Cooperative spontaneous emission of N atoms: Many-body eigenstates, the effect of virtual Lamb shift processes, and analogy with radiation of N classical oscillators

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We consider collective emission of a single photon from a cloud of N two-level atoms (one excited, $N-1$ ground state). For a dense cloud the problem is reduced to finding eigenfunctions and eigenvalues of an integral equation. We discuss an exact analytical solution of this many-atom problem for a spherically symmetric atomic cloud. Some eigenstates decay much faster then the single atom decay rate, while the others undergo very slow decay. We show that virtual processes yield a small effect on the evolution of rapidly decaying states. However, they change the long time dynamics from exponential decay into a power-law behavior which can be observed experimentally. For trapped states virtual processes are much more important yielding additional decay channels which results in a slow decay of the otherwise trapped states. We also show that quantum mechanical treatment of spontaneous emission of weakly excited atomic ensemble is analogous to emission of N classical harmonic oscillators.

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I. INTRODUCTION

Collective spontaneous emission phenomenon has been a subject of interest since the pioneering work of Dicke in 1954 [1]. In that classical paper, Dicke considered mainly two types of collective radiation phenomena: superradiance and subradiance in a collection of two-level atoms when all atoms are confined inside a volume much smaller than radiation wavelength. Later work generalized Dicke's description of superradiance to an extended system [2,3]. See especially the experimental and theoretical work of Feld and coworkers [4]. Further studies brought in the concept of superfluorescence [5], which describes the cooperative emission from a system of uncorrelated excited atoms. This process usually starts with normal spontaneous emission but later develops correlation among the system [6]. In the past half century, both types of phenomena were extensively studied theoretically and experimentally. The case of an ensemble of excited nuclei has long been, and continues to be a subject of research interest [7].

From the physical standpoint, cooperative spontaneous emission is an example of a many-body quantum problem of N atoms collectively interacting with an electromagnetic field. Emission from a weakly excited group of atoms is, in some ways, even more interesting than the case of a highly excited system. In the case of a weakly excited ensemble (e.g., one atom out of N is excited) it might be thought that the radiation rate would go as the single atom decay rate γ; however, the Dicke symmetric state of maximum cooperation radiates at a rate $\Gamma_N \propto N\gamma$.

Collective spontaneous radiation is interesting physics and also has potential applications. From the standpoint of applications, superradiance is useful as one of the methods for producing coherent emission without coherent pumping. This is especially important in those regimes, such as x-ray or γ-ray, where there are no effective mirrors which limit the use of ordinary stimulated emission process. On the other hand, with the recent advances of quantum informatics, decoherence-free subspace (DFS) [8] has been proposed to be one of the strategies to combat the effects of decoherence in quantum computation and quantum communication. A collective system of many two-level particles is one of the ideal candidates to realize DFS [8–10]. An ensemble of N two-level atoms with one excitation also plays an important role in quantum memory and quantum networking. Relevant experiments have been carried out by the groups of Lukin [11], Kimble [12], and Vuletić *et al.* [13].

Cooperative effects of N atoms in a cavity were investigated in 1980s by Cummings [14–16] and the others [17,18]. Buzek [19] studied the dynamics of an excited atom in the presence of $N-1$ atoms in the free space and predicted radiation suppression. Dynamics of the system in free space and spatial anisotropy of the emitted radiation have been re-explored in the past few years [20–37].

The problem of cooperative spontaneous emission of N atoms reduces to finding all eigenstates and their decay rates. Once they are determined, evolution of an arbitrary initial state is obtained by expanding the initial condition in terms of the set of the eigenstates. In 1969 Ernst [38] studied such an eigenproblem for a spherical atomic cloud in Weisskopf and Wigner theory disregarding the effect of virtual photons. Later Ressayre and Tallet [39], and Andreev *et al.* [40] investigated such a problem in various geometries. However, the exchange of virtual photons induces dipole-dipole interaction between atoms [25,41–44]. Recently it was shown that virtual photons modify eigenstates and eigenvalues of the system [26,27] and dramatically change the evolution of the trapped states [32]. However, virtual processes yield a small (yet interesting) effect on the evolution of rapidly decaying states [32]. This question is a subject of recent debate [45–47].

In the present paper we discuss the problem of single-photon cooperative spontaneous emission in details focusing on the issues of current interest. In particular, we clarify the effect of virtual processes and situations when the quantum N-atom problem has a classical analogy with radiation of a system of N harmonic oscillators.

II. DERIVATION OF EIGENVALUE EQUATION

We consider a system of two level (a and b) atoms, initially one of them is in the excited state a and $E_a - E_b = \hbar\omega$. Initially there are no photons. Atoms are located at positions $\mathbf{r}_j$ ($j = 1, \ldots, N$). In the dipole approximation the interaction of atoms with photons is described by the Hamiltonian (we disregard polarization effects)

$$\hat{H}_{\rm int} = \sum_{\mathbf{k}} \sum_{j=1}^{N} g_k (\hat{\sigma}_j e^{-i\omega t} + \hat{\sigma}_j^{\dagger} e^{i\omega t}) \times (\hat{a}_{\mathbf{k}}^{\dagger} e^{i\nu_k t - i\mathbf{k}\cdot\mathbf{r}_j} + \hat{a}_{\mathbf{k}} e^{-i\nu_k t + i\mathbf{k}\cdot\mathbf{r}_j}), \quad (1)$$

where $\hat{\sigma}_j$ is the lowering operator for atom j, $\hat{a}_{\mathbf{k}}$ is the photon operator, and g_k is the atom-photon coupling constant for the $\mathbf{k}$ mode [48]

$$g_k = \omega \frac{\wp}{\hbar} \sqrt{\frac{\hbar}{\epsilon_0 \nu_k V_{\rm ph}}}, \quad (2)$$

where $\wp$ is the electric-dipole transition matrix element and $V_{\rm ph}$ is the photon volume. Please note that we do not make the rotating wave approximation in Eq. (1).

We look for a solution of the Schrödinger equation for the atoms and the field as a superposition of Fock states

$$\Psi = \sum_{j=1}^{N} \beta_j(t) |b_1 b_2 \cdots a_j \cdots b_N\rangle |0\rangle + \sum_{\mathbf{k}} \gamma_{\mathbf{k}}(t) |b_1 b_2 \cdots b_N\rangle |1_{\mathbf{k}}\rangle + \sum_{m<n} \sum_{\mathbf{k}} \alpha_{mn,\mathbf{k}}(t) |b_1, b_2, \ldots a_m, \ldots a_n, \ldots b_N\rangle |1_{\mathbf{k}}\rangle, \quad (3)$$

where $\alpha_{mn,\mathbf{k}} = \alpha_{nm,\mathbf{k}}$. States in the first sum correspond to zero number of photons, while in the second sum the photon occupation number is equal to one and all atoms are in the ground state b. The third term corresponds to the presence of two excited atoms inside the cloud and one (virtual) photon with "negative" energy. Substitute of Eq. (3) into the Schrödinger equation yields the following equations for $\beta_j(t)$, $\gamma_{\mathbf{k}}(t)$, and $\alpha_{mn,\mathbf{k}}(t)$ (we put $\hbar = 1$):

$$\dot{\beta}_j(t) = -i \sum_{\mathbf{k}} g_k \gamma_{\mathbf{k}}(t) \exp[-i(\nu_k - \omega)t + i\mathbf{k}\cdot\mathbf{r}_j] - i \sum_{\mathbf{k}} g_k \sum_{j'=1, j'\neq j}^{N} \alpha_{jj',\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{r}_{j'}} e^{-i(\nu_k+\omega)t}, \quad (4)$$

$$\dot{\gamma}_{\mathbf{k}}(t) = -i g_k \sum_{j=1}^{N} \beta_j(t) \exp[i(\nu_k - \omega)t - i\mathbf{k}\cdot\mathbf{r}_j], \quad (5)$$

$$\dot{\alpha}_{mn,\mathbf{k}}(t) = -i g_k \beta_n(t) \times \exp[i(\nu_k + \omega)t - i\mathbf{k}\cdot\mathbf{r}_m] + (n \longleftrightarrow m). \quad (6)$$

Integrating Eqs. (5) and (6) over time with initial conditions $\gamma_{\mathbf{k}}(0) = 0$, $\alpha_{mn,\mathbf{k}}(0) = 0$,

$$\gamma_{\mathbf{k}}(t) = -i g_k \int_0^t dt' \sum_{j=1}^{N} \beta_j(t') \exp[i(\nu_k - \omega)t' - i\mathbf{k}\cdot\mathbf{r}_j], \quad (7)$$

$$\alpha_{mn,\mathbf{k}}(t) = -i g_k \int_0^t dt' [\beta_n(t') \exp[i(\nu_k + \omega)t' - i\mathbf{k}\cdot\mathbf{r}_m] + (n \longleftrightarrow m)], \quad (8)$$

and substituting $\gamma_{\mathbf{k}}(t)$ and $\alpha_{mn,\mathbf{k}}(t)$ into Eq. (4) we obtain an equation for $\beta_j(t)$

$$\dot{\beta}_j(t) = -\sum_{\mathbf{k}} \sum_{j'=1}^{N} \int_0^t dt' g_k^2 \beta_{j'}(t') e^{i(\nu_k-\omega)(t'-t) + i\mathbf{k}\cdot(\mathbf{r}_j - \mathbf{r}_{j'})} - \sum_{\mathbf{k}} \sum_{j'=1, j'\neq j}^{N} \int_0^t dt' g_k^2 \beta_{j'}(t') e^{i(\nu_k+\omega)(t'-t) - i\mathbf{k}\cdot(\mathbf{r}_j - \mathbf{r}_{j'})} - (N-1) \sum_{\mathbf{k}} g_k^2 \int_0^t dt' \beta_j(t') e^{i(\nu_k+\omega)(t'-t)}. \quad (9)$$

In Appendix A we derive equation for $\beta_j(t)$ assuming Markovian approximation (slow decay) which is valid provided the sate decay time is larger then the time of photon flight through the atomic cloud. The answer is given by Eq. (A7).

Next we assume that initially the system is prepared in an eigenstate and the state decays exponentially, that is

$$\beta_j(t) = \beta_j e^{-\lambda_n t}, \quad (10)$$

where $\mathrm{Re}(\lambda_n) > 0$. Substituting Eq. (10) into Eq. (A7) yields the following eigenvalue equation:

$$\lambda_n \beta_j = \gamma \beta_j - i\gamma \sum_{j'\neq j} \frac{\exp(ik_0|\mathbf{r}_j - \mathbf{r}_{j'}|)}{k_0|\mathbf{r}_j - \mathbf{r}_{j'}|} \beta_{j'}, \quad (11)$$

where

$$\gamma = \frac{k_0^3 \wp^2}{2\pi\epsilon_0\hbar} \quad (12)$$

is the single atom decay rate and $k_0 = \omega/c$.

In Appendix B we derive an eigenvalue equation in the rotating wave approximation and show that the answer substantially differs from Eq. (11). Thus counter-rotating terms in the interaction Hamiltonian (1) can play an important role for some problems.

Inclusion of light polarization changes the kernel of Eq. (11). Such general equation has been considered for the case of two identical atoms in [49,50] and for N atoms in [28,41,43,51].

III. EIGENFUNCTIONS AND EIGENVALUES FOR A DENSE CLOUD

For a dense cloud when there are many atoms in volume λ^3 ($\lambda = 2\pi c/\omega$) one can go to the continuous limit and replace summation over j' by integration. Then the eigenvalue Eq. (11)

reads

$$-i\gamma\frac{N}{V}\int d\mathbf{r}'\frac{\exp(ik_0|\mathbf{r}-\mathbf{r}'|)}{k_0|\mathbf{r}-\mathbf{r}'|}\beta(\mathbf{r}')=\lambda_n\beta(\mathbf{r}) \tag{13}$$

or

$$\gamma\frac{N}{V}\int d\mathbf{r}'[K_0(\mathbf{r},\mathbf{r}')+iK_1(\mathbf{r},\mathbf{r}')]\beta_n(\mathbf{r}')=\lambda_n\beta_n(\mathbf{r}), \tag{14}$$

where

$$K_0(\mathbf{r},\mathbf{r}')=\frac{\sin(k_0|\mathbf{r}-\mathbf{r}'|)}{k_0|\mathbf{r}-\mathbf{r}'|},\quad K_1(\mathbf{r},\mathbf{r}')=-\frac{\cos(k_0|\mathbf{r}-\mathbf{r}'|)}{k_0|\mathbf{r}-\mathbf{r}'|},$$

and $k_0=\omega/c$. We assume that atoms are uniformly distributed in a sphere of volume $V=4\pi R^3/3$, and N/V is the atomic density. Decay rate of an eigenstate n is given by $\Gamma_n=\mathrm{Re}(\lambda_n)$, while $\mathrm{Im}(\lambda_n)$ yields the frequency shift. The imaginary part of the kernel $iK_1(\mathbf{r},\mathbf{r}')$ describes contribution from virtual photons.

In Appendix C we derive analytical solution of the integral Eq. (13). The eigenfunctions are given by [26,27]

$$\beta(\mathbf{r})=j_n(ak_0r)\,Y_{nm}(\hat{r}), \tag{15}$$

where $Y_{nm}(\hat{r})\equiv Y_{nm}(\theta,\varphi)$ are spherical harmonics and

$$a=\sqrt{1-\frac{3i\gamma N}{k_0^3R^3\lambda_n}}. \tag{16}$$

Eigenvalues λ_n are determined from the following equation for a:

$$a=\frac{j_n(ak_0R)}{j_{n-1}(ak_0R)}\frac{h^{(1)}_{n-1}(k_0R)}{h^{(1)}_n(k_0R)}, \tag{17}$$

where $j_k(z)$ and $h^{(1)}_k(z)$ are the spherical Bessel functions.

For $n=0$

$$j_0(x)=\frac{\sin(x)}{x},\quad j_{-1}(x)=\frac{\cos(x)}{x}, \tag{18}$$

$$h^{(1)}_0(x)=\frac{e^{ix}}{ix},\quad h^{(1)}_{-1}(x)=\frac{e^{ix}}{x}, \tag{19}$$

and Eq. (17) reduces to

$$a=i\tan(ak_0R). \tag{20}$$

Next we analyze limiting cases of small and large atomic cloud. In the Dicke limit $k_0R\ll 1$ we have

$$\frac{h^{(1)}_{n-1}(k_0R)}{h^{(1)}_n(k_0R)}\approx i\frac{(k_0R)^{2n}}{[(2n-1)!!]^2}+\begin{cases}0, & n=0\\ \frac{k_0R}{2n-1}, & n>0\end{cases}. \tag{21}$$

We keep only the imaginary part in the right hand side of Eq. (21). Then Eq. (17) reduces to

$$aj_{n-1}(ak_0R)\approx i\frac{(k_0R)^{2n}}{[(2n-1)!!]^2}j_n(ak_0R). \tag{22}$$

Using the identity

$$\frac{d}{dx}[xj_{n-1}(x)]=nj_{n-1}(x)-xj_n(x) \tag{23}$$

we expand the left hand side of Eq. (22) near $ak_0R=A_{nl}$, where A_{nl} is a positive zero of the Bessel function $j_{n-1}(x)$, and find

$$-A_{nl}(ak_0R-A_{nl})=i\frac{(k_0R)^{2n+1}}{[(2n-1)!!]^2}. \tag{24}$$

Therefore

$$a\approx\frac{A_{nl}}{k_0R}-i\frac{(k_0R)^{2n}}{A_{nl}[(2n-1)!!]^2}. \tag{25}$$

The corresponding eigenvalues and eigenfunctions are given by [26]

$$\lambda_{nl}\approx-\frac{3i\gamma N}{A^2_{nl}k_0R}+\frac{6\gamma N(k_0R)^{2n}}{A^4_{nl}[(2n-1)!!]^2}, \tag{26}$$

$$\beta_{nlm}(\mathbf{r})=j_n\left(A_{nl}\frac{r}{R}\right)Y_{nm}(\hat{r}). \tag{27}$$

In particular, $A_{0l}=(2l-1)\pi/2$ and $A_{1l}=\pi l$, $l=1,2,3\ldots$. Equation (26) shows that in the long wavelength limit ($k_0R\ll 1$) only eigenvalues with $n=0$ have large real part and decay fast (Dicke superradiance [1]), while eigenvalues with $n>0$ are suppressed by a factor $(k_0R)^{2n}$. Those states are trapped. One should note that

$$\sum_{l=0}^{\infty}\mathrm{Re}(\lambda_{0l})=\sum_{l=0}^{\infty}\frac{6\gamma N}{A^4_{0l}}=\gamma N, \tag{28}$$

as expected from general arguments [26].

Let us now consider the large cloud limit $k_0R\gg 1$. In this case we take asymptotics of the Bessel functions in Eq. (17) ($z\gg n$)

$$j_n(z)\approx\frac{1}{z}\sin\left(z-\frac{n}{2}\pi\right),$$

$$h^{(1)}_n(z)\approx\frac{1}{z}(-i)^{n+1}e^{iz},$$

and obtain

$$a=i\tan\left(ak_0R-\frac{n}{2}\pi\right). \tag{29}$$

One can rewrite Eq. (29) as

$$i\,\mathrm{arctanh}(a)=-ak_0R+\frac{n}{2}\pi+\pi l, \tag{30}$$

where l is an integer. In logarithmic representation $\mathrm{arctanh}(a)=\frac{1}{2}\ln(\frac{1+a}{1-a})$.

For $|a|\ll 1$ we approximate $\mathrm{arctanh}(a)\approx a$, then Eq. (30) has a solution

$$a\approx\frac{\pi(n+2l)}{2k_0R}\left(1-\frac{i}{k_0R}\right), \tag{31}$$

and thus

$$\lambda_n\approx\frac{3i\gamma N}{(k_0R)^3}\left(1+\frac{\pi^2(n+2l)^2}{4(k_0R)^2}\right). \tag{32}$$

Solutions (31), (32) are valid provided $|n+2l|\ll k_0R$.

For $|a|\gg 1$ we use $\mathrm{arctanh}(a)\approx -i\pi/2+1/a$ and obtain

$$a\approx\frac{\pi(n+2l-1)}{2k_0R}-\frac{2i}{\pi(n+2l-1)}, \tag{33}$$

$$\lambda_n\approx-\frac{12i\gamma N}{\pi^2(n+2l-1)^2k_0R}+\frac{96\gamma N}{\pi^4(n+2l-1)^4}, \tag{34}$$

which is valid if $|n+2l|\gg k_0R$.

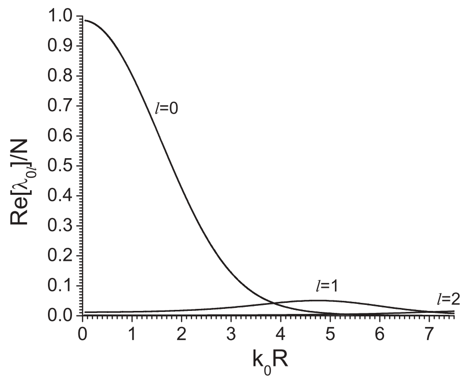

FIG. 1. Real part of λ_{0l} as a function of k_0R for $l = 0$, 1, and 2.

Finally, for $|a + 1| \ll 1$ we find with logarithmic accuracy

$$a \approx -1 + \frac{\pi}{2k_0R}\left(\frac{1}{2} + n + 2l + \left\{\frac{2k_0R}{\pi}\right\}\right) + \frac{i}{2k_0R}\ln\left(\frac{4k_0R}{\ln(4k_0R)}\right), \quad (35)$$

$$\lambda_n \approx \frac{3i\gamma N}{(k_0R)^2}\left[\pi\left(\frac{1}{2} + n + 2l + \left\{\frac{2k_0R}{\pi}\right\}\right) + i\ln\left(\frac{4k_0R}{\ln(4k_0R)}\right)\right]^{-1}, \quad (36)$$

where $\{\cdots\}$ stands for the fractional part of a number. Eigenvalues (36) are much larger then those given by Eqs. (32) and (34).

Figures 1–4 show real and imaginary parts of λ_{nl} as a function of k_0R obtained by solving Eq. (17) numerically. We plot the result for $n = 0$ and $l = 0,1,2$ in Figs. 1 and 2, while Figs. 3 and 4 show the answer for $n = 1$. One can see that the imaginary part of λ_{nl} (frequency shift) is large for small atomic samples and becomes small when $R \gg \lambda$ (λ is the wavelength of the emitted photon). It is interesting to note that states which decay slowly for $R \ll \lambda$ become super-radiant for the intermediate sample size wherein they decay faster then the $n = l = 0$ state.

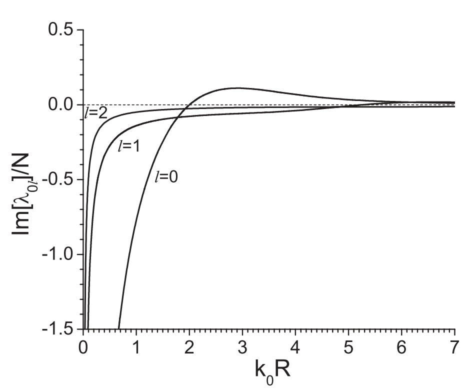

FIG. 2. Imaginary part of λ_{0l} as a function of k_0R for $l = 0$, 1, and 2.

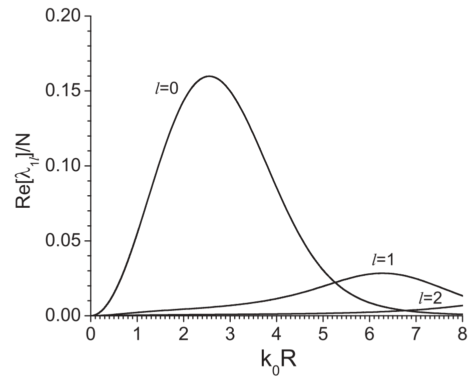

FIG. 3. Real part of λ_{1l} as a function of k_0R for $l = 0$, 1, and 2.

IV. EFFECT OF VIRTUAL PHOTONS ON STATE EVOLUTION

The imaginary part of the kernel $iK_1(\mathbf{r},\mathbf{r}')$ in Eq. (14) describes contribution from the virtual photons. For $K_1 = 0$ the integral Eq. (14) reduces to equation with sin kernel

$$\frac{\partial\beta(t,\mathbf{r})}{\partial t} = -\gamma\frac{N}{V}\int d\mathbf{r}'\frac{\sin(k_0|\mathbf{r}-\mathbf{r}'|)}{k_0|\mathbf{r}-\mathbf{r}'|}\beta(t,\mathbf{r}') \quad (37)$$

which for spherical atomic cloud was solved by Ernst in 1969 [38] and has the following solutions:

$$\beta_{nm}(\mathbf{r}) = j_n(k_0r)\,Y_{nm}(\hat{r}), \quad (38)$$

$$\lambda_{nm} = \frac{3\gamma N}{2}\left[j_n^2(k_0R) - j_{n-1}(k_0R)j_{n+1}(k_0R)\right]. \quad (39)$$

In the small sample limit $k_0R \ll 1$ Eq. (39) yields

$$\lambda_n \approx \frac{3\gamma N}{(2n+3)[(2n+1)!!]^2}(k_0R)^{2n}, \quad (40)$$

while in the opposite limit $k_0R \gg n$ we obtain

$$\lambda_n \approx \frac{3\gamma N}{2(k_0R)^2}\left[1 - \frac{(-1)^n}{2k_0R}\sin(2k_0R)\right]. \quad (41)$$

The set of eigenfunctions (38) is incomplete. There is an infinite number of functions which are orthogonal to $\beta_{nm}(\mathbf{r})$. For example, Eq. (C13) yields that functions

$$\beta_{nml}(\mathbf{r}) = j_n(A_{nl}k_0r)\,Y_{nm}(\hat{r}), \quad (42)$$

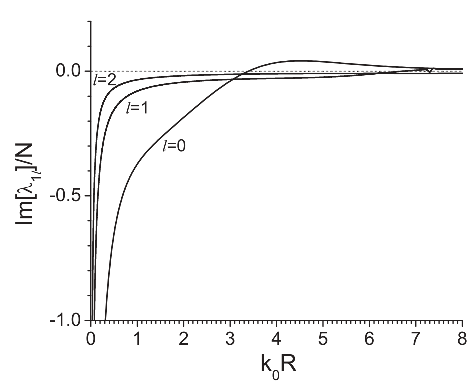

FIG. 4. Imaginary part of λ_{1l} as a function of k_0R for $l = 0$, 1, and 2.

where A_{nl} are roots of the equation (excluding the root $A = 1$)

$$A = \frac{j_n(Ak_0R)}{j_{n-1}(Ak_0R)} \frac{j_{n-1}(k_0R)}{j_n(k_0R)}, \quad (43)$$

are orthogonal to $\beta_{nm}(\mathbf{r})$. Functions (42) are also solutions of the truncated Eq. (37) but with zero eigenvalues. It is worth noting that the structure of Eq. (43) is similar to Eq. (17).

Now let us consider the effect of the K_1 term in Eq. (14). In the Dicke limit $k_0R \ll 1$ the eigenvalues of the full integral equation are given by Eq. (26). One can see that $\mathrm{Im}(\lambda_{nl})$ (frequency shift) becomes large for $k_0R \ll 1$ and Eq. (26) differs substantially from Eq. (40) obtained for the truncated kernel. Thus the K_1 term has a crucial effect on eigenvalues for a small atomic sample. For a large atomic sample $k_0R \gg 1$ the leading eigenvalues for the full and truncated kernels are given by Eqs. (36) and (41), respectively. Now the eigenvalues differ by a factor of the order of one.

To illustrate such a difference we solved numerically the matrix eigenvalue equation

$$\lambda_n \beta_i(t) = \gamma \sum_{j=1}^{N} \Gamma_{ij} \beta_j(t), \quad (44)$$

where

$$\Gamma_{ij} = K_0(\mathbf{r}_i,\mathbf{r}_j) + iK_1(\mathbf{r}_i,\mathbf{r}_j), \quad i \neq j \quad \text{and} \quad \Gamma_{ii} = 1, \quad (45)$$

for the full kernel (45) and the truncated kernel $\Gamma_{ij} = K_0(\mathbf{r}_i,\mathbf{r}_j)$. In simulations we took $N = 10\,000$ atoms randomly distributed inside a sphere of radius $R = 10\lambda$ (λ is the wavelength of the emitted photon). The results for $\mathrm{Re}(\lambda_n)$ are plotted in Fig. 5. One can see that the K_1 term modifies the eigenvalues by a factor of the order of one if the atomic sample is large.

However, despite the difference in eigenvalues and eigenfunctions, the evolution of an initial state for both kernels is close for rapidly decaying states [32]. For example, in the small sample limit $R \ll \lambda$ the Dicke symmetric state $\beta_s(\mathbf{r}) = 1/\sqrt{N}$ is an eigenstate for the sin kernel and decays at the rate $\Gamma_s = N\gamma$. For the exp kernel the fastest decaying eigenstate is [see Eqs. (27) and (26)] $\beta_0(\mathbf{r}) = \sin(\pi r/2R)/r$ and $\Gamma_0 = 96N\gamma/\pi^4 \approx 0.986N\gamma$. Because $\Gamma_0 \approx \Gamma_s$ and overlapping between states $\beta_s(\mathbf{r})$ and $\beta_0(\mathbf{r})$ is almost 1: $\langle\beta_s(\mathbf{r})|\beta_0(\mathbf{r})\rangle = 0.993$ the symmetric state $\beta_s(\mathbf{r})$ decays essentially the same way for both kernels.

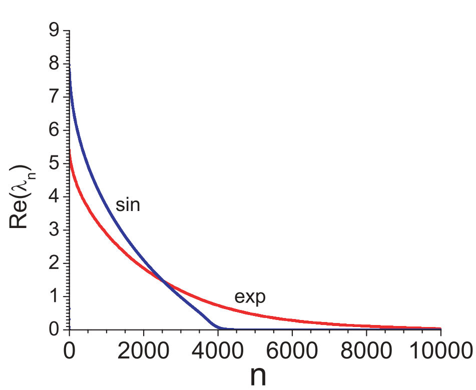

FIG. 5. (Color online) Real part of eigenvalues for a system of $N = 10\,000$ atoms randomly distributed inside a sphere of radius $R = 10\lambda$ calculated with the exp and sin kernels.

For a large atomic sample $R \gg \lambda$ the "timed" Dicke state

$$\beta(\mathbf{r}) = e^{i\mathbf{k}_0\cdot\mathbf{r}} \quad (46)$$

is an example of state which rapidly decays with a rate $\Gamma \sim N\gamma/(k_0R)^2$. State (46) is prepared by absorption of a single photon with wave vector $\mathbf{k}_0$ ($k_0 = \omega/c$) [20,21]. As shown in [32,36,37], virtual transitions modify evolution of the timed Dicke state (46) in the large sample limit by about 10%–20%. If, however, the size of the atomic cloud is very large ($R \gg c/\Gamma$) the initial state undergoes oscillations with a collective Rabi frequency [24]. In such a non-Markovian regime, virtual transitions give essentially no effect.

Next we discuss the effect of virtual photons in details in a Markovian regime.

A. Sin kernel

If we omit virtual processes the evolution of state vector is described by equation with sin kernel (37). Here we solve Eq. (37) analytically with the initial condition

$$\beta(0,\mathbf{r}) = \frac{1}{r}\sin\left(a\frac{\pi r}{2R}\right), \quad (47)$$

where a is an arbitrary parameter. We look for solution of Eq. (37) in a spherically symmetric form

$$\beta(t,\mathbf{r}) = f(t,r). \quad (48)$$

Plugging Eq. (48) into Eq. (37) and using the identity

$$\frac{\sin(k_0|\mathbf{r}-\mathbf{r}'|)}{k_0|\mathbf{r}-\mathbf{r}'|} = \sum_{m=0}^{\infty}(2m+1)P_m(\hat{r}\cdot\hat{r}')j_m(k_0r')j_m(k_0r), \quad (49)$$

where $\hat{r}$ and $\hat{k}_0$ are unit vectors in the directions of $\mathbf{r}$ and $\mathbf{k}_0$, respectively, $j_k(z)$ are the spherical Bessel functions and P_n are the Legendre polynomials, we obtain

$$\frac{\partial f(t,r)}{\partial t} = -\frac{4\pi\gamma N}{V} j_0(k_0r)\int_0^R dr' r'^2 j_0(k_0r') f(t,r'). \quad (50)$$

Taking into account that $j_0(x) = \sin(x)/x$ we find that solution of Eq. (50) satisfying the initial condition (47) is

$$\beta(t,\mathbf{r}) = \frac{1}{r}\sin\left(a\frac{\pi r}{2R}\right) + \frac{F}{r}\sin(k_0r)[1-e^{-\Gamma t}], \quad (51)$$

where

$$F = \frac{2k_0R[\pi a\cos(\pi a/2)\sin(k_0R) - 2\sin(\pi a/2)k_0R\cos(k_0R)]}{[(\pi a/2)^2 - (k_0R)^2][2k_0R - \sin(2k_0R)]} \quad (52)$$

and

$$\Gamma = \frac{3\gamma N}{2(k_0R)^2}\left[1 - \frac{\sin(2k_0R)}{2k_0R}\right]. \quad (53)$$

In the small sample limit Eq. (53) reduces to $\Gamma = N\gamma$.

Equation (51) shows that at the beginning the atomic system decays with the super-radiant rate (53) but quickly ends up in a trapped state

$$\beta(\mathbf{r}) = \frac{1}{r}\sin\left(a\frac{\pi r}{2R}\right) + \frac{F}{r}\sin(k_0 r). \quad (54)$$

Probability that atoms are excited is given by

$$P(t) = 1 - F^2\frac{\pi a[2k_0R - \sin(2k_0R)]}{2k_0R\,[\pi a - \sin(a\pi)]}[1 - e^{-2\Gamma t}]. \quad (55)$$

In the limit $k_0R \ll 1$,a Eq. (55) yields

$$P(t) = 1 - \frac{48}{(\pi a)^3}\frac{[\pi a\cos(\pi a/2) - 2\sin(\pi a/2)]^2}{[\pi a - \sin(\pi a)]}[1 - e^{-2\Gamma t}]. \quad (56)$$

Equation (56) shows that all initial population is trapped if

$$\pi a/2 = \tan(\pi a/2), \quad (57)$$

that is

$$a = 2.8606, \quad 4.9181, \quad 6.9417, \quad 8.9547, \quad \ldots. \quad (58)$$

Taking limit $a \to 0$ in Eqs. (51) and (52) we obtain formula for the time evolution of the symmetric state

$$\beta(0,\mathbf{r}) = 1, \quad (59)$$

namely,

$$\beta(t,\mathbf{r}) = 1 + 2F\frac{\sin(k_0r)}{k_0r}[1 - e^{-\Gamma t}], \quad (60)$$

where

$$F = \frac{k_0R\cos(k_0R) - \sin(k_0R)}{k_0R - \sin(k_0R)\cos(k_0R)}. \quad (61)$$

Equation (60) shows that for $t \gtrsim 1/\Gamma$ the system ends up in a trapped state

$$\beta(\mathbf{r}) = 1 + 2F\frac{\sin(k_0r)}{k_0r}. \quad (62)$$

Function (62) vanishes in the small sample limit $k_0R \ll 1$, however, for a large sample $\beta(\mathbf{r}) \approx 1$, state (59) is completely trapped. For the initial condition (59) the probability that atoms are excited is given by

$$P(t) = 1 - \frac{6\,[k_0R\cos(k_0R) - \sin(k_0R)]^2}{k_0R - \sin(k_0R)\cos(k_0R)}\frac{[1 - e^{-2\Gamma t}]}{(k_0R)^3}. \quad (63)$$

B. Exp kernel: Evolution of fast decaying state excited by single photon spherical wave

Here we solve the evolution equation with exp kernel

$$\frac{\partial\beta(t,\mathbf{r})}{\partial t} = i\gamma\frac{N}{V}\int d\mathbf{r}'\frac{\exp(ik_0|\mathbf{r} - \mathbf{r}'|)}{k_0|\mathbf{r} - \mathbf{r}'|}\beta(t,\mathbf{r}') \quad (64)$$

assuming that initially atoms are prepared in the spherically symmetric eigenstate of Eq. (37), that is

$$\beta(0,\mathbf{r}) = j_0(k_0r) = \frac{\sin(k_0r)}{k_0r}. \quad (65)$$

State (65) corresponds to $a = 2k_0R/\pi$ in Eq. (47) and can be excited by the absorption of a single photon of frequency ω with a spherical wave front.

We look for the solution of Eq. (64) in the form

$$\beta(t,\mathbf{r}) = \frac{f(t,r)}{k_0r}. \quad (66)$$

Then we use the identity (C6) and orthogonality condition (C10) and obtain the following equation for $f(t,r)$:

$$\frac{\partial f(t,r)}{\partial t} = -4\pi\gamma r\frac{N}{V}\left[h_0^{(1)}(k_0r)\int_0^r dr'r'j_0(k_0r')f(t,r') + j_0(k_0r)\int_r^R dr'r'h_0^{(1)}(k_0r')f(t,r')\right] \quad (67)$$

with the initial condition

$$f(0,r) = \sin(k_0r). \quad (68)$$

Replacing

$$j_0(z) = \frac{\sin(z)}{z}, \qquad h_0^{(1)}(z) = -i\frac{e^{iz}}{z}, \quad (69)$$

and introducing the dimensionless coordinate $x = r/R$ and $t \to \Gamma t$, where

$$\Gamma = \frac{3\gamma N}{2(k_0R)^2}, \quad (70)$$

Eq. (67) reads

$$\frac{\partial f(t,x)}{\partial t} = 2i\left[e^{ik_0Rx}\int_0^x dy\sin(k_0Ry)f(t,y) + \sin(k_0Rx)\int_x^1 dy e^{ik_0Ry}f(t,y)\right]. \quad (71)$$

1. Large atomic sample

Next we consider the large cloud limit $k_0R \gg 1$ and look for the solution of Eq. (71) in the form

$$f(t,x) = A(t,x)\sin(k_0Rx) + iB(t,x)\cos(k_0Rx), \quad (72)$$

where $A(t,x)$ and $B(t,x)$ are slowly varying functions of x. Substituting Eq. (72) into Eq. (71) and keeping the leading order terms in k_0R we obtain the following equations for $A(t,x)$ and $B(t,x)$:

$$\frac{\partial A(t,x)}{\partial t} = -\int_0^1 dyA(t,y) - \int_x^1 dyB(t,y), \quad (73)$$

$$\frac{\partial B(t,x)}{\partial t} = \int_0^x dyA(t,y) \quad (74)$$

with the initial condition

$$A(0,x) = 1, \quad B(0,x) = 0. \quad (75)$$

We solve Eqs. (73)–(75) in Appendix D using the method of Laplace transform. In dimension units the answer is given by

$$A(t,r) = \frac{1}{2}\left[J_0\left(2\sqrt{1 - \frac{r}{R}}\sqrt{\Gamma t}\right) + J_0\left(2\sqrt{1 + \frac{r}{R}}\sqrt{\Gamma t}\right)\right], \quad (76)$$

$$B(t,r) = \frac{1}{2}\left[J_0\left(2\sqrt{1-\frac{r}{R}}\sqrt{\Gamma t}\right) - J_0\left(2\sqrt{1+\frac{r}{R}}\sqrt{\Gamma t}\right)\right], \tag{77}$$

where $J_0(z)$ is the Bessel function.

The probability that atoms are excited as a function of time is

$$P(t) = \frac{\int d\mathbf{r}|\beta(t,\mathbf{r})|^2}{\int d\mathbf{r}|\beta(0,\mathbf{r})|^2}. \tag{78}$$

For integral equation with sin kernel

$$P_{\sin}(t) = e^{-2\Gamma t}. \tag{79}$$

For $\beta(t,\mathbf{r})$ given by Eq. (72) we find

$$P_{\exp}(t) = J_0^2(2\sqrt{2\Gamma t}) + J_1^2(2\sqrt{2\Gamma t}) \tag{80}$$

which for $t \gg 1/\Gamma$ yields

$$P_{\exp}(t) \approx \frac{1}{\pi\sqrt{2\Gamma t}}. \tag{81}$$

The probability that atoms are in the state (65) reads

$$P^0_{\exp}(t) = \frac{|\langle 0|\beta(t,\mathbf{r})\rangle|^2}{\langle 0|0\rangle^2} = \frac{J_1^2(2\sqrt{2\Gamma t})}{2\Gamma t} \tag{82}$$

and for $t \gg 1/\Gamma$ reduces to

$$P^0_{\exp}(t) \approx \frac{\cos^2\left(2\sqrt{2\Gamma t} + \frac{\pi}{4}\right)}{\pi(2\Gamma t)^{3/2}}. \tag{83}$$

In Fig. 6 we plot the probability that atoms are excited given by Eqs. (80) (solid line) and (79) (dash line) obtained using the exp and sin kernels in the large sample limit. Initially atoms are prepared in the state (65). Figure 7 shows $P^0_{\exp}(t)$ obtained from Eq. (82) (solid line) and compares it with those obtained from the equation with sin kernel $P^0_{\sin}(t) = e^{-2\Gamma t}$ (dashed line). The two curves are close to each other. The $P(t) - P_0(t)$ line is the probability to find the atomic system in any other state but (65).

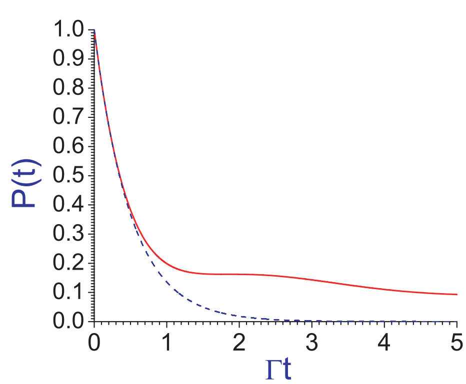

FIG. 6. (Color online) Probability that atoms are excited $P(t)$ for large atomic cloud calculated using the exp (solid line) and sin (dashed line) kernels. Initially atoms are prepared in the state (65) and $\Gamma = 3N\gamma/2(k_0R)^2$.

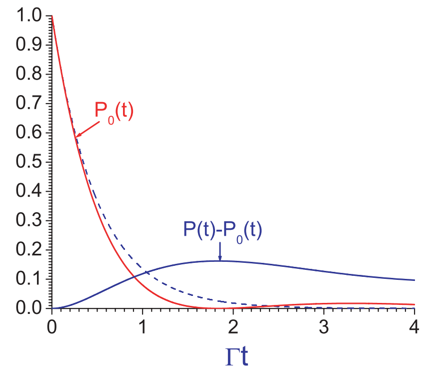

FIG. 7. (Color online) Probability that atoms are in the state (65) $P_0(t)$ obtained in the large sample limit from equation with exp (solid line) and sin (dash line) kernels. The $P(t) - P_0(t)$ curve shows probability that atoms are in any other state but (65). Initially atoms are in the state (65).

2. *Small sample limit*

In the small sample limit $R \ll \lambda$ the initial condition (65) reduces to symmetric Dicke state (59) and equation with sin kernel (37) yields Dicke result

$$\beta(t,\mathbf{r}) = e^{-\Gamma t}, \tag{84}$$

where $\Gamma = N\gamma$.

For the equation with exp kernel the state evolution can be obtained by noting that in the small sample limit

$$\beta_n(t,r) = \frac{R}{r}\sin\left[(2n+1)\frac{\pi r}{2R}\right]e^{-\lambda_n t} \tag{85}$$

are eigenfunctions of Eq. (64) with eigenvalues (26)

$$\lambda_n = -\frac{12iN\gamma}{\pi^2(2n+1)^2k_0R} + \frac{96N\gamma}{\pi^4(2n+1)^4}, \quad n = 0,1,2,\ldots. \tag{86}$$

Using the identity

$$1 = \frac{4}{\pi x}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin[(2n+1)x]$$

one can expand the initial condition $\beta(0,r) = 1$ in terms of $\beta_n(0,r)$. As a result, time evolution of the symmetric state is given by

$$\beta(t,\mathbf{r}) = \frac{8R}{\pi^2 r}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\left[(2n+1)\frac{\pi r}{2R}\right]e^{-\lambda_n t} \tag{87}$$

and the probability to find atoms excited is

$$P(t) = \frac{96}{\pi^4}\sum_{n=0}^{\infty}\frac{\exp[-2\mathrm{Re}(\lambda_n)t]}{(2n+1)^4}. \tag{88}$$

Figure 8 shows $P(t)$ given by Eq. (88) (solid line) and compares it with the answer obtained omitting virtual processes $P(t) = \exp(-2\Gamma t)$ (dashed line). The two curves are close to each other, but Eq. (88) yields a few percent of the population trapped which slowly decays with time.

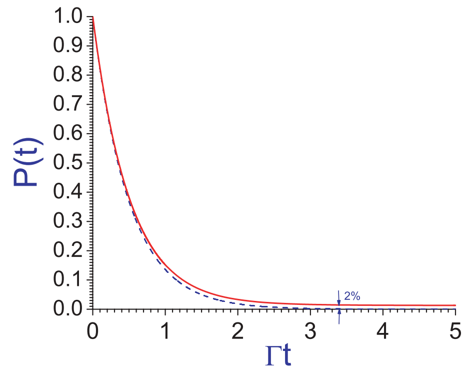

FIG. 8. (Color online) Probability that atoms are excited $P(t)$ for small atomic cloud calculated using the exp (solid line) and sin (dashed line) kernels. Initially atoms are prepared in the symmetric state (59), $\Gamma = N\gamma$.

The probability that atoms are in the symmetric state (59) is

$$P_s(t) = \frac{1}{V^2}\left|\int d\mathbf{r}\beta(t,\mathbf{r})\right|^2 = \left|\frac{96}{\pi^4}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}e^{-\lambda_n N\gamma t}\right|^2. \quad (89)$$

In Fig. 9 we plot $P_s(t)$ obtained for $R = 0.01\lambda$ from Eq. (89) (solid line) and compare it with $P(t) = \exp(-2\Gamma t)$ (dashed line). The two curves are very close meaning that the net decay rate of the symmetric state into all channels is very similar with or without virtual processes. The insert shows the probability to find atoms in any other state but the symmetric state (59) for $R = 0.01\lambda$ (solid line) and $R = 0.03\lambda$ (dash-dot line) obtained from Eqs. (88) and (89). Dependence of the imaginary part of λ_n (collective Lamb shift) on n is the reason for oscillations. The period of oscillations is proportional to k_0R. The other states are excited with a few percent probability. Thus, in the small sample limit, virtual photons also yield a small (but interesting) effect on the evolution of fast decaying states.

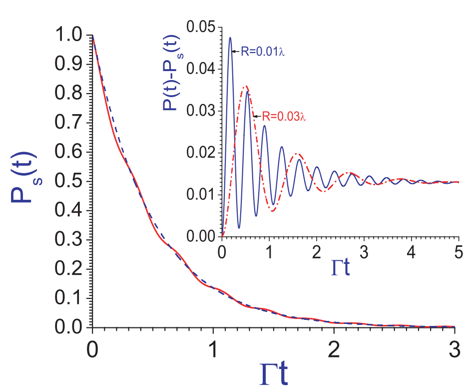

FIG. 9. (Color online) Probability that atoms are in the symmetric state $P_s(t)$ obtained using the exp (solid line) and sin (dashed line) kernels for $R = 0.01\lambda$. Initially atoms are in the symmetric state (59) and $\Gamma = N\gamma$. Insert shows probability to find atoms in any other state but symmetric state calculated for $R = 0.01\lambda$ (solid line) and $R = 0.03\lambda$ (dash-dot line) from equation with exp kernel.

C. Exp kernel: Evolution of slow decaying states

1. Symmetric state in the large sample limit

Here we solve the evolution Eq. (64) with the initial condition (59). We use Eq. (71) and look for solution in the form

$$f(t,x) = x\exp\left(-i\frac{2t}{k_0R}\right)\left[1+\frac{e^{ik_0R}}{k_0Rx}\left(1+\frac{i}{k_0R}\right)h(t,x)\right]. \quad (90)$$

Substituting Eq. (90) into Eq. (71) we obtain the following equation for $h(t,x)$:

$$\frac{1}{2}\frac{\partial h(t,x)}{\partial t} = \sin(k_0Rx) - \sin(k_0Rx)\int_0^1 dx'\sin(k_0Rx')h(t,x') + \frac{i}{k_0R}h(t,x) + i\cos(k_0Rx) \times \int_0^x dx'\sin(k_0Rx')h(t,x') + i\sin(k_0Rx)\int_x^1 dx'\cos(k_0Rx')h(t,x') \quad (91)$$

subject to the initial condition

$$h(0,x) = 0. \quad (92)$$

Here we solve Eq. (91) in the large sample limit $k_0R \gg 1$. In this limit one can represent $h(t,x)$ as

$$h(t,x) = B(t,x)\sin(k_0Rx) + iC(t,x)\cos(k_0Rx), \quad (93)$$

where B and C are real slowly varying functions of x. Substituting Eq. (93) into Eq. (91) and keeping the leading order terms in k_0R we obtain the following equations for B and C:

$$\frac{\partial B(t,x)}{\partial t} = 2 - \int_0^1 dx' B(t,x') - \int_x^1 dx' C(t,x'), \quad (94)$$

$$\frac{\partial C(t,x)}{\partial t} = \int_0^x dx' B(t,x') \quad (95)$$

with initial conditions

$$C(0,x) = 0, \quad B(0,x) = 0. \quad (96)$$

One can solve Eqs. (94) and (95) using the method of Laplace transform which in dimension units yields

$$C(t,r) = \sqrt{\frac{\Gamma t}{(1-\frac{r}{R})}}J_1\left(2\sqrt{1-\frac{r}{R}}\sqrt{\Gamma t}\right) - \sqrt{\frac{\Gamma t}{(1+\frac{r}{R})}}J_1\left(2\sqrt{1+\frac{r}{R}}\sqrt{\Gamma t}\right), \quad (97)$$

$$B(t,r) = \sqrt{\frac{\Gamma t}{(1-\frac{r}{R})}}J_1\left(2\sqrt{1-\frac{r}{R}}\sqrt{\Gamma t}\right) + \sqrt{\frac{\Gamma t}{(1+\frac{r}{R})}}J_1\left(2\sqrt{1+\frac{r}{R}}\sqrt{\Gamma t}\right), \quad (98)$$

where

$$\Gamma = \frac{3\gamma N}{2(k_0R)^2}. \quad (99)$$

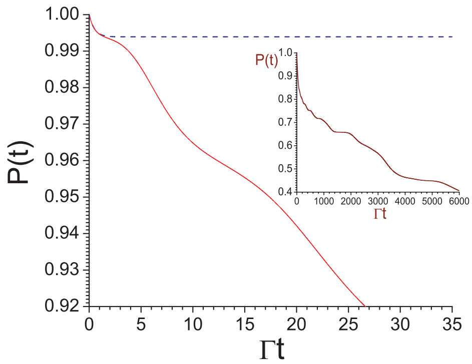

FIG. 10. (Color online) Probability that atoms are excited $P(t)$ obtained using the exp (solid line) and sin (dashed line) kernels. Initially atoms are in the symmetric state (100), $R = 5\lambda$ and Γ is given by Eq. (99).

As a result, the final answer for the state evolution in the large sample limit $k_0 R \gg 1$ is given by

$$\beta(t,\mathbf{r}) = \exp\left(-i\frac{2\Gamma t}{k_0 R}\right)\left\{1 + \frac{e^{ik_0 R}}{k_0 r}\,[B(t,r)\sin(k_0 r) + iC(t,r)\cos(k_0 r)]\right\}. \tag{100}$$

Solution (100) is valid for not very large t. For $t \gg 1/\Gamma$ one should take into account the next order terms in $k_0 R$ and Eq. (100) becomes invalid.

In Fig. 10 we plot the probability that atoms are excited $P(t)$ obtained from Eqs. (100) (solid line) and (63) (dashed line). Initially atoms are prepared in the symmetric state (59). The size of the atomic sample is $R = 5\lambda$. Analytical formula (100) is accurate up to $t \sim 20/\Gamma$. The insert shows the behavior of $P(t)$ for exp kernel on a larger time scale obtained by the numerical solution of Eq. (91). $P(t)$ exhibits interesting plateaus and oscillations. For t less then a few $1/\Gamma$ the exp and sin kernel curves are identical. For such time the real processes dominate and the initial state evolves into the state (62) which is trapped if we omit virtual processes. Virtual processes, however, result in state decay as shown by the solid curve. State (59) overlaps with many eigenstates of Eq. (64) [26]. Eigenstates which decay faster contribute to evolution at small time. As time increases $P(t)$ decays more slowly. However, eigenfunctions of Eq. (64) are not orthogonal [52] and, in addition, have different collective Lamb shifts. This makes state evolution richer.

2. *Trapped states in small sample limit*

Here we solve the evolution equation with exp kernel in the small sample limit assuming that initially the system is prepared in the state (47) with a given by Eq. (57). Such states are completely trapped if we omit virtual processes. The evolution equation with exp kernel can be solved noting that in the small sample limit, eigenfunctions and eigenvalues of Eq. (64) are given by Eqs. (85) and (86). Using the identity

$$\sin(ax) = \frac{4a}{\pi}\cos\left(\frac{\pi a}{2}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2 - a^2}\sin[(2n+1)x]$$

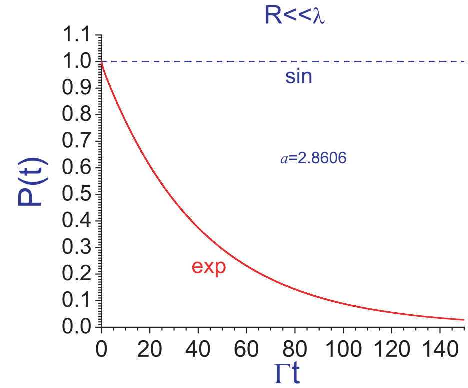

FIG. 11. (Color online) Probability that atoms are exited $P(t)$ calculated in the small sample limit taking into account virtual processes (solid line) and omitting them (dashed line). Initially atoms are prepared in the state (47) with $a = 2.8606$ and $\Gamma = N\gamma$.

one can expand the initial condition (47) in terms of $\beta_n(0,r)$. As a result, time evolution of the state (47) is given by

$$\beta(t,\mathbf{r}) = \frac{4a}{\pi r}\cos\left(\frac{\pi a}{2}\right)\sum_{n=0}^{\infty}\frac{(-1)^n e^{-\lambda_n t}}{(2n+1)^2 - a^2} \times \sin\left[(2n+1)\frac{\pi r}{2R}\right] \tag{101}$$

and probability to find atoms excited is

$$P(t) = \frac{16a^3}{\pi[\pi a - \sin(\pi a)]}\cos^2\left(\frac{\pi a}{2}\right)\sum_{n=0}^{\infty}\frac{\exp[-2\mathrm{Re}(\lambda_n)t]}{[(2n+1)^2 - a^2]^2}. \tag{102}$$

Figures 11 and 12 show $P(t)$ given by Eq. (102) (solid line) and compare it with the answer obtained omitting virtual processes (dashed line) for $a = 2.8606$ and $a = 4.9181$, respectively. For such a the initial state is trapped if we use the equation with sin kernel. Virtual processes result in the state decay as shown by the solid line. Please note that for $a = 4.9181$ the initial state decays ten times more slowly then for $a = 2.8606$.

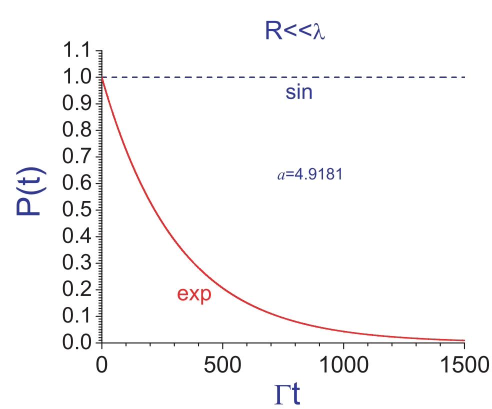

FIG. 12. (Color online) Probability that atoms are exited $P(t)$ calculated in the small sample limit taking into account virtual processes (solid line) and omitting them (dashed line). Initially atoms are prepared in the state (47) with $a = 4.9181$ and $\Gamma = N\gamma$.

V. COOPERATIVE SPONTANEOUS EMISSION OF *N* ATOMS: CLASSICAL ANALOGY

Whether or not the many-body effects of spontaneous emission of N atoms can be understood classically is a question of longstanding interest. Here we show that the quantum mechanical treatment of a single-photon emission (weak excitation) is analogous to the radiation of a system of N classical harmonic oscillators. Namely we show that Eq. (11) obtained in the quantum mechanical description is identical to those in a classical problem when the two-level atoms are treated as classical harmonic oscillators of frequency ω [53]. Let $\mathbf{r}_j(t)$ be the electron position in the oscillator j. The equation of motion for the electronic displacement is

$$\frac{\partial^2 \mathbf{r}_j(t)}{\partial t^2} + \omega^2 \mathbf{r}_j(t) = \frac{e}{m}\mathbf{E}(t,\mathbf{r}_j), \qquad (103)$$

where $\mathbf{E}(t,\mathbf{r})$ is the electric field, e is the electron charge, and m is the electron mass. Electric current density is given by

$$\mathbf{j}(t,\mathbf{r}) = \sum_{j=1}^{N} e\frac{\partial \mathbf{r}_j(t)}{\partial t}\delta(\mathbf{r}-\mathbf{r}_j). \qquad (104)$$

Next we use microscopic Maxwell's equations

$$\nabla\times\mathbf{E} = -\mu_0\frac{\partial \mathbf{H}}{\partial t}, \quad \nabla\times\mathbf{H} = \mathbf{j}+\varepsilon_0\frac{\partial \mathbf{E}}{\partial t} \qquad (105)$$

which yield ($\mu_0\varepsilon_0 = 1/c^2$)

$$-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} - \nabla\times\nabla\times\mathbf{E} = \mu_0\frac{\partial \mathbf{j}}{\partial t}.$$

Here ε_0 and μ_0 are the free space permittivity and permeability, respectively, and c is the speed of light in a vacuum. Using $\nabla\times\nabla\times\mathbf{E} = \mathrm{grad}\cdot\mathrm{div}\mathbf{E} - \Delta\mathbf{E}$ and assuming $\mathrm{div}\mathbf{E} = 0$ (the electron charge is compensated by the charge of nuclei) we obtain

$$\Delta\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \mu_0\sum_{j=1}^{N} e\frac{\partial^2\mathbf{r}_j(t)}{\partial t^2}\delta(\mathbf{r}-\mathbf{r}_j). \qquad (106)$$

We assume that electrons move in the same direction $\hat{\epsilon}$, that is $\mathbf{r}_j(t) = r_j(t)\hat{\epsilon}$. One can look for a solution of Eqs. (103) and (106) in the form

$$\mathbf{E}(t,\mathbf{r}) = \hat{\epsilon}A(t,\mathbf{r})e^{-i\omega t}, \qquad (107)$$

$$r_j(t) = \beta_j(t)e^{-i\omega t}, \qquad (108)$$

where $A(t,\mathbf{r})$ and $\beta_j(t)$ are slowly varying functions on a time scale $1/\omega$. Substituting Eqs. (107) and (108) into (103) and (106) and omitting higher order derivatives in t we obtain

$$\frac{\partial \beta_j(t)}{\partial t} = \frac{ie}{2\omega m}A(t,\mathbf{r}_j), \qquad (109)$$

$$\Delta A + k_0^2 A = -\omega^2\mu_0 e\sum_{j=1}^{N}\beta_j(t)\delta(\mathbf{r}-\mathbf{r}_j), \qquad (110)$$

where $k_0 = \omega/c$. Equation (110) has the form of Helmholt's equation

$$\left(\Delta + k_0^2\right)A(\mathbf{r}) = -f(\mathbf{r}). \qquad (111)$$

In terms of Green's function G defined as

$$\left(\Delta + k_0^2\right)G(\mathbf{r}-\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') \qquad (112)$$

the solution of Eq. (111) is

$$A(\mathbf{r}) = \int f(\mathbf{r}')G(\mathbf{r}-\mathbf{r}')d\mathbf{r}'. \qquad (113)$$

To obey the causality condition we must choose the retarded Green's function

$$G^R(\mathbf{r}-\mathbf{r}') = \frac{\exp(ik_0|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|} \qquad (114)$$

which yields

$$A(t,\mathbf{r}) = \frac{\omega^2\mu_0 e}{4\pi}\sum_{j=1}^{N}\beta_j(t)\int\delta(\mathbf{r}'-\mathbf{r}_j)\frac{\exp(ik_0|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}' \qquad (115)$$

or after integration over $\mathbf{r}'$

$$A(t,\mathbf{r}) = \frac{\omega^2\mu_0 e}{4\pi}\sum_{j=1}^{N}\beta_j(t)\frac{\exp(ik_0|\mathbf{r}-\mathbf{r}_j|)}{|\mathbf{r}-\mathbf{r}_j|}. \qquad (116)$$

Substituting this into Eq. (109) and taking out the contribution to the field from atom j we find

$$\frac{\partial \beta_j(t)}{\partial t} = -\gamma\beta_j(t) + i\gamma\sum_{j'\neq j}^{N}\beta_{j'}(t)\frac{\exp(ik_0|\mathbf{r}_j-\mathbf{r}_{j'}|)}{k_0|\mathbf{r}_j-\mathbf{r}_{j'}|}, \qquad (117)$$

where

$$\gamma = \frac{\omega^2 e^2}{8\pi\varepsilon_0 m c^3} = \frac{k_0^3 e^2 d^2}{8\pi\varepsilon_0\hbar} \qquad (118)$$

and

$$d = \sqrt{\frac{\hbar}{m\omega}}$$

is the oscillator length.

Equation (117) obtained in the classical consideration is equivalent to Eq. (11) derived in quantum mechanics (please note that equations are equivalent if in the quantum mechanical treatment we take into account virtual processes). This means that the many-body features of spontaneous emission of N two-level atoms (decay speed up and radiation trapping) can be understood in the classical model. Please note that the expression for γ obtained in the classical problem (118) is similar to those in quantum consideration (12). Namely if in Eq. (12) we replace $\wp$ by $ed/2$ we get Eq. (118).

The analogy between radiation of two-level atoms and classical harmonic oscillators takes place for weak excitation of atomic ensemble (e.g., only one atom is excited). In the opposite limit, the cooperative emission can be qualitatively different in classical and quantum treatments (see, e.g., Ref. [54]).

VI. SUMMARY

In the present paper we study the time evolution of collective N-atom states in which one atom is excited (but we do not know which one) and obtained analytical formulas for the eigenstates and eigenvalues λ_n of the system. Since

$\sum_{n=1}^{N}\lambda_n = N\gamma$ some states decay faster then the single atom decay rate γ, but some more slowly. In the small sample limit only one state decays fast with the rate $\Gamma \approx N\gamma$ (Dicke superradiance), other states undergo much slower decay or trapped. In the opposite limit many states decay with a super-radiant rate $\Gamma \sim N\gamma/(k_0R)^2$.

We also study the effect of virtual photons on the state evolution and show that they have a small effect on the evolution of rapidly decaying states. However, virtual photons change the long time dynamics from exponential decay into a power-law behavior which can be observed experimentally. For slowly decaying or trapped states the virtual processes substantially modify the state dynamics yielding new decay channels which results in a decay of the otherwise trapped states.

Finally we show that for weak atomic excitation the quantum mechanical treatment of collective spontaneous emission of N atoms is analogous to emission of N classical harmonic oscillators.

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APPENDIX A: DERIVATION OF EIGENVALUE EQUATION

We make an assumption of slow decay (Markovian approximation) and replace $\beta_j(t') \approx \beta_j(t)$ under the integral in Eq. (9). This is valid provided the state decay time is larger then the time of photon flight through the atomic cloud. Taking the remaining integral over t' and replacing summation over $\mathbf{k}$ by integration we obtain

$$\dot\beta_j(t) = \frac{iV_{\rm ph}}{(2\pi)^3}\beta_j(t)\int d^3\mathbf{k} g_k^2\left(\frac{1-e^{-i(\nu_k-\omega)t}}{\nu_k-\omega}\right) + (N-1)\frac{iV_{\rm ph}}{(2\pi)^3}\beta_j(t)\int d^3\mathbf{k} g_k^2\left(\frac{1-e^{-i(\nu_k+\omega)t}}{\nu_k+\omega}\right)$$
$$+\frac{iV_{\rm ph}}{(2\pi)^3}\int d^3\mathbf{k}\sum_{j'=1,j'\neq j}^{N} g_k^2\left[\left(\frac{1-e^{-i(\nu_k-\omega)t}}{\nu_k-\omega}\right)e^{i\mathbf{k}\cdot(\mathbf{r}_j-\mathbf{r}_{j'})} + \left(\frac{1-e^{-i(\nu_k+\omega)t}}{\nu_k+\omega}\right)e^{-i\mathbf{k}\cdot(\mathbf{r}_j-\mathbf{r}_{j'})}\right]\beta_{j'}(t), \quad \text{(A1)}$$

where $V_{\rm ph}$ is the photon quantization volume. Integration over directions of $\mathbf{k}$ gives ($\nu_k = ck$, $k_0 = \omega/c$)

$$\dot\beta_j(t) = \frac{iV_{\rm ph}}{2\pi^2c}\beta_j(t)\int_0^\infty dk k^2 g_k^2\left(\frac{1-e^{-ic(k-k_0)t}}{k-k_0}\right) + (N-1)\frac{iV_{\rm ph}}{2\pi^2c}\beta_j(t)\int_0^\infty dk k^2 g_k^2\left(\frac{1-e^{-ic(k+k_0)t}}{k+k_0}\right)$$
$$+\frac{iV_{\rm ph}}{2\pi^2c}\int_0^\infty dk k^2 g_k^2\sum_{j'\neq j}^{N}\left[\frac{1-e^{-ic(k-k_0)t}}{k-k_0}+\frac{1-e^{-ic(k+k_0)t}}{k+k_0}\right]\frac{\sin(k|\mathbf{r}_j-\mathbf{r}_{j'}|)}{k|\mathbf{r}_j-\mathbf{r}_{j'}|}\beta_{j'}(t). \quad \text{(A2)}$$

Next we replace k_0 by $k_0 + i0$ and remove exponential factors containing t. Such factors oscillate fast under integration over k and thus can be disregarded. Then we obtain

$$\dot\beta_j(t) = \frac{iV_{\rm ph}}{2\pi^2c}\beta_j(t)\int_0^\infty dk k^2 g_k^2\left(\frac{1}{k-k_0-i0}\right) + (N-1)\frac{iV_{\rm ph}}{2\pi^2c}\beta_j(t)\int_0^\infty dk k^2 g_k^2\left(\frac{1}{k+k_0+i0}\right)$$
$$+\frac{iV_{\rm ph}}{2\pi^2c}\sum_{j'\neq j}^{N}\int_0^\infty dk k^2 g_k^2\left[\frac{1}{k-k_0-i0}+\frac{1}{k+k_0+i0}\right]\frac{\sin(k|\mathbf{r}_j-\mathbf{r}_{j'}|)}{k|\mathbf{r}_j-\mathbf{r}_{j'}|}\beta_{j'}(t). \quad \text{(A3)}$$

One can rewrite the first two terms in Eq. (A3) using the relation

$$\frac{1}{x\mp i0} = {\rm P}\frac{1}{x} \pm i\pi\delta(x),$$

where P stands for the Cauchy principle part. Then taking into account Eq. (2) we find

$$\dot\beta_j(t) = \frac{i\gamma}{\pi k_0}\beta_j(t)\int_0^\infty dk\left({\rm P}\frac{1}{k-k_0}+i\pi\delta(k-k_0)\right) + \frac{i\gamma}{\pi k_0}(N-1)\beta_j(t)\int_0^\infty dk\left({\rm P}\frac{1}{k+k_0}-i\pi\delta(k+k_0)\right)$$
$$+\frac{i\gamma}{\pi k_0}\sum_{j'\neq j}^{N}\int_0^\infty dk\left[\frac{1}{k-k_0-i0}+\frac{1}{k+k_0+i0}\right]\frac{\sin(k|\mathbf{r}_j-\mathbf{r}_{j'}|)}{|\mathbf{r}_j-\mathbf{r}_{j'}|}\beta_{j'}(t), \quad \text{(A4)}$$

where $\gamma = (k_0^3\wp^2)/(2\pi\epsilon_0\hbar)$ is the single atom decay rate. The integral over dk in last term can be transformed into an integral from $-\infty$ to ∞ as

$$\int_0^\infty dk\left[\frac{1}{k-k_0-i0}+\frac{1}{k+k_0+i0}\right]\sin(k|\mathbf{r}_j-\mathbf{r}_{j'}|)$$
$$=\int_{-\infty}^\infty dk\frac{\sin(k|\mathbf{r}_j-\mathbf{r}_{j'}|)}{k-k_0-i0}$$
$$=\frac{1}{2i}\int_{-\infty}^\infty dk\left(\frac{\exp(ik|\mathbf{r}_j-\mathbf{r}_{j'}|)}{k-k_0-i0}-\frac{\exp(-ik|\mathbf{r}_j-\mathbf{r}_{j'}|)}{k-k_0-i0}\right). \tag{A5}$$

Integration over k in Eq. (A5) is performed by the contour method. For the first term we close the integration contour in the upper half-plane of complex k, while for the second term in the lower half-plane. Integration of the second term gives zero. As a result, Eq. (A4) yields

$$\dot\beta_j(t)=\frac{i\gamma}{\pi k_0}\beta_j(t)\int_0^\infty dkk\left[\mathrm{P}\frac{1}{k-k_0}+\mathrm{P}\frac{N-1}{k+k_0}\right]$$
$$-\gamma\beta_j+i\gamma\sum_{j'\neq j}^N\frac{\exp(ik_0|\mathbf{r}_j-\mathbf{r}_{j'}|)}{k_0|\mathbf{r}_j-\mathbf{r}_{j'}|}\beta_{j'}(t). \tag{A6}$$

The first term in Eq. (A6) corresponds to a frequency shift by the same value for all $\beta_j(t)$. This constant shift will be ignored in the following discussion. Finally we obtain

$$\dot\beta_j(t)=-\gamma\beta_j(t)+i\gamma\sum_{j'\neq j}\frac{\exp(ik_0|\mathbf{r}_j-\mathbf{r}_{j'}|)}{k_0|\mathbf{r}_j-\mathbf{r}_{j'}|}\beta_{j'}(t). \tag{A7}$$

APPENDIX B: INTEGRAL EQUATION IN THE ROTATING WAVE APPROXIMATION (RWA)

In the RWA the interaction of atoms with photons is described by the Hamiltonian

$$\hat H_{\mathrm{int}}=\sum_{\mathbf{k}}\sum_{j=1}^N g_k\{\hat\sigma_j\hat a_{\mathbf{k}}^\dagger\exp[i(\nu_k-\omega)t-i\mathbf{k}\cdot\mathbf{r}_j]+\mathrm{adj}\} \tag{B1}$$

and for single-atom excitation one can look for a solution of the Schrödinger equation in a form

$$\Psi=\sum_{j=1}^N\beta_j(t)|b_1b_2\cdots a_j\cdots b_N\rangle|0\rangle$$
$$+\sum_{\mathbf{k}}\gamma_{\mathbf{k}}(t)|b_1b_2\cdots b_N\rangle|1_{\mathbf{k}}\rangle. \tag{B2}$$

Substitution of Eq. (B2) into the Schrödinger equation yields the following equations for $\beta_j(t)$ and $\gamma_{\mathbf{k}}(t)$:

$$\dot\beta_j(t)=-i\sum_{\mathbf{k}}g_k\gamma_{\mathbf{k}}(t)\exp[-i(\nu_k-\omega)t+i\mathbf{k}\cdot\mathbf{r}_j], \tag{B3}$$

$$\dot\gamma_{\mathbf{k}}(t)=-ig_k\sum_{j=1}^N\beta_j(t)\exp[i(\nu_k-\omega)t-i\mathbf{k}\cdot\mathbf{r}_j]. \tag{B4}$$

Equation (B4) is the same with or without making RWA. However, Eq. (B3) obtained in the RWA has only one term in the right hand side, while Eq. (4) derived beyond the RWA contains an extra contribution.

Derivation of the eigenvalue equation based on Eqs. (B3) and (B4) is similar to those used to obtain Eq. (A4). The only difference is that now there are no counter-rotating terms with $k+k_0$ and, thus, instead of Eq. (A4), we obtain

$$\lambda_n\beta_j=-\frac{i\gamma}{\pi k_0}\beta_j\int_0^\infty dkk\left(\mathrm{P}\frac{1}{k-k_0}+i\pi\delta(k-k_0)\right)$$
$$-\frac{i\gamma}{\pi k_0}\sum_{j'\neq j}^N\int_0^\infty dk\frac{\beta_{j'}}{k-k_0-i0}\frac{\sin(k|\mathbf{r}_j-\mathbf{r}_{j'}|)}{|\mathbf{r}_j-\mathbf{r}_{j'}|}. \tag{B5}$$

The integral over dk in the last term we rewrite as $\int_0^\infty dk=\int_{-\infty}^\infty dk-\int_{-\infty}^0 dk$ and find

$$\lambda_n\beta_j=-\frac{i\gamma}{\pi k_0}\beta_j\int_0^\infty dk\mathrm{P}\frac{k}{k-k_0}+\gamma\beta_j$$
$$-i\gamma\sum_{j'\neq j}^N\frac{\exp(ik_0|\mathbf{r}_j-\mathbf{r}_{j'}|)}{k_0|\mathbf{r}_j-\mathbf{r}_{j'}|}\beta_{j'}$$
$$+\frac{i\gamma}{\pi k_0}\sum_{j'\neq j}^N\beta_{j'}\int_{-\infty}^0 dk\frac{\sin(k|\mathbf{r}_j-\mathbf{r}_{j'}|)}{|\mathbf{r}_j-\mathbf{r}_{j'}|(k-k_0)}. \tag{B6}$$

As we did during derivation of Eq. (A7) we omit the first term in Eq. (B6) and finally obtain

$$\lambda_n\beta_j=\gamma\beta_j-i\gamma\sum_{j'\neq j}^N\frac{\exp(ik_0|\mathbf{r}_j-\mathbf{r}_{j'}|)}{k_0|\mathbf{r}_j-\mathbf{r}_{j'}|}\beta_{j'}$$
$$+i\gamma\sum_{j'\neq j}^N K_2(\mathbf{r}_j,\mathbf{r}_{j'})\beta_{j'}, \tag{B7}$$

where

$$K_2(\mathbf{r}_j,\mathbf{r}_{j'})=\frac{1}{\pi k_0}\int_{-\infty}^0 dk\frac{\sin(k|\mathbf{r}_j-\mathbf{r}_{j'}|)}{|\mathbf{r}_j-\mathbf{r}_{j'}|(k-k_0)}$$
$$=\frac{1}{\pi k_0|\mathbf{r}_j-\mathbf{r}_{j'}|}[\sin(k_0|\mathbf{r}_j-\mathbf{r}_{j'}|)\mathrm{ci}(k_0|\mathbf{r}_j-\mathbf{r}_{j'}|)$$
$$-\cos(k_0|\mathbf{r}_j-\mathbf{r}_{j'}|)\mathrm{si}(k_0|\mathbf{r}_j-\mathbf{r}_{j'}|)], \tag{B8}$$

$\mathrm{ci}(x)$ and $\mathrm{si}(x)$ are the cosine and sine integrals. For $k_0|\mathbf{r}_j-\mathbf{r}_{j'}|\ll 1$, Eq. (B8) yields $K_2(\mathbf{r}_j,\mathbf{r}_{j'})\approx\ln(k_0|\mathbf{r}_j-\mathbf{r}_{j'}|)/\pi$.

One can see that the eigenvalue Eq. (B7) derived in the RWA contains an extra term compared to Eq. (A7) obtained beyond the RWA. The extra term yields an additional imaginary contribution to the kernel and appears as a consequence of improper treatment of the effect of virtual photons in the RWA.

APPENDIX C: SOLUTION OF INTEGRAL EQUATION

To find solutions of Eq. (13) we take into account that exponential kernel coincides with the retarded Green's function $G^R(\mathbf{r}-\mathbf{r}')$ of the Helmholtz equation, i.e.,

$$\left(\Delta+k_0^2\right)G^R(\mathbf{r}-\mathbf{r}')=-\delta(\mathbf{r}-\mathbf{r}'), \tag{C1}$$

where

$$G^R(\mathbf{r}-\mathbf{r}') = \frac{\exp(ik_0|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{C2}$$

Applying operator $\Delta + k_0^2$ to both sides of Eq. (13) we obtain ($V = 4\pi R^3/3$)

$$\Delta\beta(\mathbf{r}) + a^2 k_0^2 \beta(\mathbf{r}) = 0, \tag{C3}$$

where

$$a^2 = 1 - \frac{3iN\gamma}{\lambda_n (k_0 R)^3}. \tag{C4}$$

For the present spherically symmetric problem we choose solutions of the Helmholtz Eq. (C3) in the form

$$\beta(\mathbf{r}) = j_n(ak_0 r)\, Y_{nm}(\hat{r}), \tag{C5}$$

where $Y_{nm}(\hat{r}) \equiv Y_{nm}(\theta,\varphi)$ are spherical harmonics. To find parameter a (and thus the eigenvalues λ_n) we plug Eq. (C5) into the integral Eq. (13) and use the following identity [55]:

$$\frac{\exp(ik_0|\mathbf{r}-\mathbf{r}'|)}{k_0|\mathbf{r}-\mathbf{r}'|} = 4\pi i \sum_{k=0}^{\infty}\sum_{s=-k}^{k} Y_{ks}(\hat{r})Y_{ks}^*(\hat{r}') \times \begin{cases} j_k(k_0 r')h_k^{(1)}(k_0 r), & r > r' \\ j_k(k_0 r)h_k^{(1)}(k_0 r'), & r \leqslant r' \end{cases}, \tag{C6}$$

where $\hat{r}$ and $\hat{r}'$ are unit vectors in the directions of $\mathbf{r}$ and $\mathbf{r}'$, respectively, $j_k(z)$ and $h_k^{(1)}(z)$ are the spherical Bessel functions. Asymptotics of the spherical Bessel functions are

$$j_k(z) \approx \frac{\sqrt{\pi}}{(2k+1)\Gamma(k+1/2)}\left(\frac{z}{2}\right)^k, \quad h_k^{(1)}(z) \approx -i\frac{\Gamma(k+1/2)}{2\sqrt{\pi}}\left(\frac{2}{z}\right)^{k+1}, \quad z \to 0. \tag{C7}$$

If we multiply both sides of Eq. (C6) by k_0 and then take the limit $k_0 \to 0$ we obtain a familiar expansion for the Coulomb potential

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = 4\pi \sum_{k=0}^{\infty}\sum_{s=-k}^{k} Y_{ks}(\hat{r})Y_{ks}^*(\hat{r}') \times \frac{1}{(2k+1)}\begin{cases} r'^k r^{-(k+1)}, & r > r' \\ r^k r'^{-(k+1)}, & r \leqslant r' \end{cases}. \tag{C8}$$

Taking into account Eqs. (C6) and (C5) we find

$$4\pi\gamma\frac{N}{V}\int d\mathbf{r}'\, j_n(ak_0 r')Y_{nm}(\hat{r}') \times \sum_{k=0}^{\infty}\sum_{s=-k}^{k} Y_{ks}(\hat{r})Y_{ks}^*(\hat{r}')\begin{cases} j_k(k_0 r')h_k^{(1)}(k_0 r), & r > r' \\ j_k(k_0 r)h_k^{(1)}(k_0 r'), & r \leqslant r' \end{cases} = \lambda_n j_n(ak_0 r)\, Y_{nm}(\hat{r}). \tag{C9}$$

One can perform an integration over $\mathbf{r}'$ directions using the orthogonality condition for spherical harmonics

$$\int d\Omega_{r'} Y_{ls}^*(\hat{r}')Y_{nm}(\hat{r}') = \delta_{nl}\delta_{sm} \tag{C10}$$

which yields

$$4\pi\gamma\frac{N}{V}\int_0^R dr' r'^2 j_n(ak_0 r')\begin{cases} j_n(k_0 r')h_n^{(1)}(k_0 r), & r > r' \\ j_n(k_0 r)h_n^{(1)}(k_0 r'), & r \leqslant r' \end{cases} = \lambda_n j_n(ak_0 r). \tag{C11}$$

Next we introduce $x = k_0 r$ and rewrite Eq. (C11) as

$$\int_0^{k_0 R} dx' x'^2 j_n(ax')\begin{cases} j_n(x')h_n^{(1)}(x), & x > x' \\ j_n(x)h_n^{(1)}(x'), & x \leqslant x' \end{cases} = \tilde{\lambda}_n j_n(ax), \tag{C12}$$

where

$$\tilde{\lambda}_n = \frac{k_0^3 V}{4\pi N}\lambda_n = \frac{k_0^3 R^3}{3N}\lambda_n.$$

Integral in Eq. (C12) can be calculated using

$$\int dx x^2 j_n(ax) j_n(x) = \frac{x^2}{1-a^2}\left[a j_n(x) j_{n-1}(ax) - j_{n-1}(x) j_n(ax)\right], \tag{C13}$$

$$\int dx x^2 j_n(ax) h_n^{(1)}(x) = \frac{x^2}{1-a^2}\left[a h_n^{(1)}(x) j_{n-1}(ax) - h_{n-1}^{(1)}(x) j_n(ax)\right], \tag{C14}$$

and an identity

$$j_n(x)h_{n-1}^{(1)}(x) - h_n^{(1)}(x) j_{n-1}(x) = \frac{i}{x^2}. \tag{C15}$$

This results in

$$\int_0^{k_0 R} dx' x'^2 j_n(ax')\begin{cases} j_n(x')h_n^{(1)}(x), & x > x' \\ j_n(x)h_n^{(1)}(x'), & x \leqslant x' \end{cases} = \tilde{\lambda}_n j_n(ax) + i(k_0 R)^2 \tilde{\lambda}_n j_n(x) \times \left[a h_n^{(1)}(k_0 R) j_{n-1}(ak_0 R) - h_{n-1}^{(1)}(k_0 R) j_n(ak_0 R)\right]. \tag{C16}$$

The integral Eq. (13) is satisfied provided the last term in Eq. (C16) is equal to zero. This yields the following equation for the eigenvalues:

$$a = \frac{j_n(ak_0 R)}{j_{n-1}(ak_0 R)}\frac{h_{n-1}^{(1)}(k_0 R)}{h_n^{(1)}(k_0 R)}. \tag{C17}$$

APPENDIX D: SOLUTION OF EQS. (73)–(75) USING METHOD OF LAPLACE TRANSFORM

Here we solve the coupled integrodifferential equations

$$\frac{\partial}{\partial t}A(t,r) = -\frac{\Gamma}{R}\int_0^R dr' A(t,r') - \frac{\Gamma}{R}\int_r^R dr' B(t,r'), \tag{D1}$$

$$\frac{\partial}{\partial t}B(t,r) = \frac{\Gamma}{R}\int_0^r dr' A(t,r') \tag{D2}$$

with the initial conditions

$$A(0,r) = 1, \quad B(0,r) = 0. \tag{D3}$$

Adding Eqs. (D1) and (D2) yields

$$\frac{\partial}{\partial t}F(t,r) = -\frac{\Gamma}{R}\int_r^R dr' F(t,r'), \tag{D4}$$

where

$$F(t,r) = A(t,r) + B(t,r). \tag{D5}$$

Taking the Laplace transform $\mathcal{L}$ of Eq. (D4) with respect to time, we obtain

$$sQ(s,r) - F(0,r) = -\frac{\Gamma}{R}\int_r^R dr' Q(s,r'), \tag{D6}$$

where $Q(s,r) = \mathcal{L}[F(t,r)]$ and $F(0,r) = 1$. The solution of Eq. (D6) is

$$Q(s,r) = \frac{1}{s}\exp\left[-\frac{\Gamma}{s}\left(1 - \frac{r}{R}\right)\right]. \tag{D7}$$

To find the solution of Eq. (D4) as a function of time we need to take the inverse Laplace transform of Eq. (D7), which gives

$$F(t,r) = J_0(2\sqrt{1 - r/R}\sqrt{\Gamma t}). \tag{D8}$$

Equations (D1), (D2), and (D3) yield $A(t,-r) = A(t,r)$ and $B(t,-r) = -B(t,r)$. Therefore

$$\begin{aligned} A(t,r) &= \frac{1}{2}[F(t,r) + F(t,-r)] \\ &= \frac{1}{2}[J_0(2\sqrt{1 - r/R}\sqrt{\Gamma t}) + J_0(2\sqrt{1 + r/R}\sqrt{\Gamma t})], \end{aligned} \tag{D9}$$

and

$$\begin{aligned} B(t,r) &= \frac{1}{2}[F(t,r) - F(t,-r)] \\ &= \frac{1}{2}[J_0(2\sqrt{1 - r/R}\sqrt{\Gamma t}) - J_0(2\sqrt{1 + r/R}\sqrt{\Gamma t})]. \end{aligned} \tag{D10}$$

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