Stable optical vortex solitons in pair plasmas

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It is shown that the pair plasmas with small temperature asymmetry can support existence of localized as well as delocalized optical vortex solitons. Coexistence of such solitons is possible due to peculiar form of saturating nonlinearity which has a focusing-defocusing nature—for weak amplitudes being focusing becoming defocusing for higher amplitudes. It is shown that delocalized vortex soliton is stable in entire region of its existence while single- and multicharged localized vortex solitons are unstable for low amplitudes and become stable for relativistic amplitudes.

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I. MODEL

The richness of an electromagnetically active medium is often measured by the variety of structures that it can support. Such structures, in turn, are created because of the nonlinear response of the medium, for instance, to the impact of a large-amplitude electromagnetic wave. Naturally the properties of the structure (e.g., its shape, its content, its stability, and its angular momentum) are dictated by the type of nonlinearity that can arise in the medium. The discovery or identification of a new nonlinearity type, then, opens up a new era of investigation—even discovery.

In this article we work out some of the consequences of a new focusing-defocusing nonlinearity [1] belonging to the general class of saturating nonlinearities (whose magnitude tends to a constant as the wave amplitude becomes large). Saturating nonlinearities seem to appear, *inter alia*, in theories of large-amplitude wave propagation in pair plasmas (plasmas whose main constituents have equal mass and opposite charge [2–4]) in which the pair symmetry is broken by some physical mechanism. For instance, a small amount of Baryonic matter (protons) may break the symmetry of an electron-positron (*e-p*) plasma in the MEV era of the early universe [5–8]. In recently created pair ion (PI) plasmas in the laboratory, a variety of symmetry breaking mechanisms like the small contamination by a much heavier immobile ion, or a small mass difference between the two constituent species, could exist [9–13]. Asymmetries originating in small temperature differences in the constituent species may be always available for structure formation: in the laboratory such a temperature difference could be readily engineered (in a controlled way) and there are reasons to believe that species temperature differences could exist in cosmic and astrophysical settings where one encounters *e-p* plasmas. It is in this latter setting that a new type of nonlinearity

$$F(|A|^2) = \frac{\epsilon^2}{2} \frac{\kappa |A|^2}{(1+\kappa |A|^2)^2} \tag{1}$$

appeared while deriving the wave equation (in parabolic approximation) [1]

$$2i\omega_0 \frac{\partial A}{\partial t} + \frac{(2-\epsilon)}{\omega_0^2} \frac{\partial^2 A}{\partial \xi^2} + \nabla_\perp^2 A + F(|A|^2)A = 0 \tag{2}$$

describing the nonlinear evolution of the vector potential of an electromagnetic (EM) pulse propagating in an arbitrary pair plasma with temperature asymmetry. Following assumptions and notations are necessary in order to put Eqs. (1) and (2) in perspective: A is the slowly varying amplitude of the circularly polarized EM pulse $\sim A\,(\hat{\mathbf{x}}+\hat{\mathbf{y}})\exp(\mathrm{i}k_0 z - \omega_0 t)$ with mean frequency ω_0 and mean wave number k_0; $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the diffraction operator and $\xi = z - v_g t$ is the "comoving" (with group velocity v_g) coordinate.

Equation (2) is written in terms of the dimensionless quantities $A = |e|A/[mG(T_0^-)c^2]$, $r = (\omega_e/c)r$, $t = \omega_e t$; where $\omega_e = (4\pi e^2 n_0/m)^{1/2}$ is the electron Langmuir frequency and m is the electron mass. The charges $q^\pm$ and masses $m^\pm$ of positive and negative ions are assumed to be same (in this article we mainly concentrate on the specific case of pair plasma consisting of electrons and positrons, i.e., $q^+ = e^+ = q^- = -e^- = |e|$ and $m^+ = m^- = m$). The equilibrium state of the system is characterized by an overall charge neutrality $n_0^+ = n_0^- = n_0$, where n_0^+ and n_0^- are the unperturbed number densities of the positive and negative ions, respectively. The background temperatures of plasma species are $T_0^\pm$ ($T_0^+ \neq T_0^-$) and $m\,G(z^\pm) = m\,K_3(z^\pm)/K_2(z^\pm)$ is the "effective mass," [$z^\pm = mc^2/T^\pm$], where K_ν are the modified Bessel functions. For the nonrelativistic temperatures ($T^\pm \ll mc^2$) $G^\pm = 1 + 5T^\pm/2mc^2$ and for the ultrarelativistic temperatures ($T^\pm \gg m_\alpha c^2$) $G^\pm = 4T^\pm/mc^2 \gg 1$. The smallness parameter $\epsilon = [G(T_0^+) - G(T_0^-)]/G(T_0^+)$ measures the temperature asymmetry of plasma species. For the nonrelativistic temperatures $\epsilon = 5(T_0^+ - T_0^-)/2mc^2$ while in ultrarelativistic case $\epsilon = (T_0^+ - T_0^-)/T_0^+$. The numerical factor $\kappa = 1/2$ for nonrelativistic temperatures (=2/3 for ultrarelativistic temperatures). In deriving Eq. (2) with (1), we have assumed that the plasma is transparent ($\omega_0 \gg 1$, $v_g \simeq 1$) and that the longitudinal extent of the pulse is much shorter than its transverse dimensions. However, despite $\partial A/\partial\xi \gg \nabla_\perp A$, the second and the third terms in Eq. (2) can be comparable due to the transparency of the plasma ($\omega_0 \gg 1$).

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With self-evident renormalization the Eq. (2) can be written as:

$$i\frac{\partial A}{\partial t}+\frac{\partial^2 A}{\partial \xi^2}+\nabla_\perp^2 A+f(|A|^2)A=0, \tag{3}$$

where the nonlinearity function is now following [1]:

$$f(|A|^2)=\frac{|A|^2}{(1+|A|^2)^2}, \tag{4}$$

which has an unusual feature:- in the ultrarelativistic limit ($|A|^2 \gg 1$) it tends to be 0.

Note that the nonlinear refraction index for the considered system can be written as $\delta n_{nl}=f(I)$, where $I=|A|^2$ is the intensity of the EM field. The medium is a self-focusing $[d(\delta n_{nl})/dI>0]$ provided $I<1$, while for higher intensities ($I>1$) it becomes defocusing $[d(\delta n_{nl})/dI<0]$. For the localized intense EM pulse with the peak intensity $I_m>1$ the medium becomes defocusing at the peak while remaining focusing at the wings of the EM pulse intensity profile.

In Ref. [1] we have demonstrated that Eq. (3) supports existence of the stable solitonic structures for any spatial dimensions ($D=1,2,3$). Such "light-bullets" exist provided that the amplitude of the solitons is lower than certain critical values [for instance, in 1–dimensional (1D) media $A_m<A_{mcr}\simeq 1.4$]. It is important to emphasize that at $A_m \to A_{mcr}$ the profile of the central part of the soliton flattens and widens at the top. The existence of flat-top soliton can be explained by the peculiarities of our focusing-defocusing nonlinearity, implying that the pulse top part with $A>1$ entered the defocusing region has the tendency of diffraction while the wings of the soliton are in the focusing region to prevent the total spread of the pulse.

II. FORMATION OF VORTICES

In this section we explore the possibility of the formation of two-dimensional stable soliton-structures carrying a screw type of dislocation, i.e., optical vortices. The generation, propagation, and interaction of optical vortices in nonlinear media has been a subject of extensive studies (see for review Ref. [14]). In a self-defocusing medium an optical vortex soliton (OVS) is a (2 + 1)-dimensional (two transverse dimensions and time) stationary beam structure with phase singularity. An OVS is a dark spot, i.e., a zero-intensity center surrounded by a bright infinite background. Self-focusing media also support localized optical vortex soliton solutions (LOVS) with phase dislocation surrounded by a bright ring. In self-focusing medium, LOVS are unstable against symmetry breaking perturbations that lead to the breakup of rings into filaments [15].

Since the new nonlinearity (4) has both focusing-defocusing features, one could expect that both OVS and LOVS solutions are possible in such a medium. This expectation is reinforced by the results of Ref. [16] where a cubic-quintic, sign-changing nonlinearity $[f(|A|^2)=|A|^2-|A|^4]$ was postulated to model a focusing-defocusing medium. In contrast to cubic-quintic model, the focusing-defocusing saturation nonlinearity derived in Ref. [1] has the same sign for all values of $|A|$. In order to investigate further the OVS and LOVS solutions, we assume that the pulse is sufficiently long and effects related to the group velocity dispersion ($\sim\omega_0^{-2}\,\partial^2 A/\partial\xi^2$) can be ignored.

Introducing polar coordinates (r, θ) to describe the x-y plane, we look for solutions of Eq. (3) in the form

$$A=A(r)\exp(i\lambda t+im\theta), \tag{5}$$

where the integer m defines the topological charge of the vortex and λ is the nonlinear frequency shift. The ansatz (5) converts Eq. (3) to the ordinary differential equation

$$\frac{d^2A}{dr^2}+\frac{1}{r}\frac{dA}{dr}-\frac{m^2}{r^2}A-\lambda A+\frac{A^3}{(1+A^2)^2}=0. \tag{6}$$

We have used numerical methods to find localized solutions of (6). It is possible to map this equation in the A-A_r plane (phase plane) and show that it admits both OVS and LOVS solutions. LOVS can exist in the form of an infinite number of discrete bound states $A_n(r)$ ($n=1,2,\ldots$), where the radial quantum number n denotes the finite r zeros of the eigenfunction.

In what follows we consider only the lowest radial eigensolution ($n=1$) of Eq. (6). For nonzero m (the case of interest here), the ground-state LOVS has a positive amplitude and a node at the origin $r=0$, reaches a maximum, and then monotonically decreases as r increases. Such localized solution exists for $\lambda>0$ and display the following asymptotic behavior: $A_{r\to 0}\to r^{|m|}A_0$ and $A_{r\to\infty}\to \exp(-r\sqrt{\lambda})/\sqrt{r}$, where A_0 is a constant which measures the slope of A at origin. OVS solutions have the same behavior for $r\to 0$, while for $r\to\infty$ the amplitude has a nonzero value $A(r)=A_\infty+m^2/[r^2 f'(A_\infty)]$. Here, $\lambda=f(A_\infty)$; the OVS exists only if $f'(A_\infty)<0$, i.e., when the background intensity of the soliton (far beyond the vortex core) is still relativistic $A_\infty>1$. In dimensional units this condition corresponds to the negative slope of the nonlinear refractive index ($d\delta n_{nl}/dI<0$), i.e., in the asymptotic region of the solution the medium is defocusing. It is easy to demonstrate [1] that the constant background field with $A_\infty>1$ is modulationally stable.

A shooting code was used to numerically solve Eq. (6). An analogy with the nonconservative motion of a particle turns out to be useful to better understand the simulation results. Equation (6) may be written as

$$\frac{d}{dr}\left[\left(\frac{dA}{dr}\right)^2+V(A)\right]=\frac{m^2}{r^2}\frac{dA^2}{dr}-\frac{2}{r}\left(\frac{dA}{dr}\right)^2, \tag{7}$$

where the "effective potential" $V(A)=-\lambda A^2+\ln(1+A^2)-A^2/(1+A^2)$. The profile of the potential $V(A)$ for different values of λ is presented in Fig. 1. The potential has two maxima at $A=0$ and $A_{\max}=\sqrt{[1-2\lambda+\sqrt{1-4\lambda}]/2\lambda}$. The bounded solution with $A_{\max}>1$ is possible only for $0<\lambda<0.25$.

The OVS solutions correspond to a particle beginning its motion at the origin ($A=0$) with certain initial A_0 [which can be termed as a velocity (if $m=1$) or an acceleration (if $m=2$ and so on)]; it dissipates its initial energy as it asymptotically approaches the potential maximum at $A_{\max}$. The background intensity of OVS, $A_\infty=A_{\max}$, is always larger than unity and can become arbitrarily large for $\lambda\to 0$. We also found out that OVS solutions exist even for $0.25>\lambda>\lambda_{cr}\simeq 0.2162$, i.e., when $V(A_{\max})<0$ (see curve a in Fig. 1). In other words,

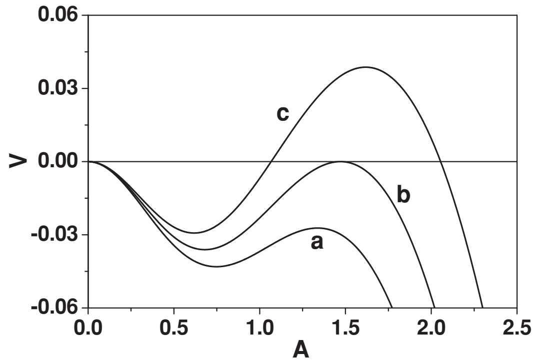

FIG. 1. The "effective potential" versus the amplitude for different values of the nonlinear frequency shift λ. Curve *a* corresponds to $\lambda > \lambda_{\rm cr} \simeq 0.2162$, curve *b* has $\lambda = \lambda_{\rm cr}$ [the value of λ for which $V(A_{\max}) = 0$], and for curve *c* $0 < \lambda < \lambda_{\rm cr}$.

the effective particle cannot cross but only asymptotically approach the lower potential maximum.

The numerical solutions of the nonlinear Eq. (6) for $m = 1,2$, and 3 are shown in Fig. 2. As expected, the soliton-like solutions go to zero as r^m for small r and reach an m-independent asymptotic value predicted earlier. In Fig. 3, curve *a* displays the dependence of the field derivative at the origin, A_0, as a function of the nonlinear frequency shift λ for $m = 1$ vortex. One can see that A_0 grows rapidly as λ goes to zero. For small λ, the position of the potential maximum "moves" to larger values of A and, consequently, "particle" needs to have larger initial "velocity" (A_0) to reach the maximum.

In contrast to OVS, the LOVS solutions correspond to the particle returning back asymptotically to the initial position at $A = 0$. It seems obvious that due to "friction" the "particle" cannot make its way back if $\lambda > \lambda_{\rm cr}$. Thus, LOVS may exist in the range $0 < \lambda < \lambda_{\rm cr}$ while its amplitude (in contrast to OVS) is a growing function of λ. Such behavior, calculated numerically, is presented in Fig. 4 for singly charged vortices ($m = 1$). One can see that the amplitude of the LOVS (A_m) is bounded from above by certain critical value for $A_{\rm cr}(\simeq 1.5)$. Thus, in contrast to OVS the localized vortex can be just moderately relativistic. Note that for $0.16 \leqslant \lambda \leqslant \lambda_{\rm cr}$ the amplitude of the LOVS (A_m) varies in the range $1 \leqslant A_m \leqslant A_{\rm cr}$. For the top part of such a solution [with $A(r) > 1$] the medium is defocusing while it tends to focus the lower intensity wings of the structure. This phenomenon is illustrated in Fig. 5 where LOVS profiles are plotted for a variety of λs. With

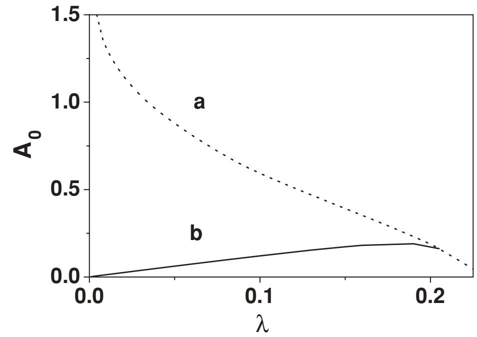

FIG. 3. A_0 versus λ for $m = 1$; curve *a* corresponds to OVS while curve *b* to LOVS.

increasing λ, the central part of the LOVS flattens and widens overlapping more and more with the OVS. In principle, it is possible to create flat-top LOVS with a large transverse width. Convergence of LOVS to OVS can also be inferred from Fig. 3 where curve *b*, corresponding to LOVS, almost coincides with curve *a* near the point $\lambda \approx \lambda_{\rm cr}$. Similar behavior of the solutions can be obtained for vortices with higher charge ($m = 2,3,\ldots$); corresponding figures are not displayed here.

III. STABILITY

The question arises concerning whether these solitonlike solutions are stable. The intensity dependent switching from the focusing to defocusing regime can have an interesting consequence for the stability properties of the solutions [17]. As is well established [14], OVSs with $m = 1$ are stable, whereas the ones with a larger value of m may decay into $m = 1$ vortices in self-defocusing media. For the particular system discussed in this article, the bulk of the OVS is always in the defocusing regime and, as mentioned earlier, the background field is always stable. However, near the vortex core the medium becomes focusing. Thus, the overall stability of an OVS for this mixed nonlinearity can not be guaranteed.

The soliton stability was examined by numerically solving Eq. (3). In the simulations (for various λs), the initial stationary OVS state was perturbed radially and azimuthally by Gaussian noise. Typical evolution of the perturbation is plotted is Fig. 6. We see that perturbations are quickly radiated away and the initial state relaxes to the ground-state solution; the OVS was found to be stable in the entire examined range.

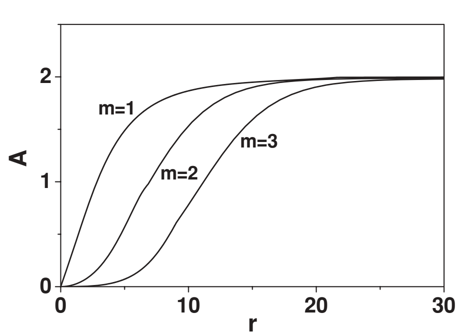

FIG. 2. Profiles of OVSs for $m = 1$, $m = 2$, $m = 3$; nonlinear frequency shift $\lambda = 0.16$.

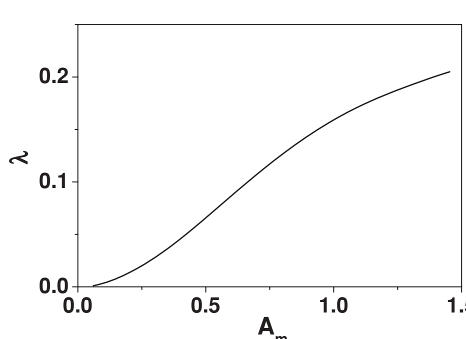

FIG. 4. The effective eigenvalue λ versus soliton amplitude A_m for $m = 1$.

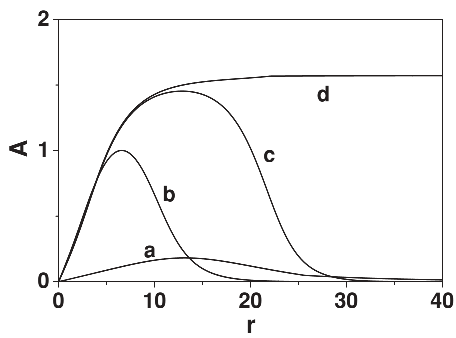

FIG. 5. Profiles of soliton solutions. Curves *a*, *b*, and *c* correspond to LOVS with $\lambda = 0.005, 0.16, 0.205$, respectively. Curve *d* corresponds to OVS for $\lambda = 0.205$.

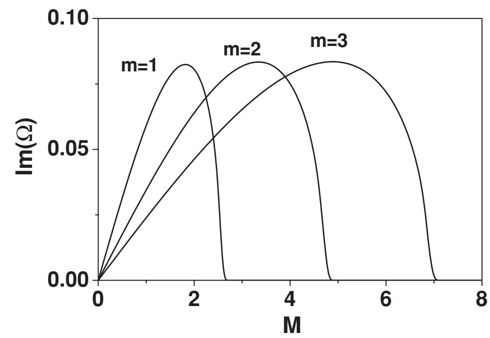

FIG. 7. Instability growth rate Im(Ω) versus M for $\lambda = 0.1$ for different topological charges m.

The stability of LOVS was conducted by following the linear stability procedure developed by Refs. [18,19] in which one considers perturbations acting along a ring of mean radius r_*, where $A(r_*) = A_m$. Assuming constant intensity and spatial uniformity for this ring, one can rewrite the diffraction operator in (3) as $\nabla_\perp^2 = r_*^{-2}\partial^2/\partial\theta^2$, and introducing azimuthal perturbation with a phase factor $\Psi = \Omega t + M\theta$ (where M is an integer) for the growth rate of instability, we derive:

$$\mathrm{Im}(\Omega) = \frac{M}{r_*}\,\mathrm{Re}\left[\frac{2\left(1 - A_m^2\right)}{\left(1 + A_m^2\right)^3} - \frac{M^2}{r_*^2}\right]. \quad (8)$$

One can see from (8) that large-amplitude LOVS with $A_m > 1$ is always stable. For the lower-amplitude case LOVS should decay into $M_{\max}$ multiple filaments, where $M_{\max}$ is an integer close to the number for which maximal growth rate is maximal.

In Fig. 7 we plot Im(Ω) versus M for $\lambda = 0.1$ and for $m = 1,2,3$. the corresponding A_m are, respectively, 0.66, 0.65,0.63 and $r_* = 6.3, 11.6, 16.9$. One should expect that instability will split the pulse into filaments (fragments) with number of filaments being, respectively, 2, 4, and 5 (or 6) for $m = 1,2,3$. As it is well known [20] the vortices have a topological sense as the branch points, where both the real and imaginary parts of the field become strictly zero, and the topological charge represents the number of intersecting pairs of zero lines of the real and imaginary parts of the field A. The circulation of the field's phase gradient is conserved along the closed path which encloses the branch point. As a consequence the vortex nested in the EM beam cannot disappear even when the EM beam undergoes a structural change. Hence, the fusion of filaments is forbidden for topological reasons. On the other hand, these filaments must conserve total angular momentum; they can eventually spiral about each other or fly off tangentially to the initial ring, generating the bright solitonic structures found, for instance, for index saturation nonlinearity [15].

Our numerical simulations for $A_m < 1$ give evidence of a quickly developing instability in agreement with predictions of linear stability analysis. Indeed, we learn from Fig. 8 that the LOVS with $m = 1$ ($m = 2$) break up into 2 (4) filaments. The filaments are running away tangentially without spiraling. All

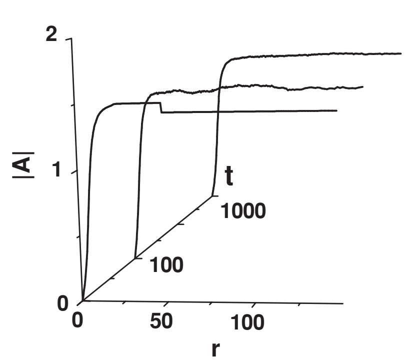

FIG. 6. The dynamics of initially perturbed OVSs; plots are chosen for different times, $t = 0,100,1000$.

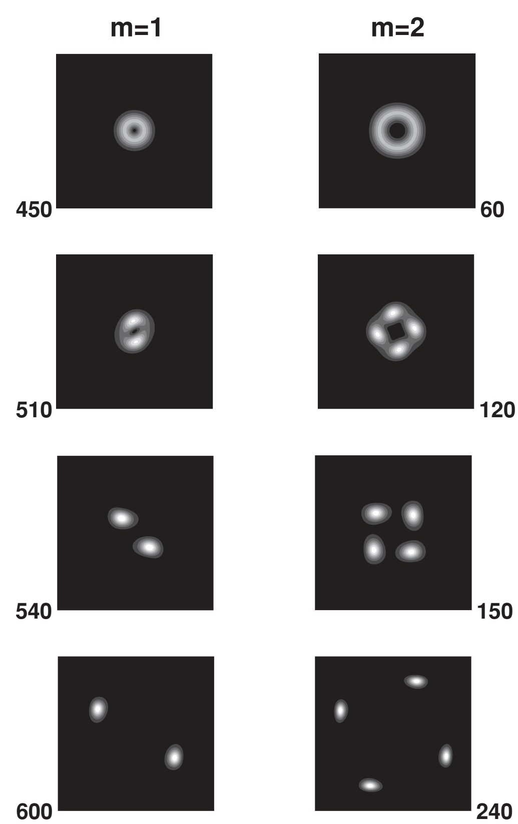

FIG. 8. Vortex dynamics (for different times) when $\lambda = 0.1$: (left) for $m = 1$, $A_{\max} = 0.66$, the vortex splits into two filaments; (right panel)- for $m = 2$, $A_{\max} = 0.6580$, the vortex splits into four filaments; the filaments run away tangentially.

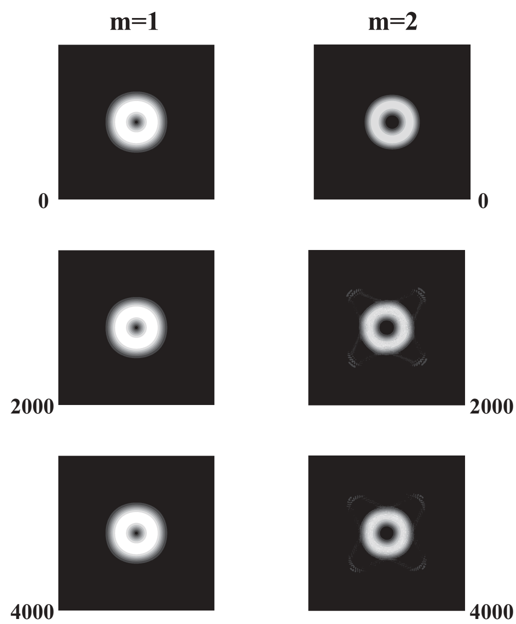

FIG. 9. Vortex dynamics (for different times) when $\lambda = 0.2$, the vortex is robust toward perturbations; (left) $m = 1$, $A_{\max} = 1.386$; (right) $m = 2, A_m = 1.3729$.

filament like spatial solitons remain stable. Most interesting is the situation where the amplitude of LOVS is larger than unity.

In Fig. 9 we again present the evolution of higher-intensity LOVS both for $m = 1$ and $m = 2$: the corresponding amplitudes for the soliton solutions are ($\lambda = 0.2$) $A_m = 1.39$ and $A_m = 1.37$, respectively. The initial input LOVS solution was modulated by a Gaussian noise (the level of noise was 5%). One can see that the LOVS maintain their fidelity; no breaking takes place. To ensure against very slow instabilities, the simulations were carried out for long time periods, $t = 4000$, i.e., for 130 soliton periods, $T_{\rm sol}$ ($=2\pi/\lambda \approx 30$). Thus, single- and multicharged large-amplitude LOVS are also demonstrated to be stable.

A word of caution should be added: Although we are confident that the single-charged LOVS are stable, one cannot be sure of the stability of multicharged LOVS as well as OVS. Indeed, from general topological reasons the multicharged vortices are supposed to be unstable and they should break into single-charged vortices. However, in our preliminary simulations we failed to detect such breaking for large-amplitude structures. It is conceivable that these structures will eventually breakup under the influence of some very slow process (subexponential or algebraic) and our simulations have unearthed only a very long-lived phase of a practically stable multicharged vortex.

The effects related to the group velocity dispersion, and the corresponding reshaping of the radiation, have been ignored in this work. Generalization of the results by keeping the term $\sim \partial^2 A/\partial \xi^2$ in Eq. (3) is quite possible [18]. In a transparent plasma, this term can affect the long time dynamics of self-guiding vortex solitons. In particular, due to weak modulation instability [21], the self-trapped beam eventually will break into a train of spatiotemporal solitons, i.e., the "light bullets" [22]. Topological reasons, however, will let the vortex line survive structural changes. We expect that "instability" will result in generation of fully localized bullets of vortex solitons (the spinning bullets). Dynamics of formation and stability of such structures is beyond the scope of the current article.

IV. CONCLUSIONS

The asymmetries originating in small temperature differences in the constituent species of an electromagnetically active medium may be always available for structure formation both in laboratory and cosmic or astrophysical settings. In present article we have shown that this asymmetry, mother to a new type of the nonlinearity (derived in Ref. [1]), imparts specific properties to the sustained structures.

We found that the pair plasmas, through the new focusing-defocusing nonlinearity generated by an "asymmetry" in initial temperatures, can support multidimensional stable large-amplitude optical vortex and localized vortex solitons. Localized structures for certain parameters may have the flat-top shapes. The coexistence of LOVS and OVS solutions and their stability in such medium is due to the specific form of saturating nonlinearity switching from the self-focusing to the self-defocusing regime and vice versa. The consequences of this nonlinearity can be, perhaps, observed in a variety of situations, in particular in the laboratory settings.

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[1] S. M. Mahajan, N. L. Shatashvili, and V. I. Berezhiani, Phys. Rev. E **80**, 066404 (2009).

[2] P. A. Sturrock, Astrophys. J. **164**, 529 (1971); M. A. Ruderman and P. G. Sutherland, *ibid.* **196**, 51 (1995); F. C. Michel, *Theory of Neutron Star Magnetospheres* (University of Chicago Press, Chicago, 1991).

[3] M. C. Begelman, R. D. Blandford, and M. J. Rees, Rev. Mod. Phys. **56**, 255 (1984).

[4] T. E. Cowan *et al.*, Laser Part. Beams **17**, 773 (1999); C. Gahn *et al.*, Appl. Phys. Lett. **77**, 2662 (2000); H. Chen, S. C. Wilks, J. D. Bonlie, E. P. Liang, J. Myatt, D. F. Price, D. D. Meyerhofer, and P. Beiersdorfer, Phys. Rev. Lett. **102**, 105001 (2009).

[5] S. Wineberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relatvity* (John Wiley & Sons, New York, 1972).

[6] M. E. Gedalin, J. G. Lominadze, L. Stenflo, and V. N. Tsitovich, Astrophys. Space Sci. **108**, 393 (1985); P. K. Shukla, N. N. Rao, M. Y. Yu, and N. L. Tsintsadze, Phys. Rep. **138**, 1 (1986).
[7] V. I. Berezhiani and S. M. Mahajan, Phys. Rev. Lett. **73**, 1110 (1994); Phys. Rev. E **52**, 1968 (1995).
[8] T. Cattaert, I. Kourakis, and P. K. Shukla, Phys. Plasmas **12**, 012310 (2005); P. K. Shukla and M. Khan, *ibid.* **12**, 014504 (2005); T. Tatsuno, M. Ohhashi, V. I. Berezhiani, and S. V. Mikeladze, Phys. Lett. A **363**, 225 (2007).
[9] W. Oohara and R. Hatakeyama, Phys. Rev. Lett. **91**, 205005 (2003); W. Oohara, D. Date, and R. Hatakeyama, *ibid.* **95**, 175003 (2005).
[10] W. Oohara, Y. Kuwabara, and R. Hatakeyama, Phys. Rev. E **75**, 056403 (2007).
[11] W. Oohara and R. Hatakeyama, Phys. Plasmas **14**, 055704 (2007).
[12] S. M. Mahajan and N. L. Shatashvili, Phys. Plasmas **15**, 100701 (2008); A. Esfandyari-Kalejahi, I. Kourakis, M. Mehdipoor, and P. K. Shukla, J. Phys. A: Math. Gen. **39**, 13817 (2006).
[13] N. L. Shatashvili, J. I. Javakhishvili, and H. Kaya, Astrophys. Space Sci. **250**, 109 (1997); N. L. Shatashvili and N. Rao, Phys. Plasmas **6**, 66 (1999).
[14] A. Desyatnikov, L. Torner, and Y. S. Kivshar, Prog. Opt. **47**, 291 (2005).
[15] W. J. Firth and D. V. Skryabin, Phys. Rev. Lett. **79**, 2450 (1997); D. V. Skryabin and W. J. Firth, Phys. Rev. E **58**, 3916 (1998).
[16] V. Skarka, V. I. Berezhiani, and R. Miklaszewski, Phys. Rev. E **56**, 1080 (1997); V. I. Berezhiani, V. Skarka, and N. B. Aleksic, *ibid.* **64**, 057601 (2001).
[17] M. Quigora-Teixeiro and H. Michinel, J. Opt. Soc. Am. B **14**, 2004 (1997).
[18] A. Vincotte and L. Berge, Phys. Rev. Lett. **95**, 193901 (2005).
[19] J. Atai, Y. Chen, J. M. Soto-Crespo, Phys. Rev. A **49**, R3170 (1994); A. Vincotte and L. Berge, Physica D: Nonlinear Phenomena **223**, 163 (2006).
[20] J. F. Nye and M. V. Berry, Proc. R. Soc. London A **336**, 165 (1974).
[21] N. A. Akhmediev and J. M. Soto-Crespo, Phys. Rev. A **47**, 1358 (1993); S. M. Mahajan, V. I. Berezhiani, and R. Miklaszewki, Phys. Plasmas **5**, 3264 (1998).
[22] B. A. Malomed, D. Mihalache, F. Wise, and L. Torner, J. Opt. B **7**, R53 (2005).