Radiation reaction at ultrahigh intensities

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Intensities of 10^{22} W cm^{-2} have been reached and it is expected that this will be increased by two orders of magnitude in the near future. At these intensities the radiation reaction force is important, especially in calculating the terminal velocity of an electron. The following briefly describes some of the problems of the existing most well-known equations and describes an approach based on conservation of energy. The resulting equation is compared to the Landau Lifshitz and Ford O'Connell equations, and laboratory tests are proposed.

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I. INTRODUCTION

In recent years laser intensities between $I = 10^{18}$ W cm^{-2} and $I = 10^{22}$ W cm^{-2} have been reached [1] and, with expectations to reach another couple of orders of magnitude, the often neglected self force becomes important. It is generally considered to be unimportant below 10^{22} W cm^{-2}, and becomes increasingly important as the intensity rises above this value. However, even at lower intensities, the correct description of physics cannot be obtained without the self force. For example, the net acceleration of a charged particle in a plane wave can only be accounted for by consideration of the radiation effects. [2] Besides this example, the self force must be taken into account above 10^{22} W cm^{-2}, and in the following these effects are shown explicitly.

When a charged particle is accelerated it creates a radiation field that acts back on the particle. This self force has been rooted at the origin of over a century's debate on this question. While some authors have claimed to have settled the issue, there is no universally accepted equation of motion that includes self forces. Perhaps it will remain this way until experimental evidence gathers enough strength to point its fateful finger to the truth. Below, an approach will be developed and it will be compared, for special realistic situations, with other theories, and may therefore be tested in the near to immediate future.

In a culmination of the work of Lorentz and Abraham, Dirac derived what is usually referred to as the LAD (or LD) equation, which is [3],

$$m\frac{dv^\mu}{d\tau} = \frac{e}{c}F^{\mu\sigma}v_\sigma + m\tau_0\left(\ddot{v}^\mu + \frac{v^\mu}{c^2}\dot{v}_\sigma\dot{v}^\sigma\right), \tag{1}$$

where $\tau_0 = 2e^2/3mc^3$ and dots indicate differentiation with respect to proper time τ. The term with the parenthesis is called the self force, or the von Laue four-vector, or the Abraham vector. The first part of it is called the Schott force,

$$F_{\rm S}^\mu = m\tau_0\ddot{v}^\mu \tag{2}$$

the second is the radiation reaction force

$$F_{\rm R}^\mu = m\tau_0\frac{v^\mu}{c^2}\dot{v}_\sigma\dot{v}^\sigma. \tag{3}$$

The most grievous transgression of this equation, which comes from the Schott force, can be seen in the low velocity limit for the case of no external force, letting $\dot{v} = a$,

$$a = \tau_0\dot{a}. \tag{4}$$

The solution, in which the particle with no external forces is accelerated to the speed of light in practically no time, is called the runaway solution. It as an egregious case of nonconservation of energy and is clearly unphysical. There is no field to begin with, and there is no known mechanism by which this charge can create some sort of inductive field to preserve our notion of conservation of energy. Even with other forces present (and also in the relativistic case), the Schott term tends to introduce unphysical runaway solutions. For these reasons the LAD equation is not considered to be a correct description of the motion, and for decades work has progressed to find a suitable replacement.

The runaway solution is an indication of a deeper problem, the problem with conservation of energy. To see this, for $v \ll c$, suppose we integrate the zero (time) component of (1) with respect to the proper time. This gives

$$mc\gamma = \frac{e}{c}\int Ev\,dt + m\tau_0\dot{v}^0 \tag{5}$$

for the case of a charged particle released from rest in an electric field. (One must be careful here. This equation is not valid when the Schott term vanishes to lowest order in $1/c$, which happens for the constant field.)

For the case that E is a monotonic function of time (and ignoring the concomitant magnetic field, (5) leads to

$$K = W_F + m\tau_0 V\dot{V}, \tag{6}$$

where W_F is the work done by the external force, $V = dx/dt$, and we used $\dot{v}^0 \sim V\dot{V}/c$. Since this last term is greater than zero, we have shown that the kinetic energy of the particle is *greater* due to radiation effects, which makes no sense. This problem is so irksome that some authors conclude that rest mass is not conserved [4,5].

This is related to a well known issue about the LAD equation. If the zero (time) component of (1) for the general case is integrated we find

$$mc^2(\gamma - \gamma_{\rm inc}) = \int \boldsymbol{F}\cdot d\boldsymbol{x} - \int P\,dt + \tau_0\left(\dot{v}^0 - \dot{v}^0_{\rm inc}\right), \tag{7}$$

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where $\boldsymbol{F} = e\boldsymbol{E}$, the power radiated is $P = -m\tau_0\dot{v}_\sigma\dot{v}^\sigma$, and x and t are the laboratory coordinates in which E is measured. In words this equation reads, the change in the kinetic energy, K, is equal to the work done by the external field, W_F, minus the energy radiated, W_R, *plus something else*. This something else is well known to be the Schott energy, but it seems very strange indeed. In fact, the Schott term is what gives rise to the runaway solutions we encountered earlier, solutions that violate conservation of energy.

Although there have been attempts to reconcile this in special cases [7], it is an abiding problem. Sorkin [6] in a recent article concerning the constant field, sums it up: "A well-known peculiarity of the radiation reaction force on a charged particle is that it vanishes when the particle accelerates uniformly. But this raises a paradox. An accelerating charge radiates, and the longer the acceleration continues, the greater the total energy radiated. If one asks where this energy comes from in the case of uniform acceleration, the usual answer is that it is 'borrowed' from the near field of the particle and then 'paid back' when the acceleration finally ceases. But this 'debt' can be arbitrarily great if the acceleration remains uniform for a long enough time. What, then, if the agent causing the acceleration decides not to repay the borrowed energy? What if, in fact, it does not even possess enough energy to pay its immense debt at that time? If we believe in conservation of energy, the respective answers must be that the accelerating agent must not be at liberty to avoid transferring the required energy and that it must always possess the necessary amount to cover its accumulated debt."

One approach that has been used to remove the runaway solutions converts the third order equation to a second order one by the use of an integrating factor. This technique is used in the nonrelativistic case but in this case the resulting equation of motion violates causality. In particular, one has [8]

$$\dot{V} = \frac{1}{m}\int_0^\infty F(t + s\tau_0)s^{-s}ds. \tag{8}$$

If we expand F in a Taylor series, $F(t + s\tau_0) = F(t) + s\tau_0\dot{F}(t)$, and neglect higher order terms, (8) becomes

$$m\dot{V} = F(t) + \tau_0\dot{F}(t). \tag{9}$$

Equivalently, by noting that the right hand side is the Taylor series of $F(t + \tau_0)$ (neglecting higher order terms) we obtain the great surprise

$$m\dot{V} = F(t + \tau_0), \tag{10}$$

or equivalently, with $\dot{V} = a$,

$$ma(t - \tau_0) = F(t). \tag{11}$$

This shows that the electron must be prophetic—it begins to accelerate at the time τ_0 before the force arrives. This violation of causality is often reluctantly accepted since it is such a tiny violation, but surely it is unphysical. For special cases, this technique has been attempted in the relativistic case [9].

Some reasonable results can be salvaged by tinkering with the LAD equation. For example, as a simple case consider (4), written as $a - \tau_0\dot{a} = 0$. Since τ_0 is so small we may recognize the left side as the Taylor series expansion, so that we have, to order τ_0, $\dot{a} = 0$, a benign equation which is exactly what we would expect. In fact, in a more general approach, assuming that the acceleration occurs at a different time than the force, the pesky runaway solution disappears [10]. For example, assuming that the self force acts a different time (by τ_0) one may derive

$$m\frac{dv^\mu}{d\tau} = \frac{e}{c}F^{\mu\sigma}v_\sigma + \frac{e\tau_0}{c}\left(\frac{d}{d\tau}(F^{\mu\sigma}v_\sigma) - \frac{v^\mu v_\gamma}{c^2}\frac{d}{d\tau}(F^{\gamma\nu}v_\nu)\right). \tag{12}$$

which was derived years before with an entirely different derivation and is called the Ford O'Connell equation [11]. In a similar approach, one may expand in terms of τ_0 [13] and obtain the Landau Lifshitz equation [12]

$$\begin{aligned}\frac{dv^\mu}{d\tau} &= (e/mc)F^{\mu\sigma}v_\sigma + \tau_0\big[(e/mc)\dot{F}^{\mu\sigma}v_\sigma \\ &\quad + (e/mc)^2\big(F^{\mu\gamma}F_\gamma{}^\phi v_\phi + F^{\nu\gamma}v_\gamma F_\nu{}^\phi v_\phi v^\mu\big)\big].\end{aligned} \tag{13}$$

Since all derivations involved some sort of expansion, they are only reliable when the self force is a small effect. But today we have situations in which laser intensities have reached 10^{22} W cm^{-2}, with expectations to increase this by at least two orders of magnitude. As we will see explicitly below, this brings us to the realm in which self force effects are not small, and may even be dominant. Thus, we cannot trust these equations that were derived in a series approach, assuming the self force is small.

It has been shown elsewhere that the FO and LL equations are equivalent to order τ_0 [13], which is a reasonable result in lieu of the fact that they have been derived by a series that utilizes the smallness of τ_0. However, there are problems with the FO and LL equations: One occurs when they are applied to the age old problem of a charged particle in a uniform electric field. The solutions are perplexing, telling us that the radiation reaction has no effect whatsoever on the equation of motion. The problem of a uniform field has been the subject of a hot and sometimes contentious debate. For a recent paper with references one may consult the literature [7].

There have been a number of attempts to overcome these problems. Rohrlich's well-known book gives many details and references and details on the history of self forces up to 1965 [14]. Later, Mo and Papas [15] add a term proportional to the acceleration, which was analyzed by Shen [16,17]. Steiger and Woods [18,19] derive a self force from the average power radiated in a cycle, Herrera considered the problem of the uniform magnetic field [20], while Moniz and Sharp consider the limit of QED [21]. Ford and O'Connell developed their equation [22–24], Hartemann and Luhmann [25] use an averaging technique over spatial integrations, Rohrlich, based on a treatment by Spohn [26], averred the LL equation is correct [27], while Bosanac assumed mass is converted to energy of the electromagnetic field [5]. More general approaches, including looking at higher dimensionality and magnetic charge were also done [28–38]. In the case of magnetic charge [28], it was shown that the magnetic dipole gives terms that are fourth order in the retarded time expansion. The method developed in the present paper is able to be generalized to this case in a straightforward manor by including the energy radiated by a magnetic dipole in the power P.

Despite this body of work, indicative of an even larger effort, no equation has been accepted by all as the correct one. I propose to obviate all of the problems outlined above by assuming energy is conserved without assuming some of it in wondrously stored in an immeasurable field. Starting with the radiation free equation, we assume there is a self force, f^μ,

$$m\frac{dv^\mu}{d\tau} = \frac{e}{c}F^{\mu\sigma}v_\sigma - f^\mu \tag{14}$$

and set out to find f^μ by assuming that the change in the kinetic energy is equal to the work done by the external field minus the energy radiated,

$$K = W_F - W_R. \tag{15}$$

In fact, Ford and O'Connell believed this so strongly that they were willing to replace the Larmor formula with a radiation formula that conserved energy, as stated above [22]. To begin our search we consider the one dimensional nonrelativistic limit,

$$m\frac{dv}{dt} = F - f, \tag{16}$$

where F is any external force. Multiplying by v and integrating with respect to time we find

$$\frac{1}{2}mv^2 = \int F dx - \int f v dt. \tag{17}$$

Now, assuming that the kinetic energy is equal to the work done by the external field minus the energy radiated, and noting that the energy radiated is $m\tau_0 \int \dot{v}^2 dt$ we find $fv = m\tau_0\dot{v}^2$. Thus we have two equation and two unknowns, v and f, but we must generalize this to three dimensions and to the relativistic realm.

To do this, we realize that the energy radiated is a scalar quantity (nonrelativistically), and assume, as is true for many forces, that it is derivable from a potential according to $\boldsymbol{f} = -\nabla\phi$. The relativistic generalization is $\nabla \to \phi^{\cdot\mu}$ and the equation of motion becomes

$$m\frac{dv^\mu}{d\tau} = \frac{e}{c}F^{\mu\sigma}v_\sigma + \phi^{\cdot\mu} - \frac{v^\mu}{c^2}\dot{\phi}, \tag{18}$$

where the last term is added to ensure $v_\sigma\dot{v}^\sigma = 0$. Thus, the radiation reaction force is given by $-f^\mu = \phi^{\cdot\mu} - v^\mu\dot{\phi}/c^2$. The $\phi^{\cdot\mu}$ are found by conservation of energy: The change in the kinetic energy is equal to the work done by the external field minus the energy radiated [13,39]. Thus, integrating the time component of (18) we find

$$f^0 = \gamma P/c, \tag{19}$$

which ensures conservation of energy. Obviously this is not a covariant statement, it holds in the laboratory frame (where we measure E, x, and t). Thus, although (18) is covariant, once we establish (19), we have chosen a frame. The idea is to treat (19) and (18) as coupled equations.

Now the question becomes: is (18) better than the LAD, FO, and LL equations, and is it correct. As far as comparison to the LAD equation, we can see that (18) does not give rise to the runaway solution LAD has, but how does it compare to the LL and FO equations?

To answer this question, and to answer the more important question 'is it correct,' we shall look at experimental situations that are being created in the laboratory, and ones that are on the horizon. These equations may be used for any situation where radiation reaction is involved, but the most important case is the ultrarelativistic case where $\gamma \gg 1$. For an electron in an electromagnetic pulse this corresponds to intensities above 10^{20} W cm^{-2}, and certainly above 10^{22} W cm^{-2}, the regime where radiation reaction begins to become important. In this case $\phi^{\cdot\sigma}v_\sigma v^\mu/c^2 \gg \phi^{\cdot\mu}$, so $f^\mu = v^\mu\dot{\phi}/c^2$, and the conservation of energy statement gives $\dot{\phi} = P$.

These results will be studied numerically for high intensity pulses and compared to the LL and FO equations for a specific case of an electromagnetic pulse containing just a few wavelengths. We nondimensionalize by letting $kx^\mu \to x^\mu$ and $\omega t \to t$, where $c = \omega/k$, $k = 2\pi/\lambda$, and we take $\lambda = 8\,000$Å, a typical approximate IR frequency. In particular the electric field is given by

$$\boldsymbol{E} = E_0 e^{-[(z-t)/w]^2}\cos[\Omega(z - t + \lambda)]\boldsymbol{x}, \tag{20}$$

where the dimensionless Ω allows us to explore frequencies other than that of the IR laser. The dimensionless w is used to describe the width of the pulse and is taken to be $2\pi/\Omega$, which maintains an envelope containing a few wavelengths. To relate the electric field to the average intensity, we use $E_0 = \sqrt{8\pi I/c}/w$.

Above an intensity of 10^{20} W cm^{-2}, $v^0 \approx v^3$, and from the symmetry (plane of polarization of the field), $v^2 = 0$, so that in the following only v^1 and v^3 are displayed. Also, we will keep things in terms of the proper time, although one may readily convert to laboratory time. Finally, it should be pointed out that to obtain ultra-high intensity, the laser beam is focused down to a thin waist, so that there is also a z component to the field. For the purposes of the comparisons made here, it is not essential to use such a pulse shape, although it presents interesting future research, and in fact, may be have a considerable effect on the radiation reaction.

We now consider numerical solutions of (13) (solutions subscripted with L), (12) (solutions subscripted with F), and (18) (solutions given no subscript). First, we check to see that the change in kinetic energy ($E_{\rm kin}$) is equal to the work done by the external force (W_f) minus the energy radiated away (W_r), which is shown in Fig. 1, where all of these quantities are divided by mc^2. For the LL or FO approach, such results are not even approximated by the solutions. It is interesting to note, in this regime, the exponential dependence of energy on intensity. In this graph we use

$$I = 10^{19+n}. \tag{21}$$

Next we consider a direct comparison of the (four) velocity as a function of time. At and below 10^{22} W cm^{-2}, the results seem to overlap (but see below), but at 10^{24} W cm^{-2} and $\Omega = 0.1$ (or 10^{25} W cm^{-2} and $\Omega = 1$) we find the result presented in Fig. 2.

The most important difference in these results is the final z component of the velocity. One may recall the Lawson-Woodward theorem that states, if radiation reaction is excluded, the particle gains no net momentum from the pulse. Since radiation reaction is being considered it is clear that

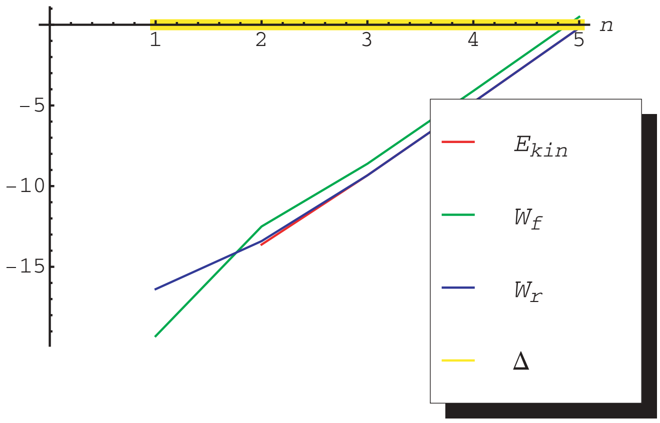

FIG. 1. (Color online) Listplot of $\ln(E_{\rm kin})$, $\ln(W_f)$, $\ln(W_r)$ and (not the ln of) $\Delta = EW_{\rm kin} - W_f + W_r$. $E_{\rm kin}$ (red curve beginning at $n = 2$) and W_r overlap at this scale.

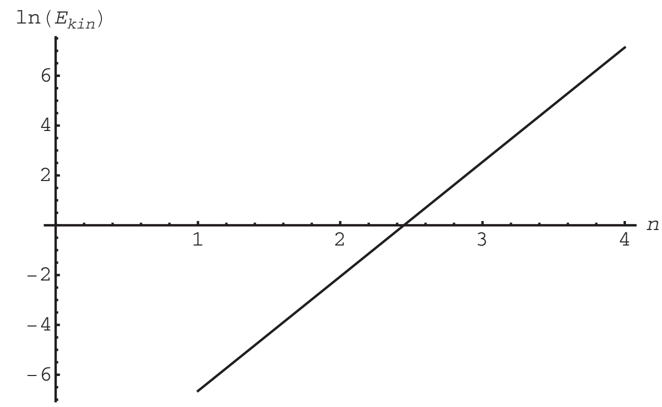

FIG. 3. The value of the ln of the kinetic energy per rest energy.

Lawson-Woodard does not apply. Even at lower intensities the z component of the velocity is substantially different for the present theory and the older theories. For example, at 10^{22} W cm^{-2} ($\Omega = 1$) the final z component of the (four) velocity is $0.13c$ while the LL and FO theories give values that are over two orders of magnitude smaller. At 10^{20} W cm^{-2} and smaller, however, all theories converge.

In order to indicate when the radiation reaction is important, one may use the final value of the z component of the velocity. It is zero if there is no radiation reaction, so its value per unit rest mass gives some indication of the effects of radiation reaction. For $\gamma \gg 1$, $v^3 \approx v^0$ so that $v^3/c \approx E_{\rm kin}$ (recall $E_{\rm kin}$ is the kinetic energy divided by mc^2). This is shown in Fig. 3 for the current theory. Between 10^{22} W cm^{-2} and 10^{23} W cm^{-2} the kinetic energy equals the rest energy, signaling he importance of radiation reaction.

In summary it has been shown that the effects of radiation reaction can be well described using the approach based on conservation of energy. It gives results quite different than existing formulations at high intensities. It was shown that the LAD equation, as is well known, leads to unphysical runaway solutions. For this reason alone one may conclude that the equation is incorrect, but it was shown that this arises from a deeper problem, explicitly, nonconservation of energy. The conventional argument is to assume that the missing energy resides in an induction field, but is is shown here that by changing this view to one in which energy is conserved without the mysterious induction field, sensible equations follow. The other two most quoted equations are the Landau Lifshitz and the Ford O'Connel equations. Since these may be derived from the LAD equation, we should not expect them to be exact. In fact, contrary to the result presented in Fig. 1, the solutions do not conserve energy. In addition, the final velocity of the electron is much greater in the current theory than that of LL and FO, and laboratory tests may be made. However, in this work the phase λ was set to zero, but the results depend quite significantly on the value of this constant. The value of this constant, called the envelope or absolute phase, is in general impossible to control. Few cycle femtosecond lasers emit pulses with varying phase: It cannot be made constant because the phase and group velocities of the beam in the laser cavity are not the same [40]. In fact, one may

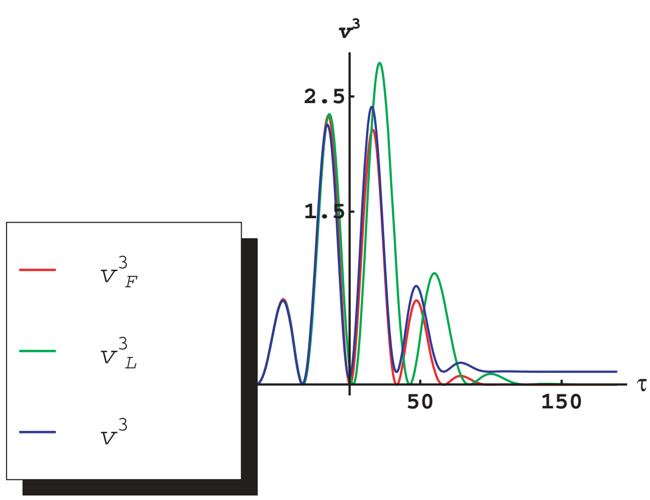

FIG. 2. (Color online) The z component of the four velocities (times 10^{-7}) for the three different methods. The highest peak (green) corresponds to v_L^3, the highest final velocity (blue) corresponds to v^3.

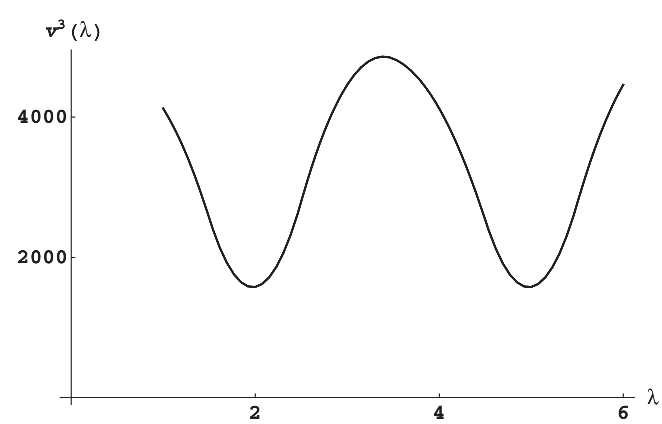

FIG. 4. The z component of the four velocities for the three different methods.

measure the final velocity to *determine* the phase. For example, for 10^{22} W cm^{-2} and $\Omega = 1$ the final velocity (z component) is plotted in Fig. 4 for λ ranging from 0 to 2π. The numerical work was done with Mathematica, the zero values reported above for $E_{\rm kin} - W_f + W_r$ were correct to five decimal places for intensities up to 10^{25} W cm^{-2}.

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