Joint measurement of two unsharp observables of a qubit

Sixia Yu,[1,2] Nai-le Liu,[1] Li Li,[1] and C. H. Oh[2]

[1]*Hefei National Laboratory for Physical Sciences at Microscale and Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China*

[2]*Centre for Quantum Technologies and Physics Department, National University of Singapore, 2 Science Drive 3, Singapore 117542*

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We derive a single inequality as the sufficient and necessary condition for two unsharp observables of a two-level system to be jointly measurable in a single apparatus and construct explicitly the joint observables of two jointly measurable observables. By introducing a generalized distinguishability as a measure of the unsharpness of an unsharp measurement, we derive a complementarity inequality, which generalizes Englert's duality inequality, from the condition of joint measurement of two orthogonal unsharp observables.

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I. INTRODUCTION

Built in the standard formalism of quantum mechanics, there are mutually exclusive but equally real aspects of quantum systems, as summarized by the complementarity principle of Bohr [1]. Mutually exclusive aspects are often exhibited via noncommuting observables, for which the complementarity is quantitatively characterized by two kinds of uncertainty relationships, namely, the preparation uncertainty relationships (PUR's) and the measurement uncertainty relationships (MUR's).

The PUR's stem from the semipositive definiteness of the density matrix describing the quantum state and characterize the predictability of two noncommuting observables in a given quantum state. To test PUR's two different projective measurements will be performed on two identically prepared ensembles of the quantum system and these measurements cannot be performed within one experimental setup on a single ensemble.

On the other hand, the MUR's characterize the trade-off between the precisions of unsharp measurements of two noncommuting observables in a single experimental setup. The very first effort of Heisenberg [2] in deriving the uncertainty relationships was based on a simultaneous measurement of the position and momentum, with the rigorous form of the MUR established recently by Werner [3]. In the interferometry the wave-particle duality between the path information and the fringe visibility of the interference pattern is characterized quantitatively by Englert's duality inequality [4], which turns out to be originated from the joint measurability of two special unsharp observables encoding the path information and the fringe visibility [5]. To establish a general MUR the condition for joint measurement has to be explored, which can be turned into some kinds of MUR's when equipped with a proper measure of the precisions or unsharpnesses (e.g., distinguishability).

In this paper we shall consider the joint measurability of two general unsharp observables of a qubit and derive a simple necessary and sufficient condition with joint observables explicitly constructed. We also present a MUR arising from the condition of joint measurement that generalizes Englert's duality inequality. In Sec. II we shall at first formulate the problem of the joint measurement, especially for two qubit observables. In Sec. III we present a geometrical necessary and sufficient condition for the joint measurability which is essential to the derivation of our single inequality as the if and only if condition in Sec. IV. In Sec. V we consider the joint measurement of a special kind of pair of unsharp observables and derive a complementarity inequality from the condition for joint measurement. Explicit joint unsharp observables are constructed in Sec. VI for two jointly measurable observables. Finally, we show explicitly that our inequality provides an analytical solution to the semidefinite programming reformulation of joint measurability. The proofs of most of our main results and the comparisons with known results are presented in the two Appendices.

II. THE PROBLEM OF JOINT MEASUREMENT

Via a measurement, we obtain a piece of classical information, that is, a probability distribution $\{p_k\}_{k=1}^K$ over possible outcomes, with the number of outcomes K being finite. Quantum mechanically, this probability is, in general, accounted for via Born's rule $p_k = \mathrm{Tr}(\rho O_k)$ where ρ is the quantum state and $\{O_k\}_{k=1}^K$ is a positive-operator valued measure (POVM), a set of positive operators summed up to the identity ($O_k \geqslant 0$ and $\sum_k O_k = I$). Given two pieces of classical information obtained via measurements of two observables there exist many joint probability distributions with these two pieces of classical information as marginal distributions. If there exists a joint distribution that can also be accounted for quantum mechanically via Born's rule independent of the quantum state, then these two pieces of information can be obtained in a single apparatus and two corresponding observables are called jointly measurable.

Mathematically formulated, a *joint measurement* of two jointly measurable observables $\{O_k\}$ and $\{O'_l\}$ is described by a *joint observable* $\{M_{kl}\}$ whose outcomes can be so grouped that

$$O_k = \sum_l M_{kl}, \quad O'_l = \sum_k M_{kl}. \tag{1}$$

Here we shall consider only the qubits, any two-level systems such as spin-half systems or two-path interferometries. A *simple* observable $\mathcal{O}(x,\vec{m})$ refers to a most general two-outcome POVM $\{O_\pm(x,\vec{m})\}$ with

$$O_\pm(x,\vec{m}) = \frac{1 \pm (x + \vec{m}\cdot\vec{\sigma})}{2}. \tag{2}$$

Here $m = |\vec{m}|$ is referred to as the *sharpness* while $|x|$ is referred to as the *biasedness*. When $|x| = 0$ the observable $\mathcal{O}(x,\vec{m})$ is called *unbiased*, in which case the outcomes of the measurement are purely random if the system is in the maximally mixed state, and when $|x| \neq 0$ the observable is referred to as *biased*, in which case *a priori* information can be employed to make better use of the outcomes of the measurement. Positivity imposes $|x| + m \leqslant 1$.

Given two simple observables $\mathcal{O}(x,\vec{m})$ and $\mathcal{O}(y,\vec{n})$, it is obvious that all possible sets of four operators satisfying the marginal constraints in Eq. (1) are

$$M_{\mu\nu}(Z,\vec{z}) = \frac{1 + \mu x + \nu y + \mu\nu Z + (\mu\nu\vec{z} + \vec{q}_{\mu\nu})\cdot\vec{\sigma}}{4}, \quad (3)$$

with $Z,\vec{z}$ being arbitrary and $\vec{q}_{\mu\nu} = \mu\vec{m} + \nu\vec{n}$ $(\mu,\nu = \pm 1)$. The problem of joint measurability becomes whether there exist $Z,\vec{z}$ such that $M_{\mu\nu}(Z,\vec{z}) \geqslant 0$ for all $\mu,\nu = \pm 1$. There are many partial results in special cases [5–7] as well as in general cases [8,9]. Here we shall derive a single inequality as the sufficient and necessary condition of joint measurement and construct explicitly the joint observables of two jointly measurable observables.

III. A GEOMETRICAL CONDITION

We shall at first present a geometrical condition for the joint measurement. Consider two simple observables $\mathcal{O}(x,\vec{m})$ and $\mathcal{O}(y,\vec{n})$ along two directions $\vec{m}$ and $\vec{n}$ and we suppose in this section that $s = |\vec{s}| > 0$ where $\vec{s} = \vec{m}\times\vec{n}$. Otherwise two observables will be trivially jointly measurable. We denote by $\mathbb{P}$ the plane spanned by two vectors $\vec{m}$ and $\vec{n}$. We have

Theorem 1. Two observables $\mathcal{O}(x,\vec{m})$ and $\mathcal{O}(y,\vec{n})$ are jointly measurable if and only if four elliptical regions in the plane $\mathbb{P}$ defined as $(\mu,\nu = \pm)$

$$\mathbb{E}_x^\mu = \left\{\vec{z}\in\mathbb{P}\,\middle|\,\sum_{\tau=\pm}|\vec{z} - \vec{q}_{\tau\mu}| \leqslant 2(1-\mu x)\right\}, \quad (4a)$$

$$\mathbb{E}_y^\nu = \left\{\vec{z}\in\mathbb{P}\,\middle|\,\sum_{\tau=\pm}|\vec{z} - \vec{q}_{\nu\tau}| \leqslant 2(1-\nu y)\right\}, \quad (4b)$$

intersect (i.e., $\mathbb{J} = \mathbb{E}_x^+ \cap \mathbb{E}_x^- \cap \mathbb{E}_y^+ \cap \mathbb{E}_y^- \neq \emptyset$).

Proof. If $\mathcal{O}(x,\vec{m})$ and $\mathcal{O}(y,\vec{n})$ are jointly measurable then there exist Z and $\vec{z}$ such that $M_{\mu\nu}(Z,\vec{z}) \geqslant 0$, that is,

$$|\mu\nu\vec{z} + \vec{q}_{\mu\nu}| \leqslant 1 + \mu x + \nu y + \mu\nu Z, \quad (5)$$

for all $\mu,\nu = \pm$. As a result $\vec{z} - (\vec{z}\cdot\vec{s})\vec{s}/s^2 \in \mathbb{J}$ with $\vec{s} = \vec{m}\times\vec{n}$. On the other hand, if there exists $\vec{z}\in\mathbb{J}$ then Eq. (5) holds true with $Z = Z(\vec{z})$ where

$$Z(\vec{z}) = \max_{\mu=\pm 1}\{|\vec{z} + \mu(\vec{m}+\vec{n})| - \mu(x+y)\} - 1, \quad (6)$$

that is, $\{M_{\mu\nu}(Z(\vec{z}),\vec{z})\}$ is a joint observable. Generally the choice of Z is not unique and therefore the joint observable is not unique either.

For later use we denote by $E_x^\pm$ and $E_y^\pm$ four elliptical ellipses that are boundaries of four elliptical regions defined in Eqs. (4a) and (4b) whose semimajor and squared semiminor axes are denoted by $A_\mu = 1 - \mu x$, $B_\nu = 1 - \nu y$, and $a_\mu = A_\mu^2 - m^2$, $b_\nu = B_\nu^2 - n^2$, respectively. Two neighboring ellipses E_x^μ and E_y^ν have one focus $Q_{\nu\mu}$ (corresponding to the vector $\vec{q}_{\nu\mu}$) in common.

IV. A SINGLE INEQUALITY AS THE IF AND ONLY IF CONDITION FOR JOINT MEASUREMENT

By transforming the previous geometrical condition into algebraic conditions we manage to prove a single inequality as the necessary and sufficient condition for the joint measurability, which is one of the main results of this paper. To this end we denote

$$F_x = \frac{1}{2}\left(\sqrt{(1+x)^2 - m^2} + \sqrt{(1-x)^2 - m^2}\right), \quad (7a)$$

$$F_y = \frac{1}{2}\left(\sqrt{(1+y)^2 - n^2} + \sqrt{(1-y)^2 - n^2}\right). \quad (7b)$$

Theorem 2. Two observables $\mathcal{O}(x,\vec{m})$ and $\mathcal{O}(y,\vec{n})$ are jointly measurable if and only if

$$\left(1 - F_x^2 - F_y^2\right)\left(1 - \frac{x^2}{F_x^2} - \frac{y^2}{F_y^2}\right) \leqslant (\vec{m}\cdot\vec{n} - xy)^2. \quad (8)$$

We shall delay the proof of the above theorem and all the following theorems in Appendix A. Instead we shall discuss some special cases in what follows. First the trivial case $s = 0$ is included. This is because due to the identities such as $x^2/F_x^2 + m^2/(1-F_x^2) = 1$ the left-hand side of Eq. (8) can be seen to be bounded above by $(mn - |xy|)^2$ which is no larger than the right-hand side of Eq. (8).

Second in the case of $x = y = 0$, Eq. (8) becomes $m^2 + n^2 \leqslant 1 + (\vec{m}\cdot\vec{n})^2$ which is exactly the if and only if condition for unbiased observables [6] and can be rewritten as

$$|\vec{m}+\vec{n}| + |\vec{m}-\vec{n}| \leqslant 2. \quad (9)$$

Thirdly when there is one unbiased observable, e.g., when $y = 0$, the condition in Eq. (8) reads

$$\sqrt{(1+x)^2 - m^2} + \sqrt{(1-x)^2 - m^2} \geqslant \frac{2|\vec{m}\times\vec{n}|}{\sqrt{m^2 - (\vec{m}\cdot\vec{n})^2}}, \quad (10)$$

which becomes simply $F_x \geqslant n$ for orthogonal observables where $\vec{m}\cdot\vec{n} = 0$ [5].

Theorem 2 is derived from the following set of conditions, which assumes a similar form as that of Eq. (9). For convenience we denote $\gamma = \vec{m}\cdot\vec{n} - xy$ and $\vec{g} = \vec{m}\alpha + \vec{n}\beta$ where

$$\alpha = \frac{1}{|\vec{m}\times\vec{n}|^2}[(y + \gamma x)n^2 - (x + \gamma y)\vec{m}\cdot\vec{n}], \quad (11a)$$

$$\beta = \frac{1}{|\vec{m}\times\vec{n}|^2}[(x + \gamma y)m^2 - (y + \gamma x)\vec{m}\cdot\vec{n}]. \quad (11b)$$

Theorem 3. Two observables $\mathcal{O}(x,\vec{m})$ and $\mathcal{O}(y,\vec{n})$ $(s \neq 0)$ are jointly measurable if and only if either $\max\{|\alpha|,|\beta|\} \geqslant 1$ or

$$\sum_{\nu=\pm}|\vec{m}+\vec{n}+\nu\vec{g}| + \sum_{\nu=\pm}|\vec{m}-\vec{n}+\nu\vec{g}| \leqslant 4. \quad (12)$$

From Lemma 2(iv) in Appendix A we see that the condition in Eq. (12) in Theorem 3 can be replaced by $R \geqslant 0$ where

$$R = 1 + x^2 + y^2 + \gamma^2 - m^2 - n^2 - |\vec{g}|^2, \quad (13)$$

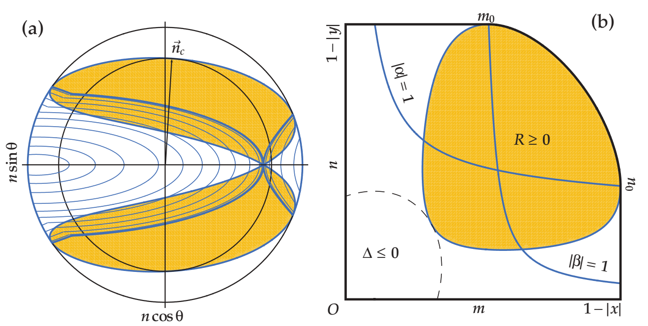

FIG. 1. (Color online) (a) The union of a blue-contoured region and a yellow-shaded region, determined by $\max\{|\alpha|,|\beta|\} \geqslant 1$ and $R \geqslant 0$, respectively, represents the admissible $\vec{n}$ in the case of $m = 0.8$, $x = -0.1$, and $y = 0.3$. The boundary lies between two circles $n = 1 - |y|$ and $n = n_c$. (b) The tradeoff curve (solid black) between sharpnesses m and n with $x = -0.1$, $y = 0.2$, and $\cos\theta = 0.3$ fixed.

which appears in an equivalent form [Eq. (55)] in [8]. Detailed comparisons with known results will be presented in Appendix B.

Theorem 3′. Two observables $\mathcal{O}(x,\vec{m})$ and $\mathcal{O}(y,\vec{n})$ ($s \neq 0$) are jointly measurable if and only if either $\max\{|\alpha|,|\beta|\} \geqslant 1$ or $R \geqslant 0$.

Given two unsharp observables $\mathcal{O}(x,\vec{m})$ and $\mathcal{O}(y,\vec{n})$, there are effectively five parameters: two biasedness x,y, two sharpness m,n, and the relative angle θ between $\vec{m}$ and $\vec{n}$ determined by $mn\cos\theta = \vec{m}\cdot\vec{n}$. In the following discussions we shall fix some parameters to see the tradeoffs among the remaining parameters imposed by the condition of the joint measurability.

At first let us examine the set of all observables $\mathcal{O}(y,\vec{n})$ with a given biasedness y that is jointly measurable with a fixed observable $\mathcal{O}(x,\vec{m})$. The admissible region of $\vec{n}$ is shown in Fig. 1(a) as the union of a blue-contoured region and a yellow-shaded region with a boundary given by Eq. (8) with equality and $|y| + n = 1$. The [blue (gray)] arcs of the circle $n = 1 - |y|$ satisfying Eq. (8) define a forward cone around $\vec{m}$ and a backward cone around $\vec{m}$ (centered on the origin) in which all $\vec{n}$ are admissible. If $1 - F_x^2 \leqslant |y|$ then Eq. (8) holds true and two cones overlap so that all $\vec{n}$ are admissible as formulated as one part of the conditions in [9].

Next we fix the biasedness x,y and the angle θ between $\vec{m}$ and $\vec{n}$ and examine the tradeoff between the sharpnesses m,n whose tradeoff curve (solid black) is plotted in Fig. 1(b). There is a critical value m_0 of the sharpness determined by Eq. (8) with equality and $|y| = 1 - n$, below which there is no constraint on n at all. In the same vein there is also a critical value for n. If

$$(1 + \mathrm{sgn}[xy]\cos\theta)(1 - |x|)(1 - |y|) \leqslant 2|xy|, \qquad (14)$$

with $\mathrm{sgn}[f] = +1$ if $f \geqslant 0$ and -1 if $f < 0$, then $m_0 \geqslant 1 - |x|$ (and $n_0 \geqslant 1 - |y|$) so that there is no tradeoff between m,n. Here n_0 is defined similarly as m_0 with (x,m) and (y,n) interchanged.

Finally we consider two fixed directions $\vec{m}$ and $\vec{n}$. If $m + n + |\vec{m} \pm \vec{n}| \leqslant 2$ every vector $\vec{g} = \vec{m}\alpha + \vec{n}\beta$ with $\max\{|\alpha|,|\beta|\} \leqslant 1$ satisfies Eq. (12) so that there is no tradeoff at all between x,y.

V. COMPLEMENTARITY INEQUALITY

As demonstrated in Ref. [5], the duality inequality is a consequence of the joint measurability of two orthogonal observables with one being unbiased. Here we shall consider a pair of observables $\mathcal{O}(x,\vec{m})$ and $\mathcal{O}(y,\vec{n})$ satisfying $\vec{m}\cdot\vec{n} = xy$ or $\gamma = 0$ with $\gamma = \vec{m}\cdot\vec{n} - xy$, which will be referred to as a pair of orthogonal unsharp observables. Obviously when the biasedness of one or two of the observables vanish we recover the usual concept of orthogonal observables.

From the if and only if condition Eq. (8) for the joint measurability we see immediately that two orthogonal observables are jointly measurable if and only if

$$F_x^2 + F_y^2 \geqslant 1. \qquad (15)$$

This is because $x^2/F_x^2 + y^2/F_y^2 < 1$ is ensured by $mn > |xy|$. In general, the condition $F_x^2 + F_y^2 \geqslant 1$ is sufficient for joint measurement since Eq. (8) is ensured because $(|xy| - mn)^2 \leqslant \gamma^2$ when $mn < |xy|$. Specifically, we refer to a pair of orthogonal observables that satisfies $\vec{n} = \vec{n}_c$ with $n_c^2/F_x^2 + y^2/(1 - F_x^2) = 1$ as a pair of *maximally orthogonal unsharp observables*. It is maximal in the sense that any observable $\mathcal{O}(y,\vec{n})$ with $n \leqslant n_c$ (regardless of its direction) is jointly measurable with $\mathcal{O}(x,\vec{m})$ while all the observables $\mathcal{O}(y,\vec{n})$ with $n > n_c$ along $\vec{n}_c$ are not jointly measurable with $\mathcal{O}(x,\vec{m})$.

As a measure for the unsharpness of an unsharp measurement, we take a linear combination of the sharpness and the biasedness, similar to the definition of the distinguishability. Explicitly for each observable we define the unsharpness as $D_1 = Q_1 m + P_1|x|$ and $D_2 = Q_2 n + P_2|y|$ where $0 \leqslant P_i \leqslant Q_i$ $(i = 1,2)$ are some constants. To measure jointly a pair of orthogonal unsharp observables there is a tradeoff between the previously defined unsharpnesses [since $D_1^2 + (Q_1^2 - P_1^2)F_x^2 \leqslant Q_1^2$]

$$D_1^2\left(Q_2^2 - P_2^2\right) + D_2^2\left(Q_1^2 - P_1^2\right) + P_1^2P_2^2 \leqslant Q_1^2Q_2^2. \qquad (16)$$

Englert's duality inequality [4] in the case of orthogonal observables with one being unbiased [5] turns out to be a special case of the above inequality if we let $Q_1 = 1$, $P_2 = 0$ so that D_1 and D_2 become the path distinguishability and the fringe visibility, respectively.

VI. JOINT UNSHARP OBSERVABLES

In previous sections we have found out the conditions under which there exists a joint observable for two unsharp observables. In this section we shall construct explicitly the joint observable for any given pair of observables that are jointly measurable.

If $s = 0$ then a joint observable of observables $\mathcal{O}(x,\vec{m})$ and $\mathcal{O}(y,\vec{n})$ is simply given by $\{O_\mu(x,\vec{m})O_\nu(y,\vec{n})\}$. If $\Delta_\tau < 0$ with

$$\Delta_\tau = (\vec{m} - \tau\vec{n})^2 - (x - \tau y)^2, \qquad (17)$$

for some $\tau=\pm$ then $O_\eta(x,\vec m)-O_{\eta\cdot\tau}(y,\vec n)\geqslant 0$ where $\eta=\mathrm{sgn}[x-\tau y]$. Therefore the POVM

$$\{0,O_{\bar\eta}(x,\vec m),O_\eta(x,\vec m)-O_{\eta\cdot\tau}(y,\vec n),O_{\eta\cdot\tau}(y,\vec n)\},\qquad(18)$$

is a joint observable ($\bar\eta=-\eta$). In the case of $s>0$ and $\Delta_\pm\geqslant 0$ considering Theorem 3′ we have:

Theorem 4. Given observables $\mathcal O(x,\vec m)$ and $\mathcal O(y,\vec n)$, (a) if $R\geqslant 0$ then $\{M_{\mu\nu}(\gamma,\vec g)\}$ is a joint observable; (b) if $R<0$ and $\max\{|\alpha|,|\beta|\}\geqslant 1$ then $\{M_{\mu\nu}(Z(\vec z_{\eta\tau}),\vec z_{\eta\tau})\}$ is a joint observable where

$$Z(\vec z)=\max_{\mu=\pm1}\{|\vec z+\mu(\vec m+\vec n)|-\mu(x+y)\}-1,\qquad(19)$$

$$\vec z_{\eta\tau}=\vec g+\frac{D_{\eta\tau}(\vec m\times\vec n)\times\vec L_{\eta\tau}}{\vec L^2_{\eta\tau}-|\vec m\times\vec n|^2},\qquad(20)$$

with $D_{\eta\tau}=\tau A_\eta\alpha+\eta B_\tau\beta+\eta\tau\gamma-1$, $\vec L_{\eta\tau}=\tau A_\eta\vec n-\eta B_\tau\vec m$, $\tau=\mathrm{sgn}[\alpha]$ and $\eta=\mathrm{sgn}[B_\tau\beta+\tau\gamma-x]$ if $|\alpha|\geqslant 1$, $\eta=\mathrm{sgn}[\beta]$ and $\tau=\mathrm{sgn}[A_\eta\alpha+\eta\gamma-y]$ if $|\beta|\geqslant 1$. Here $A_\eta=1-\eta x$ and $B_\tau=1-\tau y$.

The regions for different constructions of joint observables according to the above theorem are indicated schematically in Fig. 1(b) whenever two observables are jointly measurable. We note that $\Delta=\min\{\Delta_\pm\}<0$ infers $\max\{|\alpha|,|\beta|\}\geqslant 1$. The region where case (a) of Theorem 4 happens is the oval region defined by $R\geqslant 0$, a yellow-shaded region. Since $R=d^2_{\mu\nu}-|\vec g-q_{\mu\nu}|^2$ for all $\mu,\nu=\pm$ with $d_{\mu\nu}=1-\mu x-\nu y+\mu\nu\gamma$, we see that if $R=0$ then $4M_{\mu\nu}(\gamma,\vec g)=d_{\bar\mu\bar\nu}+(\mu\nu\vec g+\vec q_{\mu\nu})\cdot\vec\sigma$ are proportional to some projections for all $\mu,\nu=\pm$ (i.e., the joint observable is composed of projections). Outside the region $R\geqslant 0$ and inside the region that is included by the solid curve, we have either $\alpha\geqslant 1$ or $\beta\geqslant 1$ where the case (b) of Theorem 4 applies.

VII. ANALYTIC SOLUTION TO THE SEMIDEFINITE PROGRAM

In Ref. [10] the general problem of joint measurability was formulated in terms of a semidefinite program and the violation of some Bell inequality provides a sufficient and necessary condition for the joint measurability. Here we shall demonstrate explicitly that our condition Eq. (8) provides an analytical solution to the semidefinite program.

Explicitly stated in our case according to proposition 2 in [10], two observables $\mathcal O(x,\vec m)$ and $\mathcal O(y,\vec n)$ are jointly measurable [by writing $\hat Q=O_+(y,\vec n)$ and $\hat P=O_+(x,\vec m)$] if and only if the largest eigenvalue $\lambda_{\max}$ of the following 4×4 matrix

$$T=\begin{pmatrix}u^2\hat P-v^2\hat Q-u^2 & uv(\hat P+\hat Q-1)\\ uv(\hat P+\hat Q-1) & -u^2\hat P+v^2\hat Q-v^2\end{pmatrix},\qquad(21)$$

is nonpositive (i.e., $\lambda_{\max}\leqslant 0$) for all possible ϕ with $u=\cos\phi, v=\sin\phi$. As it turns out, taking into account the definition in Eq. (2), the matrix T satisfies the following identity

$$\begin{aligned}&[(1+2T)^2-u^2(m^2+x^2)-v^2(n^2+y^2)]^2\\&=4(u^2x\vec m+v^2y\vec n)^2+4u^2v^2(\vec m\times\vec n)^2,\end{aligned}\qquad(22)$$

from which four eigenvalues of T can be easily calculated. Obviously $\lambda_{\max}\leqslant 0$ is equivalent to

$$\begin{aligned}0&\leqslant[1-u^2(m^2+x^2)-v^2(n^2+y^2)]^2\\&\quad-4(u^2x\vec m+v^2y\vec n)^2-4u^2v^2(\vec m\times\vec n)^2\\&=(u^2\sqrt{g_x}-v^2\sqrt{g_y})^2+4u^2v^2(\gamma^2-f_-),\end{aligned}\qquad(23)$$

in which we have denoted $g_x=(1-m^2-x^2)^2-4x^2m^2$, $g_y=(1-n^2-y^2)^2-4y^2n^2$, $\gamma=\vec m\cdot\vec n-xy$ and

$$\begin{aligned}f_-&=m^2n^2+x^2y^2\\&\quad-\frac{(1-m^2-x^2)(1-y^2-n^2)+\sqrt{g_xg_y}}{2}.\end{aligned}\qquad(24)$$

It follows immediately from Eq. (23) that $\lambda_{\max}\leqslant 0$ for all possible u and v if and only if $\gamma^2\geqslant f_-$. As can be easily checked, f_- is exactly equal to the left-hand side of the inequality in Eq. (8) in Theorem 2, by noting some identities such as $2F_x^2=1+x^2-m^2+\sqrt{g_x}$ and $2x^2/F_x^2=1+x^2-m^2-\sqrt{g_x}$.

VIII. CONCLUSION AND DISCUSSIONS

We have derived a single inequality as the condition for the joint measurement of two simple qubit observables, based on which an example of MUR is established that generalizes the existing results. Two recent references [8,9] provide two seemingly different solutions to the same problem considered here, whose equivalency can be established in an analytical or a half-numerical and half-analytical way via our results. Also we have explicitly demonstrated that our single inequality Eq. (8) provides an analytic solution to the semidefinite program formulation of the joint measurability [10], which provides promising solution to the general problems of the joint measurability of more than two observables or observables with more than two outcomes.

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APPENDIX A: PROOFS OF THE THEOREMS

In the following proofs we shall make use of the fact that given two overlapping convex regions in a plane either their boundaries intersect or one region belongs to the other. Recalling that $\mathbb E_x^\pm$ and $\mathbb E_y^\pm$ as defined in Eqs. (4a) and (4b) are four elliptical regions on the plane spanned by $\vec m$ and $\vec n$. As shown in Fig. 2 two neighboring ellipses with intersections and the four-ellipse E (thick blue curve) are shown, in which we have denoted by

$$\mathbb E=\left\{\vec z\in\mathbb P\left|\sum_{\mu,\nu=\pm}\right|\vec z-\vec q_{\mu\nu}|\leqslant 4\right\}\qquad(A1)$$

an oval region with the boundary being a generalized ellipse E with four foci $Q_{\mu\nu}$ with $\mu,\nu=\pm$. The condition in Eq. (12) becomes simply $\vec g\in\mathbb E$. It is easy to see that $\mathbb J_x:=\mathbb E_x^+\cap\mathbb E_x^-\subset\mathbb E$, $\mathbb J_y:=\mathbb E_y^+\cap\mathbb E_y^-\subset\mathbb E$ with boundaries given by $J_x=(E_x^+\cup$

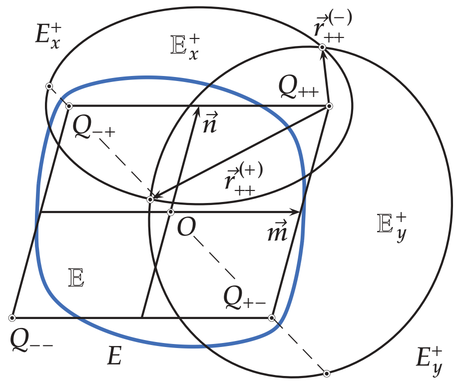

FIG. 2. (Color online) The setup for proofs. In the plane $\mathbb{P}$ spanned by $\vec{m}$ and $\vec{n}$ there are two neighboring ellipses E_x^+ and E_y^+ and the generalized ellipse E with four foci $Q_{\mu\nu}$ (thick blue curve).

$E_x^-)\cap\mathbb{E}$ and $J_y=(E_y^+\cup E_y^-)\cap\mathbb{E}$, respectively. Furthermore $E_x^+\cap E_x^-\subset E\cap J_x$, $E_y^+\cap E_y^-\subset E\cap J_y$.

Lemma 1. $\mathbb{J}\neq\emptyset$ if and only if either $E_x^\mu\cap E_y^\nu\cap\mathbb{E}\neq\emptyset$ for some $\mu,\nu=\pm$ or $E_x^\mu\cap E_y^\nu=\emptyset$ for all $\mu,\nu=\pm$.

Proof. Sufficiency. Suppose that there exists $\vec{z}\in E_x^\mu\cap E_y^\nu\cap\mathbb{E}$ for some $\mu,\nu=\pm$. From $\vec{z}\in E_x^\mu\cap\mathbb{E}$ and $\vec{z}\in E_y^\nu\cap\mathbb{E}$ it follows that $\vec{z}\in\mathbb{E}_x^{\bar\mu}$ and $\vec{z}\in\mathbb{E}_y^{\bar\nu}$ respectively, which leads to $\vec{z}\in\mathbb{J}$. If $E_x^\mu\cap E_y^\nu=\emptyset$, for all μ,ν then, taking into account Lemma 2(ii), we have either $\mathbb{E}_x^\mu\subset\mathbb{E}_y^\pm\subset\mathbb{E}_x^{\bar\mu}$ or $\mathbb{E}_y^\nu\subset\mathbb{E}_x^\pm\subset\mathbb{E}_y^{\bar\nu}$, i.e., either $\mathbb{J}=\mathbb{E}_x^\mu$ or $\mathbb{J}=\mathbb{E}_y^\nu$ for some μ,ν, which is obviously not empty.

Necessity. If $\mathbb{J}\neq\emptyset$ then two convex regions $\mathbb{J}_x$ and $\mathbb{J}_y$ overlap. As a result we have either $J_x\cap J_y\neq\emptyset$, which means $(\exists\mu,\nu)\ E_x^\mu\cap E_y^\nu\cap\mathbb{E}\neq\emptyset$, or $J_x\cap J_y=\emptyset$ with either $\mathbb{J}_x\subset\mathbb{J}_y$ or $\mathbb{J}_y\subset\mathbb{J}_x$. If $J_x\cap J_y=\emptyset$ and $\mathbb{J}_x\subset\mathbb{J}_y$, i.e., $\mathbb{J}_x$ lies totally within $\mathbb{J}_y\subset\mathbb{E}$, then $J_x\cap E=\emptyset$ which infers $E_x^+\cap E_x^-=\emptyset$, i.e., the boundaries of two overlapping regions $\mathbb{E}_x^\pm$ do not intersect. As a result $(\exists\mu)\ \mathbb{E}_x^\mu\subset\mathbb{E}_x^{\bar\mu}$ so that $(\exists\mu)\ \mathbb{E}_x^\mu=\mathbb{J}_x\subset\mathbb{E}_y^\pm$, which infers $(\exists\mu)\ E_x^\mu\cap E_y^\pm=\emptyset$ since $J_x\cap J_y=\emptyset$. In the same manner $\mathbb{J}_y\subset\mathbb{J}_x$ with $J_x\cap J_y=\emptyset$ infers $(\exists\nu)\ E_y^\nu\cap E_x^\pm=\emptyset$. In both cases, considering Lemma 2(i), we obtain $(\forall\mu,\nu)\ E_x^\mu\cap E_y^\nu=\emptyset$.

Lemma 2. (i) $E_x^\mu\cap E_y^\nu\neq\emptyset$ if and only if $\Delta_{\mu\cdot\nu}\geqslant 0$; (ii) $\mathbb{E}_x^\mu\subset\mathbb{E}_y^\nu$ infers $\mathbb{E}_y^{\bar\nu}\subset\mathbb{E}_x^{\bar\mu}$; (iii) $(\exists\mu,\nu)\ E_x^\mu\cap E_y^\nu\cap\mathbb{E}\neq\emptyset$ if and only if either $R\geqslant 0$, or $(\exists\mu,\nu,\tau)\ D_{\mu\nu}\geqslant 0$ and $\Delta_\tau\geqslant 0$; (iv) $R\geqslant 0$ if and only if $\vec{g}\in\mathbb{E}$; (v) $(\forall\mu)\ \Delta_\mu<0$ infers $(\exists\mu,\nu)\ D_{\mu\nu}>0$; (vi) provided $R<0$, $(\forall\mu,\nu)\ D_{\mu\nu}<0$ if and only if $|\alpha|<1$ and $|\beta|<1$.

Proof. (i) Consider the straight line passing through two points $Q_{\bar\nu\mu}$ and $Q_{\nu\bar\mu}$ (dashed line in Fig. 2 for the case of $\mu=\nu=+$). If $\Delta_{\mu\cdot\nu}\geqslant 0$ then one intersection of E_x^μ (or E_y^ν) with the straight line will not lie in the interior of $\mathbb{E}_y^\nu$ (or $\mathbb{E}_x^\mu$, respectively) which means neither $\mathbb{E}_x^\mu\subset\mathbb{E}_y^\nu$ nor $\mathbb{E}_y^\nu\subset\mathbb{E}_x^\mu$ and hence $E_x^\mu\cap E_y^\nu\neq\emptyset$. If $\Delta_{\mu\cdot\nu}<0$ then, e.g., $A_\mu-B_\nu>|\vec{m}-\mu\nu\vec{n}|$ and $\vec{z}\in E_y^\nu$ infers $|\vec{z}-\vec{q}_{\nu\mu}|+|\vec{z}-\vec{q}_{\bar\nu\mu}|\leqslant 2B_\nu+2|\vec{m}-\mu\nu\vec{n}|<2A_\mu$, i.e., $E_x^\mu\cap E_y^\nu=\emptyset$.

(ii) $\mathbb{E}_x^\mu\subset\mathbb{E}_y^\nu$ is equivalent to $\Delta_{\mu\cdot\nu}\leqslant 0$, i.e., $\Delta_{\bar\mu\cdot\bar\nu}\leqslant 0$, and $A_\mu\leqslant B_\nu$, i.e., $B_{\bar\nu}\leqslant A_{\bar\mu}$.

(iii) Suppose $\vec{z}\in E_x^\mu\cap E_y^\nu\cap\mathbb{E}$ for some $\mu,\nu=\pm$. Since $\vec{z}\in E_x^\mu\cap E_y^\nu$ we have $\Delta_{\mu\cdot\nu}\geqslant 0$, $r+|\vec{r}+2\nu\vec{m}|=2A_\mu$ and $r+|\vec{r}+2\mu\vec{n}|=2B_\nu$ where $\vec{r}=\vec{z}-\vec{q}_{\nu\mu}$ and $r=|\vec{r}|$. It follows that $\vec{s}\times\vec{r}=\vec{K}_{\mu\nu}-r\vec{L}_{\mu\nu}$ whose square provides a quadratic equation of r: $(L_{\mu\nu}^2-s^2)r^2-2r\vec{K}_{\mu\nu}\cdot\vec{L}_{\mu\nu}+K_{\mu\nu}^2=0$ where $\vec{K}_{\mu\nu}=\nu a_\mu\vec{n}-\mu b_\nu\vec{m}$, $L_{\mu\nu}=|\vec{L}_{\mu\nu}|$ and $K_{\mu\nu}=|\vec{K}_{\mu\nu}|$. By noticing $L_{\mu\nu}^2>s^2$ as long as $s>0$ we obtain two solutions

$$r_{\mu\nu}^{(\pm)}=d_{\mu\nu}+\frac{s^2D_{\mu\nu}\pm\sqrt{s^2a_\mu b_\nu\Delta_{\mu\cdot\nu}}}{L_{\mu\nu}^2-s^2},\tag{A2}$$

and we denote $E_x^\mu\cap E_y^\nu=\{\vec{z}_{\mu\nu}^{(+)},\vec{z}_{\mu\nu}^{(-)}\}$ with $\vec{z}_{\mu\nu}^{(\pm)}=\vec{q}_{\nu\mu}+\vec{r}_{\mu\nu}^{(\pm)}$ and $s^2\vec{r}_{\mu\nu}^{(\pm)}=(\vec{K}_{\mu\nu}-r_{\mu\nu}^{(\pm)}\vec{L}_{\mu\nu})\times\vec{s}$. The condition $(\exists\tau)\ \vec{z}_{\mu\nu}^{(\tau)}\in\mathbb{E}$ [i.e., $2(A_\mu+B_\nu)-r_{\mu\nu}^{(\tau)}+|\vec{r}_{\mu\nu}^{(\tau)}+2\vec{q}_{\nu\mu}|\leqslant 4$] is equivalent to $(\exists\tau)\ r_{\mu\nu}^{(\tau)}\geqslant d_{\mu\nu}-\min\{d_{\bar\mu\bar\nu},0\}$. Due to

$$s^2a_\mu b_\nu\Delta_{\mu\cdot\nu}=s^4D_{\mu\nu}^2+s^2R\big(L_{\mu\nu}^2-s^2\big),\tag{A3}$$

and Eq. (A2), it follows from $(\exists\tau)\ r_{\mu\nu}^{(\tau)}\geqslant d_{\mu\nu}$ that either $R\geqslant 0$, or $R<0$ and $D_{\mu\nu}\geqslant 0$. Necessity is thus proved.

If $\Delta_\pm\geqslant 0$ then $(\forall\mu,\nu)\ d_{\mu\nu}\geqslant 0$ since $2d_{\mu\nu}=\Delta_{\mu\cdot\bar\nu}+a_\mu+b_\nu$. Thus from $(\exists\mu,\nu)\ D_{\mu\nu}\geqslant 0$ and $R<0$ it follows that $r_{\mu\nu}^{(\pm)}\geqslant d_{\mu\nu}$ which infers $\vec{z}_{\mu\nu}^{(\pm)}\in\mathbb{E}$ so that $E_x^\mu\cap E_y^\nu\cap\mathbb{E}\neq\emptyset$.

If $(\exists\tau)\ \Delta_\tau<0$ and $\Delta_{\bar\tau}\geqslant 0$ then $(\forall\nu)\ E_x^\nu\cap E_y^{\tau\cdot\nu}=\emptyset$ and $(\forall\nu)\ E_x^\nu\cap E_y^{\bar\tau\cdot\nu}\neq\emptyset$. It follows that either $\mathbb{E}_x^\nu\subset\mathbb{E}_y^{\tau\cdot\nu}$ or $\mathbb{E}_y^{\tau\cdot\nu}\subset\mathbb{E}_x^\nu$, (i.e., either $\mathbb{E}_y^{\tau\cdot\bar\nu}\subset\mathbb{E}_x^{\bar\nu}$ or $\mathbb{E}_x^{\bar\nu}\subset\mathbb{E}_y^{\tau\cdot\bar\nu}$). As a result either $\mathbb{J}=\mathbb{E}_x^\nu\cap\mathbb{E}_y^{\tau\cdot\bar\nu}$ or $\mathbb{J}=\mathbb{E}_x^{\bar\nu}\cap\mathbb{E}_y^{\tau\cdot\nu}$ from which it follows that $(\exists\nu)\ E_x^\nu\cap E_y^{\tau\cdot\bar\nu}\subset\mathbb{E}$ [i.e., $(\exists\tau,\nu)\ E_x^\nu\cap E_y^{\tau\cdot\bar\nu}\cap\mathbb{E}\neq\emptyset$].

If $R\geqslant 0$ then we claim that $\Delta_\pm\geqslant 0$, from which it follows immediately that $(\forall\mu,\nu)\ E_x^\mu\cap E_y^\nu\neq\emptyset$ and $\vec{z}_{\mu\nu}^{(+)}\in\mathbb{E}$. First, if $a_\pm=0$ (or $b_\pm=0$) then $R\geqslant 0$ infers $s=0$, which is precluded. Second, if either $(\forall\mu,\nu)\ a_\mu b_\nu>0$, or $(\exists\mu)\ a_\mu=0$ and $a_{\bar\mu}>0$ and $b_\pm>0$, or $(\exists\nu)\ b_\nu=0$ and $b_{\bar\nu}>0$ and $a_\pm>0$, then the claim is obviously true due to the identity in Eq. (A3). Third, if $(\exists\mu,\nu)\ a_\mu=b_\nu=0$ and $a_{\bar\mu}b_{\bar\nu}>0$ then $R=0$, $D_{\mu\nu}=D_{\mu\bar\nu}=D_{\bar\mu\nu}=0$ with $D_{\bar\mu\bar\nu}=-4$, and $\Delta_{\mu\cdot\nu}=\Delta_{\bar\mu\cdot\bar\nu}>0$. As a result $r_{\mu\nu}^{(\pm)}=d_{\mu\nu}\geqslant 0$ which leads to $\Delta_{\mu\cdot\bar\nu}=2d_{\mu\nu}\geqslant 0$.

(iv) If $R\geqslant 0$ then $(\forall\mu,\nu)\ d_{\mu\nu}\geqslant 0$ so that $(\forall\mu,\nu)\ |\vec{g}-\vec{q}_{\nu\mu}|\leqslant d_{\mu\nu}$, which infers $\vec{g}\in\mathbb{E}$. If $\vec{g}\in\mathbb{E}$ then $(\exists\mu)\ \vec{g}\in\mathbb{E}_x^\mu$. As a result $a_\mu-A_\mu d_{\mu+}=(\vec{g}-\vec{q}_{+\mu})\cdot\vec{m}\leqslant a_\mu-|\vec{g}-\vec{q}_{+\mu}|A_\mu$ which infers either $|\vec{g}-\vec{q}_{+\mu}|\leqslant d_{\mu+}$ (i.e., $R\geqslant 0$) or $A_\mu=0$ which leads to $R=4y^2\geqslant 0$.

(v) $\Delta_\pm<0$ infers $R<0$ [i.e., $(1\pm\gamma)^2<\Delta_\mp+|\vec{g}|^2$] and thus $|\vec{g}|>1+|\gamma|$. Let $\eta=\mathrm{sgn}[\beta]$ and $\tau=\mathrm{sgn}[\alpha]$ then $|\vec{g}|\leqslant A_\eta|\alpha|+B_\tau|\beta|\leqslant D_{\eta\tau}+1+|\gamma|$ which means $D_{\eta\tau}>0$.

(vi) If $(\forall\mu,\nu)\ D_{\mu\nu}<0$ then $|\alpha|<1$ and $|\beta|<1$ since $\max\{D_{-+},D_{+-}\}+\max\{D_{--},D_{++}\}<0$. If $|\alpha|<1$ and $|\beta|<1$ then $|\vec{g}-\vec{q}_{\nu\mu}|\leqslant A_\mu(1-\nu\alpha)+B_\nu(1-\mu\beta)$ which, together with $(\forall\mu,\nu)\ d_{\mu\nu}<|\vec{g}-\vec{q}_{\nu\mu}|$ inferred from $R<0$, leads to $(\forall\mu,\nu)D_{\mu\nu}=d_{\mu\nu}-A_\mu(1-\nu\alpha)-B_\nu(1-\mu\beta)<0$.

Proof of Theorem 3. From Lemmas 1 and 2 and statements (i), (iii), and (v) of Lemma 2 it follows that two observables are jointly measurable if and only if either $R\geqslant 0$ or $(\exists\mu,\nu)\ D_{\mu\nu}\geqslant 0$ and Theorem 3 is an immediate result of statements (iv) and (vi) of Lemma 2.

Proof of Theorem 4. (a) $R\geqslant 0$ is equivalent to $(\forall\mu,\nu)\ |\vec{g}-\vec{q}_{\nu\mu}|\leqslant d_{\mu\nu}$, which means $(\forall\mu,\nu)\ M_{\mu\nu}(\gamma,\vec{g})\geqslant 0$.

(b) From $\Delta_\pm\geqslant 0$ it follows that $(\forall\mu,\nu)\ E_x^\mu\cap E_y^\nu\neq\emptyset$ and $d_{\mu\nu}\geqslant 0$ and from $\max\{|\alpha|,|\beta|\}\geqslant 1$ and the choice of η,τ as in Theorem 4(b) it follows that $D_{\eta\tau} \geqslant 0$. As a result $\{\vec{z}^{(\pm)}_{\eta\tau}\} = E^\eta_x \cap E^\tau_y \subset \mathbb{E}$ so that $\vec{z}^{(\pm)}_{\eta\tau} \in \mathbb{J}$ (Lemma 1). Since $\mathbb{J}$ is convex we obtain $\vec{z}_{\eta\tau} = (\vec{z}^{(+)}_{\eta\tau} + \vec{z}^{(-)}_{\eta\tau})/2 \in \mathbb{J}$ and $M_{\mu\nu}(Z(\vec{z}_{\eta\tau}),\vec{z}_{\eta\tau})$ is a joint observable (Theorem 1).

From now on s may be 0. For simplicity we denote by Π_i ($i = 1,2,3,4$) four functions $s^2(\pm\alpha - 1)$ and $s^2(\pm\beta - 1)$ and $\Pi = \max_i\{\Pi_i\}$. A set of if and only if conditions for joint measurement reads $s^2R \geqslant 0$ or $\Pi \geqslant 0$. We have

Lemma 3. $\Pi = 0$ infers $s^2R \geqslant 0$.

Proof. (a) If $s > 0$ then $\Pi = 0$ infers $\max\{|\alpha|,|\beta|\} = 1$ (e.g., $|\alpha| = 1$ and $|\beta| \leqslant 1$). Thus $|\vec{g} - \vec{q}_{\nu\mu}| = (1 - \mu\beta)n \leqslant (1 - \mu\beta)B_\nu \leqslant d_{\mu\nu}$ which is exactly $R \geqslant 0$. Here $\nu = \mathrm{sgn}[\alpha]$ and $\mu = \mathrm{sgn}[B_\nu\beta + \nu\gamma - x]$. (b) If $s = 0$ then $\Pi = 0$ infers $s^2\alpha = s^2\beta = 0$ and thus $s^2R = 0$.

Proof of Theorem 2. We have only to prove that Eq. (8) is equivalent to either $s^2R \geqslant 0$ or $\Pi \geqslant 0$. From the identity $s^2R = (\gamma^2 - f_-)(f_+ - \gamma^2)$, where f_- is the left-hand side of Eq. (8) and $f_+ = f_- + \sqrt{a_+a_-b_+b_-}$, it follows that $s^2R \geqslant 0$ is equivalent to $f_- \leqslant \gamma^2 \leqslant f_+$. Thus we have only to show that $\gamma^2 \geqslant f_+$ infers $\Pi \geqslant 0$ and that $\Pi \geqslant 0$ infers $\gamma^2 \geqslant f_-$. We notice first of all that Π_i are four quadratic (or linear) functions of $c = \vec{m}\cdot\vec{n}$ by regarding x,y,m,n as parameters and Π is continuous. Case (a) $F_x^2 + F_y^2 \leqslant 1$. In this case $mn \geqslant |xy|$ and $f_\pm \geqslant 0$ and $\Pi \leqslant 0$ for $c = xy$ since $|y| \leqslant F_y^2$ and $F_y^2(n^2 - x^2) \leqslant m^2n^2 - x^2y^2$. Now that $\Pi \geqslant 0$ for $c = \pm mn$, there exist $-mn \leqslant c_- \leqslant xy \leqslant c_+ \leqslant mn$ such that $\Pi = 0$ for $c = c_\pm$, which infers $xy \pm \sqrt{f_\mp} \leqslant c_\pm \leqslant xy \pm \sqrt{f_\pm}$ (Lemma 3). If $\gamma^2 \geqslant f_+$ then $c \leqslant c_-$ or $c \geqslant c_+$, which ensures $\Pi \geqslant 0$ since all the coefficients of c^2 of Π_i ($i = 1,2,3,4$) are nonnegative. If $\Pi \geqslant 0$ then $c \leqslant c_-$ or $c \geqslant c_+$, which infers $\gamma^2 \geqslant f_-$. Case (b) $F_x^2 + F_y^2 \geqslant 1$. In this case $\gamma^2 \geqslant f_-$ always and we have only to show that $\gamma^2 \geqslant f_+$ infers $\Pi \geqslant 0$. If $\Pi = 0$ has no solution then $\Pi > 0$ for all c since $\Pi > 0$ for $c = \pm mn$. Let $c_- \leqslant c_+$ be its two solutions and it follows that $(c_\pm - xy)^2 \leqslant f_+$. As a result if $\gamma^2 \geqslant f_+$ then $c \geqslant c_+$ or $c \leqslant c_-$, which ensures $\Pi \geqslant 0$.

APPENDIX B: COMPARISON WITH KNOWN RESULTS

Here we shall formulate those results in [8,9] in our notations and and examine the boundary of admissible $\vec{n}$ by fixing y,x,m. The same boundary means the equivalency.

The Stano-Reitzner-Heinosaari (SRH) Theorem [9]. Two observables $\mathcal{O}(x,\vec{m})$ and $\mathcal{O}(y,\vec{n})$ are jointly measurable if and only if either

(C1) $\sqrt{1-|y|} \leqslant F_x$; or
(C2) $\sqrt{1-|y|} > F_x$ and $|\gamma| \geqslant l$; or
(C3) $\sqrt{1-|y|} > F_x$ and $|\gamma| < l$ and $\sqrt{a_+h_-} + \sqrt{a_-h_+} \geqslant 2s$.

Here $s = |\vec{m}\times\vec{n}|$, $\gamma = \vec{m}\cdot\vec{n} - xy$, $a_\pm = (1 \mp x)^2 - m^2$, $h_\pm = m^2 - (\gamma \pm y)^2$, and

$$l = \sqrt{y^2 + m^2 - |y|(1 - x^2 + m^2)}.$$

Remarks. The corresponding boundary is plotted in Fig. 3(a) [with the same parameters as in Fig. 1(a)]. If (C1) then $F_x^2 + F_y^2 \geqslant 1$ so that Eq. (8) holds always true. If (C2) then, by noticing that the left-hand side f_- of Eq. (8) can be

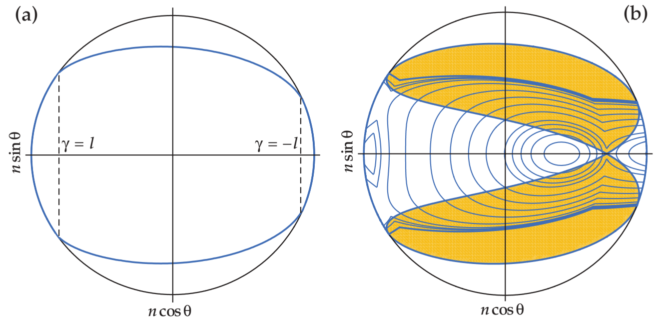

FIG. 3. (Color online) The boundaries of admissible $\vec{n}$ arising from (a) SRH conditions and (b) BS conditions with fixed $m = 0.8$, $x = -0.1$, and $y = 0.3$.

rewritten as

$$f_- = \frac{(a_+ + 2x)(b_+ + 2y) - \sqrt{a_-a_+b_-b_+}}{2} + m^2 + n^2 - 1, \tag{B1}$$

we have $f_- \leqslant l$ so that Eq. (8) holds true. If $1 - |y| > F_x^2$ and $|\gamma| < l$ then $|\gamma| < m - |y|$ so that Lemma 4(a) applies and Eq. (B2) coincides with Eq. (8). Thus we have reproduced the boundary in [9] analytically.

The Busch-Schmidt (BS) Theorem [8]. Two observables $\mathcal{O}(x,\vec{m})$ and $\mathcal{O}(y,\vec{n})$ are jointly measurable if and only if either

(53) $4\Delta_+s^2 \leqslant a_+b_+(\vec{L}^2_{--} - s^2)$; or
(54) $4\Delta_+s^2 \leqslant a_-b_-(\vec{L}^2_{++} - s^2)$; or
(55) $4\Delta_+s^2 \leqslant 2(A_+B_+ - c)(A_-B_- - c)(s^2 - \vec{L}_{++}\cdot\vec{L}_{--}) - (A_+B_+ - c)^2(\vec{L}^2_{--} - s^2) - (A_-B_- - c)^2(\vec{L}^2_{++} - s^2)$.

Here we have denoted $s = |\vec{m}\times\vec{n}|$, $\Delta_+ = (\vec{m} - \vec{n})^2 - (x - y)^2$, $a_\pm = (1 \mp x)^2 - m^2$, and $b_\pm = (1 \mp y)^2 - n^2$ together with

$$\vec{L}_{\mu\mu} = \mu(1 - \mu x)\vec{n} - \mu(1 - \mu y)\vec{m},$$

$A_\pm = 1 \mp x$, $B_\pm = 1 \mp y$, and $c = \vec{m}\cdot\vec{n}$.

Remarks. Despite the facts that we have identified Eq. (55) with $R \geqslant 0$ [Lemma 4(b)] and that the boundaries $R = 0$, $|y| + n = 1$, and

$$4\Delta_+s^2 = \max_{\mu=\pm}\left\{a_\mu b_\mu\left(\vec{L}^2_{\bar\mu\bar\mu} - s^2\right)\right\},$$

intersect at exactly where $\max\{|\alpha|,|\beta|\} = 1$ and that numerical evidences indicate that BS conditions also give rise to the same boundary, we fail to work out an analytical proof for the equivalency so far. The corresponding boundary is plotted in Fig. 3(b). The yellow-shaded region comes from $R \geqslant 0$ while the blue-contoured region comes from the conditions in Eqs. (53) and (54).

Lemma 4. (a) Either $R \geqslant 0$ or $\{|\beta| \geqslant 1$ and $h_\pm \geqslant 0\}$ if and only if

$$\sqrt{a_+h_-} + \sqrt{a_-h_+} \geqslant 2s. \tag{B2}$$

(b) The condition in Eq. (55) is equivalent to $R \geqslant 0$.

Proof. (a) If $R \geqslant 0$ then $\vec{g} \in \mathbb{E}^+_x \cap \mathbb{E}^-_x$ so that $h_\pm \geqslant 0$ and $|(1 \pm \beta)s| \leqslant \sqrt{a_\mp h_\pm}$ which infers Eq. (B2). If $|\beta| \geqslant 1$ and $h_\pm \geqslant 0$ since $4\beta s^2 = h_+a_- - h_-a_+$, then $4s^2 \leqslant h_+a_- + h_-a_+$ and Eq. (B2) follows. On the other hand, if Eq. (B2) holds true then $h_\pm \geqslant 0$ and $(\exists\mu)\,(1 - \mu\beta)|s| \leqslant \sqrt{a_\mu h_{\bar\mu}}$ which infers either $|\beta| \geqslant 1$ or $\vec{g} \in \mathbb{E}^\mu_x$ (i.e., $R \geqslant 0$).

(b) It follows from the identities $A_+B_+ + A_-B_- - 2c = 2(1-\gamma)$ and $(A_+B_+ - c)\vec{L}_{--} + (A_-B_- - c)\vec{L}_{++} = 2(y+\gamma x)\vec{n} - 2(x+\gamma y)\vec{m}$ whose length squared is equal to $4s^2|\vec{g}|^2$ and $R = (1-\gamma)^2 - |\vec{g}|^2 - \Delta_+$.

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