Solution and entanglement dynamics of a cavityless optomechanical system with Gaussian states

Shao-Hua Xiang, Wei Wen, Zheng-Gang Shi, and Ke-Hui Song

Department of Physics and Information Engineering, Huaihua University, Huaihua, Hunan 418008, People's Republic of China

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We use the symplectic transformation of the known linear optical elements to investigate the dynamical evolution of a three-mode cavityless optomechanical system and give explicit formulas for the input and output covariance matrices of the three mode fields when the initial state is Gaussian. Unlike the conventional approaches, the present one does not necessitate solution of the equation of motion of such a system. We study the dynamical behavior of bipartite entanglement in this system when the mirror vibrational mode is initially in a single-mode squeezed vacuum state and each of the two reflected optical sideband modes is in the vacuum state. It is found that the entanglement configuration alternates between a fully separable state and a continuous-variable entangled state during evolution.

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Quantum entanglement is one of the most striking properties of the quantum world and has played a crucial role in some of the studies connected with the foundations of quantum mechanics [1] and in quantum computing and quantum information processing [2], for which an important goal is to generate entangled states of discrete- and continuous-variable quantum systems. It is well known that quantum continuous variables are intrinsically easier to manipulate experimentally than their discrete counterparts by use of linear optical elements. In continuous-variable settings, the information is encoded in the two quadrature phase amplitudes of quantum systems such as optical fields and atomic ensembles, which can be efficiently measured by homodyne detection. Therefore, there has recently been increasing interest in the generation of entangled continuous-variable systems. A known example is optomechanical entanglement in a cavityless optomechanical system where the radiation pressure of an optical beam incident on a mirror can realize an effective coupling between a mirror vibrational mode and the two reflected optical sideband modes of the incident carrier beam [3]. There are two different approaches to analysis of the dynamical evolution of such a system. One approach is to derive the expressions for the time-dependent boson operators of the quantum system from the Heisenberg evolution of motion. The other approach is the phase-space one, that is, the master equation of the density operators is mapped into the resulting Fokker-Plank equation using the standard operator correspondence. However, in such methods, the sets of coupled differential equations for the time-dependent coefficients need to be considered, and this is difficult. Hence, the results in previous work have been limited to some rather special cases, for example, the two sideband modes in the vacuum state and the mirror vibrational mode in the thermal state. No explicit formula for the general input states is given there.

Gaussian states are essential in quantum information processing with continuous variables. Experimental schemes for generation of multimode entangled states have already been proposed and demonstrated [4]. In particular, there have been many studies of the production of multipartite entanglement via beam splitters using many different input Gaussian states [5]. It is, therefore, of interest to find the solution of the evolution equations for a cavityless optomechanical system when the initial input states are Gaussian. In this Brief Report, we are going to use a distinct method. That is, we first change the interaction Hamiltonian of the studied system into the beam-splitter Hamiltonian, including a collective mode and one vibrational mode of the mirror applying the canonical transformation. Then, by means of the symplectic transformation of the beam splitter, we can obtain an explicit formula for the covariance matrix of the output state if the initial state of the system is a rather general Gaussian state. Compared with the methods mentioned previously, the advantage of our method is that there is no need to solve the sets of coupled differential equations for the time-dependent coefficients, and accordingly it is simple and efficient.

Our model system consists of a perfectly reflecting mirror and intense quasimonochromatic laser beam incident on its surface (see Fig. 1 in Ref. [3]). When the laser beam impinges on the mirror, it is reflected into an elastic carrier mode characterized by the annihilation operator $\hat{b}$ and two additional weak inelastic sideband modes characterized by the annihilation operators $\hat{a}_1$ and $\hat{a}_2$, respectively. The effective Hamiltonian describing the interaction can be written as ($\hbar = 1$) [3]

$$\hat{H}^{\text{eff}} = -i\mu(\hat{a}_1\hat{b} - \hat{a}_1^\dagger\hat{b}^\dagger) - i\nu(\hat{a}_2\hat{b}^\dagger - \hat{a}_2^\dagger\hat{b}), \tag{1}$$

where μ and ν are coupling constants proportional to $\sqrt{P}$, with P the incident laser power. In what follows, we derive the covariance matrix of the evolved state by means of the symplectic transformation of the known beam splitter. With this aim, we need to introduce a 6×6 matrix,

$$\mathbf{B} = \frac{1}{\mu\Omega}\begin{pmatrix} 0 & -\mu^2 & -\nu^2 & \nu\Omega \\ -\mu^2 & 0 & \nu\Omega & -\nu^2 \\ -\mu\nu & 0 & \mu\Omega & -\mu\nu \\ 0 & -\mu\nu & -\mu\nu & \mu\Omega \end{pmatrix} \bigoplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2}$$

with $\Omega^2 = \nu^2 - \mu^2$, such that the old operators are related to the new operators by

$$(\hat{a}_1,\hat{a}_1^\dagger,\hat{a}_2,\hat{a}_2^\dagger,\hat{b},\hat{b}^\dagger)^T = \mathbf{B}(\hat{c}_1,\hat{c}_1^\dagger,\hat{c}_2,\hat{c}_2^\dagger,\hat{b},\hat{b}^\dagger)^T. \tag{3}$$

It is easy to verify that these new operators satisfy the commutation relations $[\hat{c}_i,\hat{c}_j^\dagger] = \delta_{ij}$ $(i,j = 1,2)$. Under this canonical transformation, the effective Hamiltonian Eq. (1) can be given in the form

$$\hat{H}^{\mathrm{eff}} = i\Omega(\hat{c}_1\hat{b}^\dagger - \hat{c}_1^\dagger\hat{b}). \tag{4}$$

One sees that the time-evolution operator $\hat{U}(t)$ for such a Hamiltonian is equivalent to a beam-splitter operator with the field mode b and the reduced mode c_1 as its input ports, and is given by

$$\hat{U}(t) = \exp[\Theta(\hat{c}_1^\dagger\hat{b} - \hat{c}_1\hat{b}^\dagger)], \tag{5}$$

where $\Theta = \Omega t$ determines the transmissivity $\tau = \cos^2\Theta$. It is shown that, under the action of the beam splitter, the output states preserve their Gaussian form if the input states are Gaussian. Then, the input-output covariance matrix of a two-mode Gaussian state through a beam splitter is found as [6] $\sigma_{\mathrm{out}} = \mathbf{S}_{\mathrm{BS}}^T\sigma_{\mathrm{in}}\mathbf{S}_{\mathrm{BS}}$, where the symplectic transformation $\mathbf{S}_{\mathrm{BS}}$ associated with the evolution operator (5) of the beam splitter can be written, in terms of phase-space quadrature variables, as

$$\mathbf{S}_{\mathrm{BS}} = \begin{pmatrix} \sqrt{\tau}\mathbf{I}_2 & \mathbf{0} & \sqrt{1-\tau}\mathbf{I}_2 \\ \mathbf{0} & \mathbf{I}_2 & \mathbf{0} \\ -\sqrt{1-\tau}\mathbf{I}_2 & \mathbf{0} & \sqrt{\tau}\mathbf{I}_2 \end{pmatrix}, \tag{6}$$

where $\mathbf{I}_2$ is the 2×2 identity matrix, and for the sake of completeness of notation later, the mode c_2 has been taken into account in Eq. (6).

We assume that the three modes are initially in a pure Gaussian state represented by the characteristic function $\chi(\{\xi_1,\xi_2,\xi_b\},0) = \exp[-\frac{1}{2}\mathbf{\Lambda}^T\mathbf{V}_3(0)\mathbf{\Lambda}]$ with $\mathbf{V}_3(0)$ being the initial covariance matrix. Using Eqs. (2) and (6) and by a simple calculation, we find that the time-dependent matrix of the three modes is given by

$$\mathbf{V}_3(t) = \mathbf{G}^T\mathbf{V}_3(0)\mathbf{G}, \tag{7}$$

where $\mathbf{G} = \mathbf{B}\mathbf{E}\mathbf{S}_{\mathrm{BS}}\mathbf{E}^{-1}\mathbf{B}^{-1}$ is given by

$$\mathbf{G} = \begin{pmatrix} f_1 & 0 & 0 & g_1 & 0 & -g_2 \\ 0 & f_1 & g_1 & 0 & -g_2 & 0 \\ 0 & -g_1 & f_2 & 0 & -g_3 & 0 \\ -g_1 & 0 & 0 & f_2 & 0 & -g_3 \\ 0 & -g_2 & g_3 & 0 & f_3 & 0 \\ -g_2 & 0 & 0 & g_3 & 0 & f_3 \end{pmatrix}, \tag{8}$$

where $f_1 = [\nu^2 - \mu^2\cos(\Omega t)]/\Omega^2$, $f_2 = [\nu^2\cos(\Omega t) - \mu^2]/\Omega^2$, $f_3 = \cos(\Omega t)$, $g_1 = \mu\nu[\cos(\Omega t) - 1]/\Omega^2$, $g_2 = \mu\sin(\Omega t)/\Omega$, $g_3 = \nu\sin(\Omega t)/\Omega$, and $\mathbf{E} = \bigoplus_{n=1}^3 \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & i \\ 1 & -i \end{smallmatrix}\right)$.

It is seen that the calculation becomes an easy task and our method provides an intuitive picture of the evolution process governed by the Hamiltonian (1). That is, the total transformation $\mathbf{G}$ consists of the linear canonical transformation $\mathbf{B}$, the unitary transformation $\mathbf{E}$, and the symplectic transformation $\mathbf{S}_{\mathrm{BS}}$ of the beam splitter, but the last plays a central role in the derivation of the matrix (7). It is interesting to note that the effective Hamiltonian (1) describes in fact two interlinked bilinear interactions occurring among the three modes. One is a nondegenerate parametric amplifier (two-mode squeezer) via which continuous-variable entanglement can be produced between modes 1 and b. The other is a parametric converter (beam splitter) that can lead to the possibility of exchange of quantum information between modes 2 and b. Therefore, one might expect that the total transformation $\mathbf{G}$ should be a mixture of the symplectic transformations of the two-mode squeezer and the beam splitter. In order to see this point clearly, let us apply the canonical transform $\mathbf{E}$ on the matrix (8), leading to

$$\mathbf{G} = \begin{pmatrix} f_1 & 0 & g_1 & 0 & g_2 & 0 \\ 0 & f_1 & 0 & -g_1 & 0 & -g_2 \\ -g_1 & 0 & f_2 & 0 & g_3 & 0 \\ 0 & g_1 & 0 & f_2 & 0 & g_3 \\ g_2 & 0 & -g_3 & 0 & f_3 & 0 \\ 0 & -g_2 & 0 & -g_3 & 0 & f_3 \end{pmatrix}. \tag{9}$$

We can easily see that, for the case of $\mu = 0$, the matrix (9) corresponds to the symplectic transformation of a beam splitter with transmissivity $\tau = \cos^2(\nu t)$, whereas $\nu = 0$ means that it is the symplectic transformation of a two-mode squeezer with squeezing $r = \mu t$. So we call the matrix (9) *the symplectic transformation of the Hamiltonian* (1). In addition, it is interesting to find that the same symplectic transformation (9) may be available in terms of the symplectic transformation of a two-mode squeezer by setting $\hat{c}_1 = (1/\Omega)(\mu\hat{a}_1 - \nu\hat{a}_2^\dagger)$ and $\hat{c}_2 = (1/\Omega u)[u(\Omega\hat{a}_2^\dagger + \nu\hat{a}_2) - \nu(\nu\hat{a}_1^\dagger + \Omega\hat{a}_1)]$.

As an example of our approach, let us consider the situation in which the mirror is initially in a squeezed vacuum state and the two other modes are in the vacuum states, namely, the initial covariance matrix of the system can be written as

$$\mathbf{V}_3(0) = \bigoplus_{n=1,2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bigoplus \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix}, \tag{10}$$

with the parameter $r \in \mathbb{R}$. Substituting Eq. (10) into Eq. (7) and using a local linear unitary Bogolinbov operation [7], we find the time-dependent covariance matrix for the three modes:

$$\mathbf{V}_3(t) = \begin{pmatrix} V_{11} & 0 & V_{13} & 0 & V_{15} & 0 \\ 0 & V_{22} & 0 & V_{24} & 0 & V_{26} \\ V_{13}^* & 0 & V_{33} & 0 & V_{35} & 0 \\ 0 & V_{24}^* & 0 & V_{44} & 0 & V_{46} \\ V_{15}^* & 0 & V_{35}^* & 0 & V_{55} & 0 \\ 0 & V_{26}^* & 0 & V_{46}^* & 0 & V_{66} \end{pmatrix}, \tag{11}$$

where the asterisk means the complex conjugate, and the matrix elements are given by

$$V_{11} = \frac{x^4 + \cos^2(\Omega t) + h_1}{(x^2-1)^2} + \frac{e^r\sin^2(\Omega t)}{|\sqrt{x^2-1}|^2}, \tag{12a}$$

$$V_{22} = \frac{x^4 + \cos^2(\Omega t) + h_1}{(x^2-1)^2} + \frac{e^{-r}\sin^2(\Omega t)}{|\sqrt{x^2-1}|^2}, \tag{12b}$$

$$V_{33} = \frac{x^4\cos^2(\Omega t) + h_1 + 1}{(x^2-1)^2} + \frac{x^2e^r\sin^2(\Omega t)}{|\sqrt{x^2-1}|^2}, \tag{12c}$$

$$V_{44} = \frac{x^4\cos^2(\Omega t) + h_1 + 1}{(x^2-1)^2} + \frac{x^2e^{-r}\sin^2(\Omega t)}{|\sqrt{x^2-1}|^2}, \tag{12d}$$

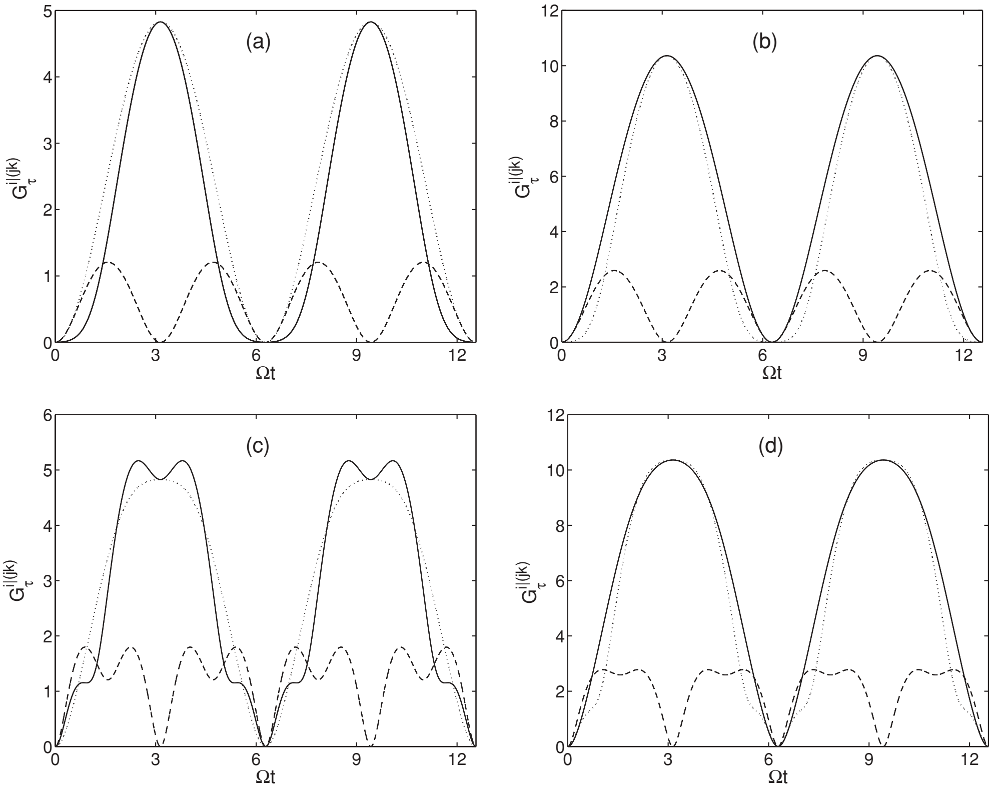

FIG. 1. Time evolution of $G_{i|(jk)}$ for four different cases: (a) $x = 0.5$, $r = 0.3$, (b) $x = 1.5$, $r = 0.3$, (c) $x = 0.5$, $r = 2$, and (d) $x = 1.5$, $r = 2$. In all subplots the solid line refers to $G_{1|(2b)}$, the dotted to $G_{2|(1b)}$, and the dashed to $G_{b|(12)}$.

$$V_{13} = -\frac{x(1+x^2)[1-\cos(\Omega t)]^2}{(x^2-1)^2} - \frac{xe^r \sin^2(\Omega t)}{|\sqrt{x^2-1}|^2}, \quad (12e)$$

$$V_{15} = \frac{-2x^2\sin(\Omega t)+h_2}{(x^2-1)^{3/2}} - \frac{e^r \sin(\Omega t)\cos(\Omega t)}{(\sqrt{x^2-1})^*}, \quad (12f)$$

$$V_{24} = \frac{x(1+x^2)[1-\cos(\Omega t)]^2}{(x^2-1)^2} + \frac{xe^{-r}\sin^2(\Omega t)}{|\sqrt{x^2-1}|^2}, \quad (12g)$$

$$V_{26} = \frac{2x^2\sin(\Omega t)-h_2}{(x^2-1)^{3/2}} + \frac{e^{-r}\sin(\Omega t)\cos(\Omega t)}{(\sqrt{x^2-1})^*}, \quad (12h)$$

$$V_{35} = \frac{-x[h_2 - 2\sin(\Omega t)]}{(x^2-1)^{3/2}} - \frac{xe^r\sin(\Omega t)\cos(\Omega t)}{(\sqrt{x^2-1}|)^*}, \quad (12i)$$

$$V_{46} = \frac{-x[h_2 - 2\sin(\Omega t)]}{(x^2-1)^{3/2}} - \frac{xe^{-r}\sin(\Omega t)\cos(\Omega t)}{(\sqrt{x^2-1})^*}, \quad (12j)$$

$$V_{55} = \frac{(1+x^2)\sin^2\Omega t}{|\sqrt{x^2-1}|^2} + \cos^2(\Omega t)e^r, \quad (12k)$$

$$V_{66} = \frac{(1+x^2)\sin^2\Omega t}{|\sqrt{x^2-1}|^2} + \cos^2(\Omega t)e^{-r}, \quad (12l)$$

where $h_1 = [1+\cos^2(\Omega t) - 4\cos(\Omega t)]x^2$, $h_2 = (1+x^2)\sin(\Omega t)\cos(\Omega t)$, and $x = \nu/\mu$.

We now turn to the investigation of the entanglement dynamics of the tripartite system under consideration. It has been shown that the nonpositive partial transposition criterion is necessary and sufficient for $1 \times N$ bipartite continuous-variable (CV) Gaussian states [8]. Recently, Adesso *et al.* put forward the Gaussian contangle 1×2 $G_{i|(jk)}$ as an entanglement measure of 1×2 partition of the pure three-mode Gaussian state [9]. The 1×2 $G_{i|(jk)}$ is defined as [9]

$$G_{i|(jk)} := \text{arcsinh}^2\left(\sqrt{m^2_{i|(jk)} - 1}\right). \quad (13)$$

Here $m_{i|(jk)} = \text{Det}(\sigma_i)$ with σ_i being the reduced CM of the mode i obtained from Eq. (11) by tracing over the degrees of freedom of the two other modes.

In Fig. 1 we plot the time evolution of the 1×2 entanglement $G_{i|(jk)}$ under different conditions. It is seen that, in a closed system of three interacting oscillators, the dynamics of bipartite entanglement of 1×2 partitions oscillates periodically. The periods of $G_{1|(2b)}$ and $G_{2|(1b)}$ are twice as large as that of $G_{b|(12)}$. According to the classification scheme of three-mode CV states in Ref. [10], we find that the evolved state of our tripartite system may belong to three different entanglement classes: (1) fully separable states, (2) one-mode biseparable states, and (3) fully inseparable states (or genuine tripartite entanglement). At the times $t = (2n+1)\pi/\Omega$ for $n = 0,1,2,\ldots,$ mode b is disentangled from the modes 1 and 2, whereas the modes 1 and 2 are prepared in a two-mode squeezed state with the two-mode squeezing parameter $\text{arcsinh}[4x(1+x^2)/(x^2-1)^2]$. That is, the evolved state becomes a one-mode biseparable state. When the interaction time is $t = 2n\pi/\Omega$ for $n = 0,1,2,\ldots,$ our tripartite system is prepared in a mixture of the product states. It is not difficult to see from Eq. (11) that in this case our system evolves back to its initial state, that is, the modes 1 and 2 are in vacuum states and mode b in the squeezed vacuum state. We also find that, except at these isolated time instants, the 1×2 entanglement $G_{i|(jk)}$ is always greater than zero, implying that each mode is in an entangled state with the remaining two modes taken as a whole for any nonzero value of the ratio between coupling constants x. Thus, this state is said to be fully inseparable,that is, it contains genuine tripartite entanglement.

Finally, we would to make some remarks on the method presented here. We note that quantum dynamical evolution in some physical systems can be described by the same Hamiltonian as Eq. (1); these systems include a Bose-Einstein condensate driven by a far-off-resonant pump laser and interacting with a single mode of an optical ring [11], the interlinked bilinear interactions taking place in a single-$\chi^{(2)}$ nonlinear medium [12], the cascaded nonlinear interaction in an optical cavity with quansiperiodic superlattice [13], and the coherent interaction between a laser-driven single trapped atom and an optical high-finesse resonator [14]; therefore, their dynamical evolutions can be solved exactly using our approach if the initial states are of Gaussian form. We also see that the method can be exploited to investigate the dynamics of a quadrature quantum system in which the bilinear transformations leave the Gaussian character invariant. Nevertheless, to date, the symplectic transformations of only a few linear optical elements have been found [15]. Hence, our investigation provides a direct and effective way to construct the symplectic transformations associated with quantum systems such as multisplitter [16] and multimode squeezing [17]. In summary, we have presented an elegant way to solve the equation of motion for a system in which a laser field impinges on a mirror and obtained the input-output relation of the evolved state if the initial state is Gaussian. We have studied the bipartite entanglement dynamics in such a system when the mirror vibrational mode is initially in a single-mode squeezed vacuum state and two reflected optical sideband modes are in vacuum states, and found that the evolved state may belong to three different entanglement classes during the evolution: (i) fully inseparable, (ii) one-mode biseparable, and (iii) fully separable states.

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