qquad (A10)$$

$$Q_- = \cos\theta x_1 - \sin\theta x_2, \qquad (A11)$$

while, for the momentum,

$$P_+ = \cos\theta p_2 + \sin\theta p_1, \qquad (A12)$$

$$P_- = \cos\theta p_1 - \sin\theta p_2. \qquad (A13)$$

Notice that, for $\omega_1 = \omega_2, \theta = \pi/4$. This implies $\alpha_{11} = \alpha_{22}$ and $\beta_{11} = \beta_{22}$.

Since the two baths are identical and are uncorrelated, we have $C_{1,1}(\tau) = C_{2,2}(\tau) = C(\tau)$ and $C_{1,2}(\tau) = C_{2,1}(\tau) = 0$. From the knowledge of the density of states $J(\Omega)$, it is possible to obtain the explicit expression of $C^C(\tau)$ and $C^A(\tau)$:

$$C^C(\tau) = -i\int_0^\infty d\Omega J(\Omega)\sin\Omega\tau, \qquad (A14)$$

$$C^A(\tau) = \int_0^\infty d\Omega J(\Omega)\cos\Omega\tau\coth\frac{\Omega}{2k_BT}. \qquad (A15)$$

The master equation reads

$$\frac{d\rho}{dt} = -i[H_S,\rho] - \int_0^t d\tau \sum_{i=1,2} \left\{ [x_i,\{x_i(-\tau),\rho\}]\frac{C^C(\tau)}{2} + [x_i,[x_i(-\tau),\rho]]\frac{C^A(\tau)}{2} \right\}. \quad \text{(A16)}$$

By using the expressions in Eqs. (A4)–(A7), it can be written as in Eq. (4), with

$$\epsilon_{ij}^2 = -i\int_0^t d\tau \alpha_{ij} C^C(\tau), \quad \text{(A17)}$$

$$D_{ij} = \int_0^t d\tau \alpha_{ij} C^A(\tau), \quad \text{(A18)}$$

$$F_{ij} = \int_0^t d\tau \beta_{ij} C^A(\tau), \quad \text{(A19)}$$

$$\Gamma_{ij} = i\int_0^t d\tau \beta_{ij} C^C(\tau). \quad \text{(A20)}$$

The considerations about symmetry given in Sec. II can be understood in terms of the explicit form of the master equation. The substitution $\lambda \to -\lambda$ is equivalent to sending θ to $-\theta$. Then, α_{12} and β_{12} change their sign. However, since these coefficients multiply one operator (position or momentum) of the first oscillator and one operator of the second operator, the canonical transformation U compensates the change of sign, and the evolution of ρ is unchanged.

The explicit form of $\epsilon_{ij}, D_{ij}, \Gamma_{ij}$, and F_{ij}, in the Markovian case, can be calculated using the equalities

$$\lim_{t\to\infty} \int_0^t d\tau e^{-i(\omega-\omega_0)\tau} = \pi\delta(\omega-\omega_0) + i\frac{\mathrm{P}}{\omega-\omega_0}, \quad \text{(A21)}$$

where P denotes the Cauchy principal value, and

$$\coth\frac{\omega}{2k_BT} = 2k_BT \sum_{n=-\infty}^{+\infty} \frac{\omega}{\omega^2+\nu_n^2}, \quad \text{(A22)}$$

with $\nu_n = 2\pi n k_B T$.

From the master equation, a set of closed equations of motion for the average values of the second moments can be derived $(i,j=1,2)$:

$$\frac{d\langle x_i x_j\rangle}{dt} = \langle p_i x_j + p_j x_i\rangle, \quad \text{(A23)}$$

$$\begin{aligned}\frac{d\langle p_i p_j\rangle}{dt} =& -\frac{1}{2}\big(\omega_i^2+\epsilon_{ii}^2\big)\langle\{x_i,p_j\}\rangle - \frac{1}{2}\big(\omega_j^2+\epsilon_{jj}^2\big)\langle\{x_j,p_i\}\rangle \\ & - \frac{1}{2}\big(\lambda+\epsilon_{12}^2\big)\sum_{k=1}^{2}[\langle\{x_k,p_i\}\rangle(1-\delta_{k,j}) \\ & + \langle\{x_k,p_j\}\rangle(1-\delta_{k,i})] - (\Gamma_{ii}+\Gamma_{jj})\langle p_i p_j\rangle \\ & - \Gamma_{12}\sum_{k=1}^{2}[\langle p_k p_i\rangle(1-\delta_{k,j}) \\ & + \langle p_k p_j\rangle(1-\delta_{k,i})] + D_{ij}, \end{aligned} \quad \text{(A24)}$$

$$\begin{aligned}\frac{d\langle\{x_i,p_j\}\rangle}{dt} =& 2\langle p_i p_j\rangle - 2\big(\omega_j^2+\epsilon_{jj}^2\big)\langle x_i x_j\rangle - 2\big(\lambda+\epsilon_{12}^2\big) \\ & \times \big[\big\langle x_i^2\big\rangle(1-\delta_{ij}) + \delta_{ij}\langle x_1 x_2\rangle\big] \\ & - \Gamma_{jj}\langle\{x_i,p_j\}\rangle - \Gamma_{12}\langle\{x_i,p_i\}\rangle + F_{ij}. \end{aligned} \quad \text{(A25)}$$

B. Common bath

In this case, $C_{1,1}(\tau) = C_{2,2}(\tau) = C_{1,2}(\tau) = C_{2,1}(\tau)$. Then, the master equation can be written as

$$\frac{d\rho}{dt} = -i[H_S,\rho] - \int_0^t d\tau \sum_{i,j=1,2} \left\{ [x_i,\{x_j(-\tau),\rho\}]\frac{C^C(\tau)}{2} + [x_i,[x_j(-\tau),\rho]]\frac{C^A(\tau)}{2} \right\}, \quad \text{(A26)}$$

and, by using Eqs. (A17)–(A20), assumes the form given in Eq. (9) with $\bar{D}_{ii} = D_{ii} + D_{12}$ and similarly for ϵ^2, Γ, and F. The equations of motion for the second moments read $(i,j=1,2)$

$$\frac{d\langle x_i x_j\rangle}{dt} = \langle p_i x_j + p_j x_i\rangle, \quad \text{(A27)}$$

$$\begin{aligned}\frac{d\langle p_i p_j\rangle}{dt} =& -\sum_{k=1}^{2}\big(\omega_k^2+\bar{\epsilon}_{kk}^2\big)(\langle x_k p_j\rangle\delta_{ik} + \langle p_i x_k\rangle\delta_{jk}) \\ & - \frac{1}{2}\sum_{k=1}^{2}\big(\lambda+\bar{\epsilon}_{ii}^2\big)[\langle\{x_k,p_i\}\rangle(1-\delta_{kj}) \\ & + \langle\{x_k,p_j\}\rangle(1-\delta_{ki})] + \frac{1}{2}(\bar{D}_{ii}+\bar{D}_{jj}) \\ & - (\bar{\Gamma}_{ii}+\bar{\Gamma}_{jj})\langle p_i p_j\rangle - \sum_{k=1}^{2}\bar{\Gamma}_{kk}[\langle p_i p_k\rangle(1-\delta_{kj}) \\ & + \langle p_j p_k\rangle(1-\delta_{ki})], \end{aligned} \quad \text{(A28)}$$

$$\begin{aligned}\frac{d\langle\{x_i,p_j\}\rangle}{dt} =& 2\langle p_1 p_2\rangle - 2\big(\omega_j^2+\bar{\epsilon}_{jj}^2\big)\langle x_i x_j\rangle \\ & - 2\sum_{k=1}^{2}\big(\lambda+\bar{\epsilon}_{kk}^2\big)\langle x_i x_k\rangle(1-\delta_{ik})\delta_{ij} \\ & - 2\big(\lambda+\bar{\epsilon}_{ii}^2\big)\big\langle x_i^2\big\rangle(1-\delta_{ij}) - \bar{\Gamma}_{jj}\langle\{x_i,p_j\}\rangle \\ & - \sum_{k=1}^{2}\bar{\Gamma}_{kk}\langle\{x_i,p_k\}\rangle(1-\delta_{ik}) + \bar{F}_{ii}. \end{aligned} \quad \text{(A29)}$$

[1] R. Loudon, *The Quantum Theory of Light* (Oxford University Press, New York, 2000).

[2] S. Haroche and J.-M. Raimond, *Exploring the Quantum: Atoms, Cavities, and Photons* (Oxford University Press, New York, 2006).

[3] A. Naik, O. Buu, M. D. LaHaye, A. D. Armour, A. A. Clerk, M. P. Blencowe, and K. C. Schwab, Nature (London) **443**, 193 (2006).

[4] T. Rocheleau, T. Ndukum, C. Macklin, J. B. Hertzberg, A. A. Clerk, and K. C. Schwab, Nature (London) **463**, 72 (2010).

[5] F. Marquardt and S. M. Girvin, Phys. **2**, 40 (2009), and references therein; A. Schliesser *et al.*, Nat. Phys. **5**, 509 (2009); Y.-S. Park and H. Wang, *ibid.* **5**, 489 (2009); S. Groblacher *et al.*, *ibid.* **5**, 485 (2009).
[6] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008).
[7] M. Srednicki, Phys. Rev. Lett. **71**, 666 (1993); K. Audenaert, J. Eisert, M. B. Plenio, and R. F. Werner, Phys. Rev. A **66**, 042327 (2002); M. B. Plenio, J. Hartley, and J. Eisert, New J. Phys. **6**, 36 (2004); R. G. Unanyan and M. Fleischhauer, Phys. Rev. Lett. **95**, 260604 (2005).
[8] J. Eisert, M. B. Plenio, S. Bose, and J. Hartley, Phys. Rev. Lett. **93**, 190402 (2004).
[9] F. Galve and E. Lutz, Phys. Rev. A **79**, 032327 (2009).
[10] K.-L. Liu and H.-S. Goan, Phys. Rev. A **76**, 022312 (2007).
[11] J. P. Paz and A. J. Roncaglia, Phys. Rev. Lett. **100**, 220401 (2008).
[12] J. P. Paz and A. J. Roncaglia, Phys. Rev. A **79**, 032102 (2009).
[13] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).
[14] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[15] K. Zyczkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Phys. Rev. A **58**, 883 (1998).
[16] C. K. Hong, Z. Y. Ou, and L. Mandel, Phys. Rev. Lett. **59**, 2044 (1987).
[17] C. Santori, D. Fattal, J. Vucaronkovic, G. S. Solomon, and Y. Yamamoto, Nature (London) **419**, 594 (2002).
[18] A. J. Bennett, R. B. Patel, C. A. Nicoll, D. A. Ritchie, and A. J. Shields, Nat. Phys. **5**, 715 (2009).
[19] P. D. Drummond and Z. Ficek, *Quantum Squeezing* (Springer, Berlin, 2004).
[20] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. **77**, 513 (2005).
[21] P. Leaci and A. Ortolan, Phys. Rev. A **76**, 062101 (2007).
[22] S. Gröblacher, K. Hammerer, M. R. Vanner, and M. Aspelmeyer, Nature (London) **460**, 724 (2009).
[23] J. Anders, Phys. Rev. A **77**, 062102 (2008).
[24] See review, T. Yu, and J. H. Eberly, Science **323**, 598 (2009), and references therein.
[25] J. S. Prauzner-Bechcicki, J. Phys. A **37**, L173 (2004).
[26] M. Schlosshauer, *Decoherence and the Quantum-to-Classical Transition* (Springer, Berlin, 2007); X. L. Feng, C. J. White, A. Hajimiri, and M. L. Roukes, Nat. Nanotech. **3**, 342 (2008); T. J. Kippenberg and K. J. Vahala, Science **321**, 1172 (2008).
[27] S. Maniscalco, S. Olivares, and M. G. A. Paris, Phys. Rev. A **75**, 062119 (2007).
[28] R. Vasile, S. Olivares, M. G. A. Paris, and S. Maniscalco, Phys. Rev. A **80**, 062324 (2009).
[29] T. Zell, F. Queisser, and R. Klesse, Phys. Rev. Lett. **102**, 160501 (2009).
[30] N. Hashitsume, F. Shibata, and M. Shingu, J. Stat. Phys. **17**, 155 (1977).
[31] B. L. Hu, J. P. Paz, and Y. Zhang, Phys. Rev. D **45**, 2843 (1992).
[32] C.-H. Chou, T. Yu, and B. L. Hu, Phys. Rev. E **77**, 011112 (2008).
[33] A. O. Caldeira and A. J. Leggett, Physica A **121**, 587 (1983).
[34] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[35] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[36] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[37] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. Lett. **92**, 087901 (2004).
[38] G. Giorgi, F. Galve, and R. Zambrini (unpublished).
[39] M. D. Reid and D. F. Walls, Phys. Rev. A **34**, 1260 (1986).
[40] U. Weiss, *Quantum Dissipative Systems*, 2nd ed. (World Scientific, Singapore, 1999).
[41] R. Vasile, e-print arXiv:1002.0973.
[42] C. W. Gardiner and P. Zoller, *Quantum Noise*, 2nd ed. (Springer, Berlin, 2000).