PHYSICAL REVIEW A **81**, 060307(R) (2010)

Strong entanglement causes low gate fidelity in inaccurate one-way quantum computation

Tomoyuki Morimae*
Laboratoire Paul Painlevé, Université Lille 1, F-59655 Villeneuve d'Ascq Cedex, France

(Received 4 February 2010; published 28 June 2010)

We study how entanglement among the register qubits affects the gate fidelity in the one-way quantum computation if a measurement is inaccurate. We derive an inequality that shows that the mean gate fidelity is upper bounded by a decreasing function of the magnitude of the error of the measurement and the amount of the entanglement between the measured qubit and other register qubits. The consequence of this inequality is that, for a given amount of entanglement, which is theoretically calculated once the algorithm is fixed, we can estimate from this inequality how small the magnitude of the error should be in order not to make the gate fidelity below a threshold, which is specified by a technical requirement in a particular experimental setup or by the threshold theorem of the fault-tolerant quantum computation.

DOI: 10.1103/PhysRevA.81.060307 PACS number(s): 03.67.Lx, 03.67.Mn

I. INTRODUCTION

The one-way quantum computation [1] is a novel scheme of quantum computation often contrasted with the traditional circuit model of quantum computation [2]. It is believed to be one of the most promising approaches to the realization of scalable quantum computers, and indeed, small-size cluster states have already been created in laboratories [3]. Some important quantum algorithms, such as the Deutsch algorithm [4] and the Grover search algorithm [5], have also been demonstrated experimentally.

One great advantage of the one-way quantum computation over the circuit model is that the preparation of the resource (entanglement) and the consumption of it are clearly separated with each other. This fact has prompted many researchers to explore lower bounds or upper bounds for the proper amount of resource entanglement for the one-way quantum computation [6–9]. For example, it was shown [6] that a certain amount of entanglement is necessary for any universal resource state for the one-way quantum computation. On the other hand, it was shown [7,8] that a state that has too much entanglement is useless for the one-way quantum computation. These important results and further research based on them will ultimately enable us to pin down the exact amount of resource entanglement, which is neither too small nor too large for the one-way quantum computation.

If the proper amount of resource entanglement for the one-way quantum computation is determined, the next goal is to clarify how such proper entanglement affects the gate fidelity of the one-way quantum computation. Because a highly entangled state is often fragile [10–13], we cannot make the most of the power of entanglement if the one-way quantum computation itself is unstable. Of course, a one-way quantum computer is, like the circuit model of a quantum computer, finally stabilized to some extent by embedding a quantum error-correcting code as shown in Ref. [14]. However, it is still very important to investigate the stability of a bare one-way quantum computer for several reasons [15]. First, it gives valuable feedback for the study of general fault-tolerant schemes. Second, it helps the development of made-to-measure error-correcting codes. Third, what experimentalists are now interested in is not the gigantic fully fledged quantum computer but a bare elementary gate between a couple of qubits. Finally, and most importantly, although the stability of the final result of the computation is guaranteed by the threshold theorem, we must verify the stability of each gate independently, because the crucial assumption of the threshold theorem is that the fidelity of each gate is larger than a certain threshold [14].

In this Rapid Communication, we study how the gate fidelity of the one-way quantum computation is affected by the amount of entanglement between the measured qubit and other register qubits if the measurement is inaccurate in the sense that the direction to which the qubit is projected is slightly deviated from the ideal one. As the resource state that has a proper amount of entanglement, we adopt the cluster state [16]. Our main result is

$$F \leqslant 1 - S \sin^2 \frac{\epsilon}{2},$$

which shows that the mean gate fidelity F ($0 \leqslant F \leqslant 1$) is upper bounded by the decreasing function of the amount S ($0 \leqslant S \leqslant 1$) of entanglement and the magnitude ϵ ($0 \leqslant \epsilon \ll 1$) of the deviation. The main consequence of this inequality is that, for a given amount S of entanglement, which is theoretically calculated once the algorithm is fixed and is *often very large* (see Refs. [11,17–19] and Sec. IV), we can estimate, from this inequality, how small ϵ should be in order not to make the gate fidelity F below a threshold, which is specified by an experimentalist implementing the one-way quantum computation on his or her particular experimental instruments or by the threshold theorem of the fault-tolerant quantum computation.

II. SETUPS

Before showing our main result, some setups are necessary. As a universal set of quantum gates, we adopt the set of single-qubit rotations about the x and z axes and the controlled-NOT (CNOT) gate between two qubits [2]. This is a universal gate set, since, according to the Euler decomposition, any single-qubit rotation can be written as a combination of these two types of rotations. We denote the Pauli x, y, and z operators

*morimae@gmail.com

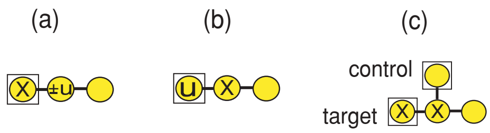

FIG. 1. (Color online) Circles represent qubits, bonds represent the controlled-Z (CZ) interaction $|0\rangle\langle 0|\otimes\hat{1}+|1\rangle\langle 1|\otimes\hat{Z}$, and squares represent the input state. X represents the measurement in the $\hat{X}$ basis. $\pm u$ represents the adaptive measurement in the $(\cos u\hat{X}\mp\sin u\hat{Y})$ basis according to the result of the previous measurement $\pm$, respectively. u represents the measurement in the $(\cos u\hat{X}-\sin u\hat{Y})$ basis. Output states are modified according to the measurement history [20]. (a) The single-qubit rotation $e^{-i(u/2)\hat{X}}$ by u about the x axis. (b) The single-qubit rotation $e^{-i(u/2)\hat{Z}}$ by u about the z axis. (c) The CNOT gate.

acting on the ith qubit by $\hat{X}_i$, $\hat{Y}_i$, and $\hat{Z}_i$, respectively. We also define eigenvectors of $\hat{X}_i$ and $\hat{Z}_i$ by $\hat{X}_i|\pm\rangle_i=\pm|\pm\rangle_i$ and $\hat{Z}_i|z\rangle_i=(-1)^z|z\rangle_i$ $(z=0,1)$, respectively. Let us be reminded [20] that the single-qubit rotation $e^{-i(u/2)\hat{X}}$ by u about the x axis, the single-qubit rotation $e^{-i(u/2)\hat{Z}}$ by u about the z axis, and the CNOT gate are realized in the one-way scheme as Figs. 1(a)–1(c), respectively.

Let us also be reminded that there are two possibilities for the implementation of the one-way quantum computation. One possibility is one that appeared in the original proposal [1] of the one-way quantum computation, where the whole cluster state was created before the onset of adaptive measurements. The other, which is called the one-buffered implementation [14], is the repetition of the addition of a single column of the cluster state to the register column and the measurement of register qubits (see Fig. 2). We will adopt the one-buffered implementation.

Let $|\psi\rangle$ be an N-qubit state, which is considered as the quantum register. We assume that one of the three operations in Figs. 1(a)–1(c) is applied to $|\psi\rangle$ in the one-buffered implementation as shown in Figs. 3(a)–3(c). We are interested in the fidelity of these operations by assuming that a measurement is inaccurate.

It is easy to see that we have only to consider the fidelity of the process in Fig. 3(d) for the study of Figs. 3(a)–3(c). First, any of three operations, Figs. 3(a)–3(c), is a combination of the two elementary processes in Figs. 3(d) and 3(e) (see Fig. 4). Therefore, the study of the fidelity of Figs. 3(a)–3(c) is reduced to that of Figs. 3(d) and 3(e). Second, in the process of Fig. 3(e), the measurement on the first qubit [which is

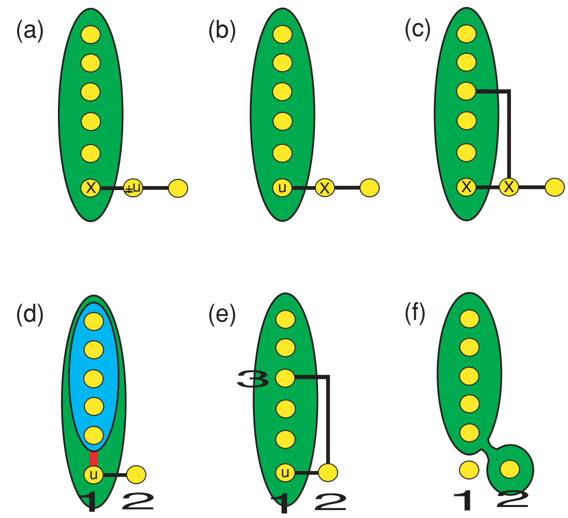

FIG. 3. (Color online) (a) The register state $|\psi\rangle$ is represented by the green ellipse. The rotation of the bottommost qubit of $|\psi\rangle$ by u about the x axis. (b) The rotation of the bottommost qubit of $|\psi\rangle$ by u about the z axis. (c) The CNOT gate between the bottommost qubit and a qubit of $|\psi\rangle$. (d) The red bond (bond from u to the blue inner ellipse) represents entanglement between the first qubit and other register qubits [which are in the blue (inner) ellipse]. Processes (a) and (b) can be written as a combination of (d). (e) Process (c) can be written as a combination of (d) and (e). (f) The state after the measurement on the first qubit in (d).

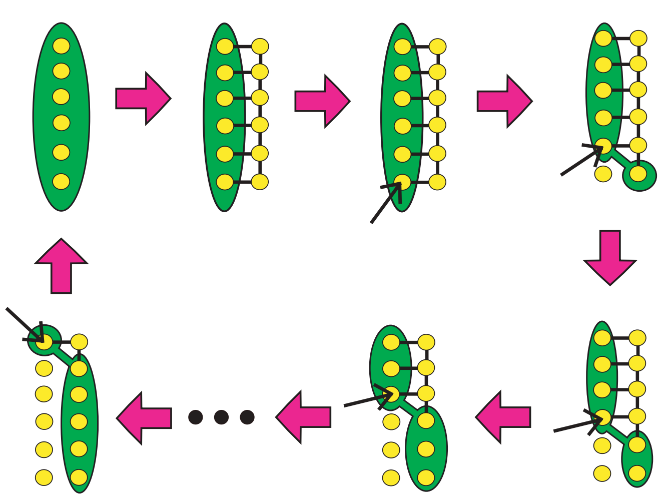

FIG. 2. (Color online) The one-buffered implementation [14] of the one-way quantum computation. The green ellipse represents the register state. The black solid (thin) arrow represents the measurement.

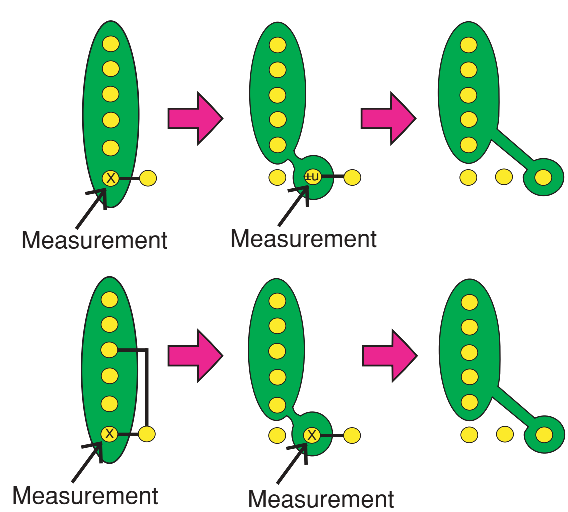

FIG. 4. (Color online) Top line: The rotation of the bottommost qubit in $|\psi\rangle$ about the x axis. The rotation about the z axis is given in a similar way. Bottom line: The CNOT gate between the bottommost qubit and another qubit in $|\psi\rangle$.

labeled as 1] commutes with the CZ interaction between the second qubit and the third qubit [which are labeled as 2 and 3, respectively]. Therefore, the study of Fig. 3(e) is reduced to that of Fig. 3(d). In summary, we have only to consider the fidelity of the process of Fig. 3(d) for our purpose.

III. MAIN RESULT

Therefore, let us calculate the fidelity of the process in Fig. 3(d). We assume that the measurement on the first qubit [which is labeled as 1 in Fig. 3(d)] is inaccurate in the sense that the direction to which the qubit is projected is slightly deviated from the ideal one. In other words, the measurement is not the ideal one $\{|u_+\rangle, |u_-\rangle\}$, where

$$|u_\pm\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle \pm e^{-iu}|1\rangle), \qquad (1)$$

but the slightly deviated one $\{|\tilde{u}_+\rangle, |\tilde{u}_-\rangle\}$, where

$$|\tilde{u}_+\rangle \equiv \cos\frac{\epsilon}{2}|u_+\rangle + e^{-i\delta}\sin\frac{\epsilon}{2}|u_-\rangle,$$
$$|\tilde{u}_-\rangle \equiv \sin\frac{\epsilon}{2}|u_+\rangle - e^{-i\delta}\cos\frac{\epsilon}{2}|u_-\rangle.$$

[If the measurement is performed in the $\hat{X}$ basis, we have only to put $u = 0$ in Eq. (1).] It is easy to see that the degree of the deviation is parametrized by ϵ and δ: $|\tilde{u}_+\rangle$ ($|\tilde{u}_-\rangle$) is the vector obtained by rotating $|u_+\rangle$ ($|u_-\rangle$) by ϵ about the z axis and by $\frac{\pi}{2} - \delta$ about the $|u_+\rangle$ axis. This kind of inaccuracy is ubiquitous in quantum physics. For example, in the measurement model of von Neumann [21], the direction to which the primary state is projected is deviated in this way if there is an inaccuracy in the control of the coupling constant or the coupling time between the primary system and the apparatus, or if the projection measurement on the apparatus is inaccurate.

After the measurement of the first qubit in Fig. 3(d), the entanglement between the first qubit and the other qubits is broken. Then, Fig. 3(d) changes into Fig. 3(f). Let the register state after this measurement [i.e., the state of qubits in the green (outer) ellipse in Fig. 3(f)] be $|\phi_{\epsilon,\delta}\rangle$. If the measurement was accurate, this is $|\phi_{0,0}\rangle$. Then, we can show that

$$F \equiv \mathbb{E}[|\langle\phi_{0,0}|\phi_{\epsilon,\delta}\rangle|^2] \leqslant 1 - S\sin^2\frac{\epsilon}{2}, \qquad (2)$$

where $\mathbb{E}[\cdot]$ means the average over all measurement histories,

$$S \equiv 2\left[1 - \mathrm{Tr}\left(\hat{\rho}_1^2\right)\right]$$

is the entanglement between the first qubit and the other register qubits [which is indicated by the red bond (bond from u to blue inner ellipse) in Fig. 3(d)], and $\hat{\rho}_1 \equiv \mathrm{Tr}_1(|\psi\rangle\langle\psi|)$ is the reduced density operator of the first qubit (Tr_1 is the trace over all qubits except for the first qubit). If the first qubit and the other register qubits are not entangled, $S = 0$, whereas if they are maximally entangled, $S = 1$. Equation (2) is our main result.

Proof of Eq. (2): Let us examine Fig. 3(d). The register state $|\psi\rangle$ [which is represented by the green (outer) ellipse] is written as

$$|\psi\rangle = \alpha|0\rangle_1 \otimes |\eta_0\rangle_b + \beta|1\rangle_1 \otimes |\eta_1\rangle_b,$$

where $|0\rangle_1$ and $|1\rangle_1$ are states of the first qubit [labeled as 1 in Fig. 3(d)], and $|\eta_0\rangle_b$ and $|\eta_1\rangle_b$ are states of the other register qubits [represented by the blue (inner) ellipse in Fig. 3(d)]. $|\eta_0\rangle_b$ and $|\eta_1\rangle_b$ are not necessarily orthogonal with each other. Let us add the second qubit $|+\rangle_2$ [labeled as 2 in Fig. 3(d)] to $|\psi\rangle$ and perform the CZ interaction between the first qubit and the second qubit:

$$|\psi\rangle \otimes |+\rangle_2 \to \alpha|0\rangle_1 \otimes |\eta_0\rangle_b \otimes |+\rangle_2 + \beta|1\rangle_1 \otimes |\eta_1\rangle_b \otimes |-\rangle_2.$$

As we have assumed, the first qubit is measured in $\{|\tilde{u}_+\rangle, |\tilde{u}_-\rangle\}$. Then, Fig. 3(d) changes into Fig. 3(f). Let the states of the green ellipse in Fig. 3(f) be $|\phi^\pm_{\epsilon,\delta}\rangle$ if the result of the measurement is $\pm$, respectively. By a straightforward calculation, the probabilities $P_\pm$ of obtaining $|\phi^\pm_{\epsilon,\delta}\rangle$ are

$$P_\pm = \tfrac{1}{2}(1 \pm \xi\sin\epsilon\cos\delta),$$

respectively, where $\xi \equiv \mathrm{Tr}(\hat{\rho}_1\hat{Z}_1)$. The fidelity for each output is also calculated as

$$F_\pm \equiv |\langle\phi^\pm_{0,0}|\phi^\pm_{\epsilon,\delta}\rangle|^2 = \frac{1 \pm \xi\sin\epsilon\cos\delta - (1-\xi^2)\sin^2\frac{\epsilon}{2}}{2P_\pm},$$

respectively, and the mean fidelity is, therefore,

$$F_+P_+ + F_-P_- = 1 - (1-\xi^2)\sin^2\frac{\epsilon}{2}.$$

Our goal, Eq. (2), is obtained by applying the relation,

$$1 - \mathrm{Tr}^2(\hat{\rho}_1\hat{Z}_1) \geqslant S,$$

which is shown as follows. Let $\hat{\rho}_1 = \lambda_0|\tau_0\rangle_1\langle\tau_0| + \lambda_1|\tau_1\rangle_1\langle\tau_1|$, where $\lambda_0 \geqslant 0$, $\lambda_1 \geqslant 0$, $\lambda_0 + \lambda_1 = 1$, and

$$|\tau_0\rangle_1 = \cos\frac{\mu}{2}|0\rangle_1 + e^{-i\nu}\sin\frac{\mu}{2}|1\rangle_1,$$
$$|\tau_1\rangle_1 = \sin\frac{\mu}{2}|0\rangle_1 - e^{-i\nu}\cos\frac{\mu}{2}|1\rangle_1.$$

Then, we obtain $1 - \mathrm{Tr}^2(\hat{\rho}_1\hat{Z}_1) = 1 - (\lambda_0 - \lambda_1)^2\cos^2\mu \geqslant 1 - (\lambda_0 - \lambda_1)^2 = S$.

IV. DISCUSSION

If S was always 0 during any quantum computation, Eq. (2) would be of no use. However, in fact, S often becomes very large during a quantum computation. For example, in Ref. [17], it was shown that if an N-qubit register state $|\psi\rangle$ is decomposed as the tensor product of inseparable states $|\psi\rangle = \bigotimes_i |\psi_i\rangle$, at least one of these inseparable states $\{|\psi_i\rangle\}_i$ must have an unboundedly increasing size during a quantum computation if the quantum computation offers an exponential speedup over a classical one. This result is not changed even if a weak entanglement is established among $|\psi_i\rangle$'s. Therefore, there is a high probability that the measured qubit has a sufficiently strong entanglement with other register qubits during a quantum computation. Moreover, in Refs. [18,19], it was shown that the register state has a superposition of macroscopically distinct states during the execution of the Shor factoring algorithm and the Grover search algorithm. According to the result of Ref. [11], a randomly chosen single qubit is strongly entangled with other qubits with a high probability if the state has such a macroscopic superposition.

In short, S is often very large in a quantum computation; and, therefore, Eq. (2) offers a meaningful upper bound for the gate fidelity of the inaccurate one-way quantum computation.

The error model studied here is not an atypical one. This type of error is indeed often considered in many studies of fault-tolerant quantum computations including Ref. [14], where the possibility of the fault-tolerant one-way quantum computation is shown. Therefore, the effect of our error is recoverable to some extent, and the whole quantum computation can be performed successfully. However, as mentioned in Sec. I, the study of the stability of a bare one-way quantum computation is very important. This is where our result can contribute.

In addition to the inaccurate measurement considered here, there are many other possibilities for errors in the one-way quantum computation. For example, if the one-way quantum computation is implemented with the discrete-variable linear-optics schemes [22], we must also consider the imperfection of the CZ gate, since, in this case, the entangling operation is not deterministic. To consider other error models would lead to interesting generalizations of the present work. It is left for a future study.

ACKNOWLEDGMENTS

The author thanks T. Rudolph, M. S. Tame, A. Douglas K. Plato, Y. Omar, and J. Kahn for discussions, and the French Agence Nationale de la Recherche (ANR) for support under Grant No. StatQuant (JC07 07205763).

[1] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).
[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, UK, 2000).
[3] M. Yukawa, R. Ukai, P. van Loock, and A. Furusawa, Phys. Rev. A **78**, 012301 (2008); X. Su, A. Tan, X. Jia, J. Zhang, C. Xie, and K. Peng, Phys. Rev. Lett. **98**, 070502 (2007); Y. Tokunaga, S. Kuwashiro, T. Yamamoto, M. Koashi, and N. Imoto, *ibid.* **100**, 210501 (2008); G. Vallone, E. Pomarico, P. Mataloni, F. De Martini, and V. Berardi, *ibid.* **98**, 180502 (2007); R. Ceccarelli, G. Vallone, F. De Martini, P. Mataloni, and A. Cabello, *ibid.* **103**, 160401 (2009).
[4] M. S. Tame, R. Prevedel, M. Paternostro, P. Böhi, M. S. Kim, and A. Zeilinger, Phys. Rev. Lett. **98**, 140501 (2007).
[5] P. Walther, K. J. Resch, T. Rudolph, E. Schenck, H. Weinfurther, V. Vedral, M. Aspelmeyer, and A. Zeilinger, Nature (London) **434**, 169 (2005).
[6] M. Van den Nest, A. Miyake, W. Dür, and H. J. Briegel, Phys. Rev. Lett. **97**, 150504 (2006).
[7] D. Gross, S. T. Flammia, and J. Eisert, Phys. Rev. Lett. **102**, 190501 (2009).
[8] M. J. Bremner, C. Mora, and A. Winter, Phys. Rev. Lett. **102**, 190502 (2009).
[9] D. Gross, J. Eisert, N. Schuch, and D. Perez-Garcia, Phys. Rev. A **76**, 052315 (2007).
[10] A. Shimizu and T. Miyadera, Phys. Rev. Lett. **89**, 270403 (2002).
[11] T. Morimae, Phys. Rev. A **81**, 010101(R) (2010).
[12] T. Morimae, Phys. Rev. A **81**, 022304 (2010).
[13] T. Morimae, A. Sugita, and A. Shimizu, Phys. Rev. A **71**, 032317 (2005).
[14] M. A. Nielsen and C. M. Dawson, Phys. Rev. A **71**, 042323 (2005).
[15] M. S. Tame, M. Paternostro, M. S. Kim, and V. Vedral, Phys. Rev. A **72**, 012319 (2005); K. Kieling, T. Rudolph, and J. Eisert, Phys. Rev. Lett. **99**, 130501 (2007); K. Kieling, D. Gross, and J. Eisert, New J. Phys. **9**, 200 (2007); P. P. Rohde and S. D. Barrett, *ibid.* **9**, 198 (2007); D. Jennings, A. Dragan, S. D. Barrett, S. D. Bartlett, and T. Rudolph, Phys. Rev. A **80**, 032328 (2009).
[16] To consider other resource states, such as those studied in Ref. [9], would be an interesting subject for a future study.
[17] R. Jozsa and N. Linden, Proc. R. Soc. London, Ser. A **459**, 2011 (2003).
[18] A. Ukena and A. Shimizu, Phys. Rev. A **69**, 022301 (2004).
[19] T. Morimae and A. Shimizu, Phys. Rev. A **74**, 052111 (2006).
[20] P. Aliferis and D. W. Leung, Phys. Rev. A **70**, 062314 (2004).
[21] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955).
[22] M. A. Nielsen, Phys. Rev. Lett. **93**, 040503 (2004).