Quantum discord, local operations, and Maxwell's demons

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Quantum discord was proposed as a measure of the quantumness of correlations. There are at least three different discordlike quantities, two of which determine the difference between the efficiencies of a Szilard's engine under different sets of restrictions. The three discord measures vanish simultaneously. We introduce an easy way to test for zero discord, relate it to the Cerf-Adami conditional entropy and show that there is no simple relation between the discord and the local distinguishability.

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I. INTRODUCTION

We learn about the external world from correlations between our measuring devices and the systems we study. The information flow is studied in (classical) information theory through the analysis of correlations [1]. Their buildup and propagation are at the core of measurement theory [2,3]. The research of their behavior with respect to space-time locality led to the identification of quantum entanglement [3,4]. All correlations—classical and quantum—are important in statistical thermodynamics [5], which, in turn, influences the entanglement theory [6]. Initial correlations modify the dynamics of open systems [7,8], and a significant part of quantum information processing is devoted to preservation and manipulation of useful correlations, while trying to mitigate the effects of the unwanted ones.

A boundary between quantum and classical correlations is sharp for pure states, which are either simply separable product states or entangled. It becomes less clear for mixed states, particularly for systems larger than a pair of qubits. We investigate this boundary through a characteristic of quantum discord [9–11] (more precisely, we review and define three similar, but subtly different, discord measures).

First, we recall some basic definitions [1] and set the notation. The (Shannon) entropy of a classical discrete probability distribution $p(a) \equiv p_a$ over a random variable A is defined by

$$H(A) = -\sum_a p_a \log p_a, \tag{1}$$

where the logarithm has, either a natural ($\log e \equiv \ln e = 1$) or a binary ($\log 2 = 1$) base. Both bases work with obvious adjustments in constants. However, when it is necessary to compare quantities of interest with the results of quantum information protocols, we express them in qubits and adapt the binary basis (end of this section, Sec. II C, and Sec. III).

Entropy of the joint probability distribution $p(a,b)$ over AB, $H(AB)$ is defined analogously. The Bayes theorem relates it to the conditional probabilities,

$$p(a,b) = p(a|b)p(b) = p(b|a)p(a), \tag{2}$$

where $p(a|b)$ is a conditional probability of $A = a$ given that $B = b$. The conditional entropy of A,

$$H(A|B) = \sum_b p_b H(A|b) = -\sum_{a,b} p(a,b) \log p(a|b) \tag{3}$$

is a weighted average of the entropies of A given a particular outcome of B.

Correlations between two probability distributions are measured by the symmetric mutual information. It has two equivalent expressions,

$$I(A:B) = H(A) + H(B) - H(A,B), \tag{4}$$

and

$$J(A:B) = H(A) - H(A|B) = H(B) - H(B|A). \tag{5}$$

Quantum-mechanical (von Neumann) entropy [3] is defined as

$$S(\rho) = -\mathrm{tr}\rho\ \log \rho. \tag{6}$$

It minimizes the Shannon entropy of probability distributions that result from rank-1 positive operator-valued measures (POVMs) that are applied to the state ρ on the Hilbert space $\mathcal{H}_A$. The minimum is actually reached on a probability distribution A that results from a projective measurement $\Pi = \{\Pi_a, a = 1, \ldots, d\}$, $\sum_a \Pi_a = \mathbb{1}$, $\Pi_a \Pi_b = \delta_{ab} \Pi_a$, which is constructed from the eigenprojectors of ρ,

$$S(\rho) = \min_\Pi H\big(A_\rho^\Pi\big), \tag{7}$$

that is,

$$S(\rho) = H\big(A_\rho^{\Pi^*}\big), \qquad \rho = \sum_a p_a \Pi_a^*, \quad p_a \geqslant 0, \quad \sum_a p_a = 1. \tag{8}$$

The expression A_ρ^Λ stands for a classical probability distribution (of a measured parameter A) that is obtained from the state ρ under the POVM Λ.

Quantum channels are abstracted as maps from the initial states ρ_A to the final states ρ_X, where the space $\mathcal{H}_X$ may be either the same space $\mathcal{H}_A$ or a different one [4]. Any orthonormal basis $\{|a\rangle\}$ of $\mathcal{H}_A$ defines a dephasing channel P,

$$\rho \mapsto \rho' = \mathsf{P}(\rho) = \sum_a \Pi_a \rho \Pi_a, \quad \Pi_a := |a\rangle\langle a|. \tag{9}$$

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Hence, taking the weighted average over the outcomes of a measurement $\Pi = \{\Pi_a\}_{a=1,\dots,d_A}$ is equivalent to sending the initial state through a dephasing channel with a superoperator $\mathsf{P} = \mathsf{P}_\Pi$.

The information function that expresses knowledge [6] about a system of dimension d in the state ρ is a variety of the negentropy,

$$K(\rho) = \log d - S(\rho). \tag{10}$$

It has two related operational interpretations. A Maxwell's demon can draw $K(\rho)$ units of work from a single heat bath using a Szilard engine [12]. We will discuss the demons in Sec. III.

On the other hand, $K(\rho)$ determines a conversion rate between pure and mixed states. By allowing arbitrary unitary operations, by adding of maximally mixed ancillas, and by taking partial traces [and thinking in terms of qubits (i.e., $\log 2 = 1$)], it is possible to perform two tasks with asymptotically perfect fidelity. First, given n copies of the state ρ, one can obtain $nK(\rho)$ qubits in a predetermined pure state. This is done by performing the usual quantum data compression, but instead of discarding the then-redundant pure qubits, it is the signal block that is discarded [6]. Second, by taking $nK(\rho)$ pure qubits and by diluting them with ancillas in the maximally mixed state, one produces n copies of ρ [13].

Quantum discord stems from the fact that the classical mutual information can be extended to quantum states in two inequivalent ways, following either Eq. (4) or Eq. (5). In Sec. II, we introduce the three discord measures and discuss some of their properties. Their role in the efficiency of different Szilard's engines is explored in Sec. III, and, in Sec. IV, we discuss their relationship with the local distinguishability of orthogonal states.

II. QUANTUM DISCORD

The first expression for mutual information has an obvious quantum generalization,

$$I(\rho_{AB}) := S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{11}$$

and represents the total amount of quantum and classical correlations [6,14,15].

A. Conditional state definition

To obtain a quantum version of $J(A\!:\!B)$, it is necessary to determine a conditional state of the subsystem B. If the objective is to preserve the equivalence of two definitions in the quantum domain $I(\rho_{AB}) \equiv J(\rho_{AB})$, then the conditional entropy can be introduced [16] as

$$S(\rho_{B|A}) = -\mathrm{tr}\rho_{AB} \log \rho_{B|A}. \tag{12}$$

The positive operator $\rho_{B|A}$ is defined through

$$\begin{aligned}\rho_{B|A} &:= \lim_{n\to\infty} \left[\rho_{AB}^{1/n}(\rho_A \otimes \mathbb{1}_B)^{-1/n}\right]^n \\ &= \exp(-\log \rho_A \otimes \mathbb{1}_B + \log \rho_{AB}),\end{aligned} \tag{13}$$

where the inverse of ρ_A is defined on its support. It does not usually have a unit trace, and when $\rho_A \otimes \mathbb{1}_B$ commutes with ρ_{AB} it reduces to

$$\rho_{B|A} = \rho_{AB}(\rho_A \otimes \mathbb{1}_B)^{-1}. \tag{14}$$

We will return to this quantity at the end of this section.

B. Three versions of the discord

Given a complete projective measurement Π on A, a quantum definition of J follows its interpretation as the information gained about the system B from the measurement on A [9],

$$J^{\Pi^A}(\rho_{AB}) := S(\rho_B) - S(\rho_B|\Pi^A), \tag{15}$$

where the conditional entropy is now given by

$$S(\rho_B|\Pi^A) := \sum_a p_a S(\rho_{B|\Pi_a}). \tag{16}$$

The postmeasurement state of B that corresponds to the outcome $A = a$ is

$$\rho_{B|\Pi_a} = (\Pi_a \otimes \mathbb{1}_B \rho_{AB} \Pi_a \otimes \mathbb{1}_B)/p_a, \quad p_a = \mathrm{tr}\rho_A\Pi_a. \tag{17}$$

The state of B remains unchanged,

$$\rho_B = \mathrm{tr}_A\, \rho_{AB} = \sum_a p_a \rho_{B|\Pi_a}. \tag{18}$$

Unlike their classical counterparts, the quantum expressions are generally inequivalent, and $I(\rho_{AB}) \geqslant J^{\Pi^A}(\rho_{AB})$ [9–11]. The quantum discord as defined in Ref. [9] is the difference between these two quantities,

$$D_1^{\Pi^A}(\rho_{AB}) := S(\rho_A) + S(\rho_B|\Pi^A) - S(\rho_{AB}). \tag{19}$$

Its dependence on the measurement procedure is removed by minimizing the result over all possible sets of Π,

$$D_1^A(\rho_{AB}) := \min_{\Pi^A} D_1^{\Pi^A}(\rho_{AB}). \tag{20}$$

Similarly,

$$J_1^A(\rho_{AB}) := \max_{\Pi^A} J^{\Pi^A}(\rho_{AB}). \tag{21}$$

This definition of the discord has its origins in the studies of the measurement procedure and pointer bases, thus, projective measurements are natural in this context. It is possible to define the discord when the difference is minimized over all possible POVM Λ^A [10]. However, unless stated otherwise, we restrict ourselves to the projective measurements of rank 1.

An explicit form of a postmeasurement state will be useful in the following text. We denote this state as $\rho'_X \equiv \rho_X^{\Pi^A}$, where the subscript X stands for A, B, or AB, and use the former expression if it does not lead to confusion. After a projective measurement Π^A, the state of the system becomes

$$\rho'_{AB} = \sum_a p_a \Pi_a \otimes \rho_B^a, \tag{22}$$

where p_a and $\rho_B^a \equiv \rho_{B|\Pi_a}$ are given by Eq. (17), and the states of the subsystems are

$$\rho_A' = \sum_a p_a \Pi_a, \quad \rho_B = \rho_B' = \sum_a p_a \rho_B^a, \tag{23}$$

respectively.

The discord of the state ρ_{AB} is zero if and only if it is a mixture of products of arbitrary states of B and projectors on A [9],

$$\rho_{AB} = \sum_a p_a \Pi_a \otimes \rho_B^a, \quad p_a \geqslant 0, \quad \sum_a p_a = 1. \tag{24}$$

By using this decomposition and properties of the entropy of block-diagonal matrices [17], we can identify

$$J^{\Pi^A}(\rho_{AB}) \equiv I(\rho_{AB}'), \tag{25}$$

because $S(\rho_A') = H(A_\rho^\Pi)$ and

$$S(\rho_{AB}') = H\big(A_\rho^\Pi\big) + S(\rho_B|\Pi^A). \tag{26}$$

The discord is not a symmetric quantity: It is possible to have states with $0 = D_1^A(\rho_{AB}) \neq D_1^B(\rho_{AB})$. A subclass of separable states that satisfy $D_1^A = D_1^B = 0$ is of the form

$$\rho_{AB}^c = \sum_{ab} w_{ab}\, \Pi_a^A \otimes P_b^B, \tag{27}$$

where P^B is a set of projectors on $\mathcal{H}_B$, and consists of classically correlated states in the sense of Ref. [18].

Another possibility is to set

$$\begin{aligned} J_2^{\Pi^A} &:= S(\rho_A) + S(\rho_B) - \big(H\big(A_\rho^\Pi\big) + S(\rho_B|\Pi^A)\big) \\ &= (\rho_A) + S(\rho_B) - S(\rho_{AB}'), \end{aligned} \tag{28}$$

arriving to the quantum discord as defined in Ref. [19],

$$D_2^A(\rho_{AB}) := \min_\Pi \big(H\big(A_\rho^\Pi\big) + S(\rho_B|\Pi^A)\big) - S(\rho_{AB}), \tag{29}$$

where the quantity to be optimized is a sum of postmeasurement entropies of A and B. By using Eq. (7), we see that $D_1 \leqslant D_2$. It is also easy to see that $D_1 = 0 \Leftrightarrow D_2 = 0$. By using Eqs. (22) and (26), we obtain a different expression for D_2:

$$D_2^{\Pi^A}(\rho_{AB}) = S\big(\rho_{AB}^{\Pi^A}\big) - S(\rho_{AB}). \tag{30}$$

Since the definition of the discord(s) involves optimization, the analytic expressions are known only in some particular case [9,11,20]. Moreover, typically, it is important to know whether the discord is zero or not, while the numerical value itself is less significant.

It follows from Eq. (24) that if the spectrum of a reduced state $\rho_A = \sum_a p_a \Pi_a$ is nondegenerate, then its eigenbasis gives a unique family of projectors Π that results in the zero discord for ρ_{AB}. Hence, a recipe for testing states for zero discord and for finding the optimal basis is to trace out a subsystem that is left alone (B), to diagonalize ρ_A, and to calculate the discord in the resulting eigenbasis.

If the state ρ_A is degenerate, a full diagonalization should be used. For the state of Eq. (24), each of the reduced states ρ_B^a can be diagonalized as

$$\rho_B^a = \sum_b r_b^a P_a^b, \quad P_a^b P_a^{b'} = \delta^{bb'} P_a^b. \tag{31}$$

The eigendecomposition of the state ρ_{AB} then easily follows. Writing it as

$$\rho_{AB} = \sum_{a,b} w_a r_b^a \Pi_a \otimes P_a^b, \tag{32}$$

it is immediate to see that its eigenprojectors are given by $\Pi_a \otimes P_a^b$. Hence, if ρ_B has a degenerate spectrum, but ρ_{AB} has not, the structure of its eigenvectors reveals if it is of a zero or nonzero discord. Hence, we established

Property 1. The eigenvectors of a zero discord state $D_1^A(\rho_{AB}) = 0$ satisfy

$$\rho_{AB}|ab\rangle = r_{ab}|ab\rangle \Rightarrow |ab\rangle\langle ab| = \Pi_a \otimes P_a^b. \tag{33}$$

■

This consideration leads to the simplest necessary condition for zero discord (first noticed in Ref. [21]):

Property 2. If $D_1^A(\rho_{AB}) = 0$, then

$$[\rho_A \otimes \mathbb{1}_B, \rho_{AB}] = 0. \tag{34}$$

Hence, a nonzero commutator implies $D_1^A(\rho_{AB}) > 0$. ■

Naturally, if the state has a zero discord, and the eigenbasis is only partially degenerated, we can use it to reduce the optimization space. On the other hand, the diagonalizing basis Π_* is not necessarily the optimal basis $\hat\Pi$ or $\check\Pi$ that enters the definition of D_1 or D_2, respectively. Consider, for example, a two-qubit state

$$\rho_{AB} = \tfrac{1}{4}\big(\mathbb{1}_{AB} + b\sigma_A^z \otimes \mathbb{1}_B + c\sigma_A^x \otimes \sigma_B^x\big), \tag{35}$$

where σ_X^a are Pauli matrices on the relevant spaces $X = A, B$, and the constants b and c are restricted only by the requirements that ρ_{AB} is a valid density matrix. For this state, $\rho_B = \mathbb{1}/2$ and $\rho_A = \mathrm{diag}(1+b, 1-b)/2$. After the measurement in the diagonalizing basis $\Pi^z = \big((\mathbb{1}+\sigma^z)/2, (\mathbb{1}-\sigma^z)/2\big)$, the conditional state of B becomes

$$\rho_{B|\Pi_\pm^z} = \mathbb{1}/2, \tag{36}$$

and the conditional entropy is maximal, $S(\rho_B|\Pi^z) = \log 2$.

On the other hand, in the basis $\Pi^x = \big((\mathbb{1}+\sigma^x)/2, (\mathbb{1}-\sigma^x)/2\big)$, the probabilities of the outcomes are equal, $p_+ = p_- = 1/2$, but the postmeasurement states of B are different from the maximally mixed one,

$$\rho_{B|\Pi_\pm^x} = \tfrac{1}{2}(\mathbb{1} \pm c\sigma^x), \tag{37}$$

so the entropy $S(\rho_B|\Pi^z) \geqslant S(\rho_B|\Pi^x)$.

This discrepancy motivates us to define a new version of the discord, which is useful if the eigenvectors of ρ_A are not degenerate,

$$D_3^A(\rho_{AB}) := S(\rho_A) - S(\rho_{AB}) + S(\rho_B|\Pi_*^A), \tag{38}$$

where Π_*^A is the set of eigenprojectors of ρ_A. Otherwise, it can be introduced using the continuity of entropy in finite-dimensional systems [17]. By applying Eq. (7) to the subsystem A, we find that for D_3, simultaneously hold the analog of Eq. (25),

$$J_3(\rho_{AB}) \equiv I\big(\rho_{AB}^{\Pi_*^A}\big), \tag{39}$$

and the analog of Eq. (30),

$$D_3(\rho_{AB}) \equiv S\big(\rho_{AB}^{\Pi_*^A}\big) - S(\rho_{AB}). \tag{40}$$

We also arrive at the following ordering of the discord measures:

$$D_1^A \leqslant D_2^A \leqslant D_3^A. \tag{41}$$

There are several important cases when the measures of discord coincide. Most importantly, they vanish simultaneously:

Property 3. $D_1 = 0 \Leftrightarrow D_2 = 0 \Leftrightarrow D_3 = 0$.

The proof follows from Eqs. (24) and (41). ■

On pure states, the discord is equal to the degree of entanglement,

$$D_i^A(\phi_{AB}) = S(\phi_A) = E(\phi_{AB}), \quad i = 1,2,3. \tag{42}$$

Discord is also independent of the basis of measurement if the state is invariant under local rotations [9]. Finally, if A is in a maximally mixed state, then $D_1^A = D_2^A$.

These coincidences make it interesting to check when the discords D_1 and D_2 are different. By returning to the measurement-dependent versions of the discords, we see that

$$D_1^{\Pi^A}(\rho_{AB}) = D_2^{\Pi^A}(\rho_{AB}) - \big(H\big(A_\rho^\Pi\big) - S(\rho_A)\big). \tag{43}$$

Assume that $D_2^{\Pi^A}(\rho_{AB})$ reaches the minimum on the set of projectors $\check{\Pi}$, which are not the eigenprojectors of ρ_A. In this case, $H(A_\rho^{\check{\Pi}}) - S(\rho_A) > 0$, so we can conclude that the strict inequality $D_1^A < D_2^A$ holds, because

$$\begin{aligned} D_1^A(\rho_{AB}) &\leqslant D_1^{\check{\Pi}}(\rho_{AB}) \\ &= D_2^A(\rho_{AB}) - \big(H\big(A_\rho^{\check{\Pi}}\big) - S(\rho_A)\big) < D_2^A(\rho_{AB}). \end{aligned} \tag{44}$$

For example, the state of Eq. (35), with $b = c = \frac{1}{2}$, satisfies $D_1^A \approx 0.05$, $D_2^A \approx 0.20$, and $D_3^A \approx 0.21$.

C. Relations with other quantities

Quantum discord $D_1^{\Lambda^A}(\rho_{AB})$ is a concave function over the set of all POVMs Λ^A [11], and the minimum is reached on a rank-1 POVM that consists of linearly independent operators [22]. The easiest way to obtain this result is to note that the set of all POVMs is convex, and the minimum of a concave function over a convex set is obtained on its boundary.

The states of zero discord are nowhere dense anywhere in the set of all states [21]. Nevertheless, it is obvious that even double-zero discord states of Eq. (32) convexly span the set of all states.

The operator $\rho_{B|A}$ has a closed form on the states of zero discord. Property 1 allows us to write Eq. (14) as

$$\begin{aligned} \rho_{B|A} &= \left(\sum_a w_a \Pi_a \otimes \rho_B^a\right)\left(\sum_b \frac{1}{w_b}\Pi_b \otimes \mathbb{1}_B\right) \\ &= \sum_a \Pi_a \otimes \rho_B^a, \end{aligned} \tag{45}$$

which indeed results in the conditional entropy:

$$S(\rho_{B|A}) = \sum_a w_a S\big(\rho_B^a\big) = S(\rho_B|\Pi^A). \tag{46}$$

For general states, the definitions imply

$$S(\rho_{B|A}) = S(\rho_B|\Pi^A) - D_1^{\Pi^A}(\rho_{AB}), \tag{47}$$

in any basis, not only in the optimal one.

In the paradigm of closed local operations [6], Alice and Bob are allowed to perform arbitrarily local unitary operations and projective measurements, and Alice can send her system to Bob via a dephasing channel. In the one-way version, only a single use of the channel is allowed. Since, at the end of the operation, both systems are accessible to Bob, the discord D_2^A was identified with one-way quantum deficit [6],

$$\Delta^{\rightarrow}(\rho_{AB}) = \min_{\Pi^A} S\big(\rho_{AB}^{\Pi^A}\big) - S(\rho_{AB}) = D_2^A(\rho_{AB}). \tag{48}$$

Operationally, it expresses the fact that a simple one-way purification strategy consists of Alice performing the measurement $\check{\Pi}$ and announcing her results to Bob (or, equivalently, sending her part of the state individually through the channel $\mathsf{P}_{\check{\Pi}}$). In this case, the purification rate is given by $K_2(\rho_{AB}) = (\log d_A d_B) - S(\rho_{AB}^{\check{\Pi}^A})$.

However, it is the discord D_1^A that gives the optimal efficiency of a purification in this context. If Alice is allowed to borrow pure states (that are returned at the completion of the protocol) and to use block encoding prior to (individually) sending her particles to Bob, then [23] the optimal rate is

$$K^{\rightarrow}(\rho_{AB}) = (\log d_A d_B) + I\big(\rho_{AB}^{\hat{\Pi}}\big) - S(\rho_A) - S(\rho_B), \tag{49}$$

so, by using Eq. (25), we see that $K^{\rightarrow}(\rho_{AB}) = (\log d_A d_B) + J_1^A(\rho_{AB}) - S(\rho_A) - S(\rho_B)$, or

$$K(\rho_{AB}) - K^{\rightarrow}(\rho_{AB}) = D_1^A(\rho_{AB}). \tag{50}$$

The additivity of J_1^A was shown to be equivalent to that of several other quantities [23], including the Holevo capacity of quantum channels. The additivity of the latter was disproved [24], so a block processing will improve the distributed purification efficiency for entangled ρ_{AB}.

A symmetrized version of J_3 involves (projective) measurements on both sides [25] and is given by $J_3^{\rm sym}(\rho_{AB}) = I(\rho_{AB}^{\Pi_*^A \otimes \Pi_*^B})$. The symmetrized discord,

$$D_3^{\rm sym} := I(\rho_{AB}) - J_3^{\rm sym}(\rho_{AB}) \tag{51}$$

is called the measurement-induced disturbance and serves as another upper bound on D_1 and D_2.

III. LOCAL MAXWELL'S DEMONS

Maxwell's demon [26] is a being whose facilities are so sharpened as to enable him to challenge the second law of thermodynamics. Modern exorcism mostly focuses on his information-processing ability, with information erasure cost balancing the books and keeping the second law intact. Quantum logic and quantum correlations introduce new subtleties into this discussion [12].

A typical setting is provided by a (quantum) Szilard's model [27], in whose original form the demon operates a heat engine with one-particle working fluid. For our purposes, it is enough to consider only the work-extracting phase of the cycle and to ignore the resetting of the demon's memory. The optimal work extracted from a system of a dimension d in a known state ρ at the temperature T is, on average,

$$W^+ = kT\big(\log d - S(\rho)\big) = kTK(\rho), \tag{52}$$

where k is the Boltzmann constant adjusted to the base of the logarithm.

For a bipartite state ρ_{AB}, the benchmark performance $W^+(\rho_{AB})$ is achieved by a fully quantum (nonlocal) demon Charlie that can perform arbitrary quantum operations on the system. We compare his performance with actions of two local goblins of lesser powers. Alice and Bob are goblins that can perform only local operations on their subsystems. They may have only partial information about the state ρ_{AB} and may not be able to communicate freely.

The work that is extracted by Alice and Bob that are aware of their respective states $\rho_{A,B}$ but are not allowed to communicate is

$$W_L = kT\big((\log d_A d_B) - S(\rho_A) - S(\rho_B)\big), \qquad (53)$$

so the difference of the extracted work by a global demon and local noncommunicating goblins is given by the mutual information,

$$\Delta_L W := W^+ - W_L = kT\,I(\rho_{AB}). \qquad (54)$$

A much more interesting scenario was proposed in Ref. [19]. In this setting, both Alice and Bob know the state ρ_{AB}, and Alice can communicate to Bob the results of her measurement. She chooses her measurement Π in such a way as to maximize the extracted work,

$$\begin{aligned} W_2 &= \left(\log d_A - H\left(A_\rho^\Pi\right)\right) + \big(\log d_B - S(\rho_B|\Pi^A)\big) \\ &= (\log d_A d_B) - S\left(\rho_{AB}^{\Pi^A}\right), \end{aligned} \qquad (55)$$

through steering of Bob's state to $\rho_{B|\Pi_a}$, which, on average, makes up for a higher entropy of Alice's $\rho_A^{\Pi^A} = \sum_a \Pi_a \rho_A \Pi_a$. Hence, the minimal difference between the work extracted by the goblins and the work extracted by the demon is given by

$$\Delta_2 W = kT\,D_2^A(\rho_{AB}). \qquad (56)$$

In this setting, Alice and Bob are essentially performing the purification protocol from Sec. II C.

An operational meaning of D_3 is clarified in the setting where Alice is still able to report her results to Bob, but has less knowledge than in the original example. Namely, Alice knows only ρ_A, while Bob is aware of the entire state ρ_{AB}. In this case, the best Alice can do is to perform the measurement in the eigenbasis of ρ, and tell her result to Bob. Then, on average, the gain is

$$W_3 = kT\big(\log d_A - S(\rho_A)\big) + kT\big(\log d_B - S(\rho_B|\Pi_*^A)\big), \qquad (57)$$

so the difference in the extracted work is now:

$$\Delta_3 W = kT\,D_3^A(\rho_{AB}). \qquad (58)$$

IV. LOCAL DISTINGUISHABILITY AND DISCORD

Since zero discord is thought to represent the absence of classical correlations, it is interesting to investigate the following question. Consider a set of pure orthogonal bipartite states, each of which may have a different prior probability, with the ensemble density matrix ρ_{AB}. Does the value of $D(\rho_{AB})$ tell us something about the ability to perfectly distinguish these states by local operations and classical communication (LOCC)?

TABLE I. Local measurability vs discord.

States	Discord	Locally measurable
Nine 3×3 product orthogona states, equal weights	$D^A = D^B = 0$	No
Two product bi-orthogonal states	$D^A = D^B = 0$	Yes
Two entangled orthogonal states	$D_1^A > 0$	Yes
Nine 3×3 product orthogonal states, unequal weights	$D_1^A > 0$	No

As exhibited in Table I, there is no relation between $D(\rho_{AB})$ and local distinguishability. First, it was shown in Ref. [28] that a certain set of nine 3×3 product orthogonal states cannot be perfectly distinguished by LOCC. These states $|\psi_1\rangle, \ldots, |\psi_9\rangle$ are

$$\begin{aligned} &|1\rangle \otimes |1\rangle, \quad |0\rangle \otimes |0 \pm 1\rangle/\sqrt{2}, \quad |2\rangle \otimes |1 \pm 2\rangle/\sqrt{2}, \\ &\qquad |1 \pm 2\rangle \otimes |0\rangle/\sqrt{2}, \quad |0 \pm 1\rangle|2\rangle/\sqrt{2}. \end{aligned} \qquad (59)$$

Their equal mixture is the maximally mixed state $\rho_{AB} = \mathbb{1}/9$ that obviously has a zero discord $D^A(\rho_{AB}) = D^B(\rho_{AB}) = 0$.

Second, it was shown [29] that any two orthogonal (entangled or not) states can be perfectly distinguished by LOCC. Consider a mixture of the Bell states $|\Psi^\pm\rangle$,

$$\rho_{AB} = a|\Psi^+\rangle\langle\Psi^+| + (1-a)|\Psi^-\rangle\langle\Psi^-|, \quad 0 < a < 1. \qquad (60)$$

The discord of such a state can be calculated analytically. It equals

$$D_{1,2}^{A,B}(a) = a\log_2 a - (1-a)\log_2 a + 1, \qquad (61)$$

which vanishes only for the equal mixture $a = \frac{1}{2}$.

Two other cases easily follow. A pair of bi-orthogonal product states, such as $|0\rangle|0\rangle$ and $|1\rangle|1\rangle$, results in a zero discord. Mixing the states of Eq. (59) with different weights may result in a nonzero discord. For example, giving $|\psi_9\rangle$ and $|\psi_7\rangle$ weights, which are twice as high as the rest of the states results in a mixture ρ_{AB} for which $[\rho_A \otimes \mathbb{1}_B, \rho_{AB}] \neq 0$, thus, implying a nonzero discord.

V. SUMMARY AND OUTLOOK

There are at least three useful one-way measures of the quantumness of states, namely, the three discords. They vanish simultaneously, and it is easy to check if this is the case by using Properties 1 and 2. The discord measures D_2 and D_3 have a natural physical interpretation in terms of the work extracted by a pair of Maxwell's demons that operate on a bipartite system under different sets of restrictions. On the other hand, depending on the imposed restrictions, it is either D_1 or D_2 that can serve as a measure of a quantum deficit in the state.

Despite its intuitive appeal, quantum discord $D(\rho_{AB})$ is not an indicator of a local distinguishability of the states making an ensemble ρ_{AB}.

Zero quantum discord allows for a completely positive dynamics of an open system even if there are initial correlations with the environment [8]. In a forthcoming paper [30], it will be shown that it is not enough for practical quantum tomography.

We discussed the discord in the states on finite-dimensional spaces. One obvious difficulty in the generalization for the continuous variables is in the infinite-dimensional optimization that is involved in the definition of D_1 and D_2. It is possible that the measure D_3 is more suitable in the latter case.

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