Solitons in curved space of constant curvature

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We consider spatial solitons as, for example, self-confined optical beams in spaces of constant curvature, which are a natural generalization of flat space. Due to the symmetries of these spaces we are able to define respective dynamical parameters, for example, velocity and position. For positively curved space we find stable multiple-hump solitons as a continuation from the linear modes. In the case of negatively curved space we show that no localized solution exists and a bright soliton will always decay through a nonlinear tunneling process.

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I. INTRODUCTION

A remarkable effect in nonlinear systems such as Bose-Einstein condensates or optical fields, which interact with matter [1–3], is the formation of self-localized states conveniently termed solitons. Although of completely different origin, they are often described by the same evolution equation, called the Gross-Pitaevskii equation [1] for Bose-Einstein condensates or the nonlinear Schrödinger equation in optics [3,4]. One key feature of solitons is that they behave like particles. They can be characterized by constants of motion such as momentum, center of mass, and energy, which arise from the symmetries of flat space. Each symmetry introduces one parameter of the soliton. Galilean invariance, for example, gives rise to the velocity of the soliton. Owing to these symmetries we can study collisions of *similar* solitons. Unfortunately this nice picture breaks down when one considers nonlinear solutions in an external potential as in the case of Bose-Einstein condensates, which are naturally confined in an external potential [1]. This potential breaks, in general, the translational invariance of flat space. Hence the standard notion of solitons is lost. This leads to the question, Is there a way to introduce a nontrivial external potential without breaking translational invariance? This is in fact possible in some cases. One considers the nonlinear Schrödinger equation with inhomogeneous coefficients and then transforms it to the standard nonlinear Schrödinger equation with constant coefficients [5,6]. In this paper we present a geometric approach to maintain translational invariance.

In fact curved space can play the role of an effective potential. In cosmology, space is curved in general, but the cosmological principle states that space is homogeneous and isotropic. Thus translational invariance is not broken [7], and an internal observer, like us in our universe, cannot physically distinguish two different points by any measurement. In this sense, these spaces are a natural generalization of the standard flat space. As we will see, although translational invariance is not broken, the evolution of fields in these spaces is almost equivalent to the evolution in an effective external potential. Therefore we are able to introduce a nontrivial potential and maintain translational invariance and thus the notion of solitons.

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II. SPACES OF CONSTANT CURVATURE

Homogeneous and isotropic spaces have a constant curvature [7]. If we neglect topology, there are three spaces of constant curvature in cosmology. They differ by the sign of the scalar curvature K. The three-dimensional metric of these spaces can be written in spherical coordinates as [7]

$$ds^2 = \frac{dr^2}{1-Kr^2} + r^2(\sin^2\theta d\xi^2 + d\theta^2). \tag{1}$$

Unfortunately there is no practical way to study three-dimensional curved space in the laboratory. Therefore we restrict ourselves to the two-dimensional case $\theta = \pi/2$. However, this restriction is not too strong, since the nontrivial effects of curvature result mainly from the radial dependence.

Now for $K = 0$ we find the usual two-dimensional flat space in polar coordinates. For $K \neq 0$ we define the new coordinate η through $r = R\cos_\sigma \eta$ for $K = \sigma/R^2$ and $\sigma = \mathrm{sgn}(K)$. For convenience we defined $\cos_1 := \cos$ and $\cos_{-1} := \cosh$. Therefore we find the two metrics,

$$ds_\sigma^2 = R^2\left(d\eta^2 + \cos_\sigma^2 \eta d\xi^2\right), \quad \sigma = \pm 1, \tag{2}$$

for the two cases $K > 0$ and $K < 0$. The coordinate ranges are $\xi \in [0,2\pi)$ and $\eta \in (-\pi/2,\pi/2)$ for positive and $\eta \in \mathbb{R}$ for negative curvature. Especially, to discuss flat space as a limiting case ($R \to \infty$), it is also useful to define the real lengths $x = \eta R$ and $z = \xi R$.

Here we consider the solitonic solution of light in spaces of constant, positive and negative, curvature. Experimentally this could be achieved by a nonlinear waveguide, which is attached to the surface of a three-dimensional body and therefore models the corresponding curved spaces (see Fig. 1). For a detailed discussion of these surfaces and the ideas, see [8] and [9]. Therefore a curved waveguide on the surface of these bodies is able to reproduce the metric in (2).

However, the presented results are universal in the sense that they are valid for any self-interacting scalar field and do not depend on the particular experimental realization.

Now the evolution of a nonlinear scalar field ψ, in particular light in a fixed polarization state, can be modeled by the nonlinear Helmholtz equation [8],

$$-\Delta_\sigma \psi = \beta^2(1+\gamma_0|\psi|^2)\psi, \quad \sigma = \pm 1. \tag{3}$$

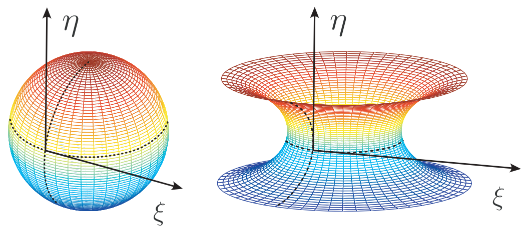

FIG. 1. (Color online) Illustration of the considered spaces and coordinates. (a) An example of a positively curved space is the sphere. (b) A negatively curved space.

Here $\beta = kR$, $k = n\omega_0/c$ is the wave number, n the refractive index of the wave guiding layer, and ω_0 the frequency. The operator

$$\Delta_\sigma = \partial_\eta^2 - \sigma \tan_\sigma \eta \partial_\eta + \frac{\partial_\xi^2}{\cos_\sigma^2 \eta}, \tag{4}$$

is the generalized Laplace operator [7,10] with respect to the metric, (2). Note that there is no coordinate transformation for $K \neq 0$ that transforms Eq. (3) into one with constant coefficients, that is, to the flat case $K = 0$. This reflects the fact that spaces with different signs of curvature or curvature 0 are not isometric [9,10].

III. SCHRÖDINGER AND FLAT SPACE LIMIT

For the physical interpretation of Eq. (3) it is suggested to study the corresponding nonlinear Schrödinger evolution equation. We consider propagation in the ξ direction and make the ansatz

$$\psi(\eta,\xi) = (\cos_\sigma \eta)^{-1/2} U_0 U(\eta,\xi) \exp(i\beta\xi).$$

As in flat space the assumption of propagation in the ξ direction is no restriction, since we know that no direction is preferred. Similarly to the flat case [4] there are different length scales involved in the problem. The diffraction length $\ell_d = kw_0^2$, with w_0 the typical size of the soliton solution, the nonlinear length scale $\ell_{\rm nl} = 2/(|\gamma_0 U_0^2|k)$, associated with the peak amplitude U_0, and, in addition, the curvature scale R. Due to the additional curvature length scale we are able to define the dimensionless parameters $\omega = \ell_{\rm d}/R$, $\delta = 1/(kR)$ and the geometric mean $\epsilon = \sqrt{\omega\delta} \ll 1$. Furthermore, we define the scaled coordinates X and Z through $\eta = \epsilon X$ and $\xi = \omega Z$. Within the paraxial approximation we obtain, for the lowest order in ϵ, the nonlinear Schrödinger equation:

$$i\frac{\partial U}{\partial Z} = -\frac{1}{2}\frac{\partial^2 U}{\partial X^2} + \frac{1}{2}\sigma\omega^2 X^2 U \pm N^2 |U|^2 U, \tag{5}$$

with an attractive ($\sigma = 1$) or repulsive ($\sigma = -1$) harmonic potential and $N^2 = \ell_{\rm d}/\ell_{\rm nl}$. We find that the flat space limit is obtained not only for $R \to \infty$, but also for $\ell_{\rm d},\ell_{\rm nl} \to 0$, by keeping N^2 constant. Therefore the high-amplitude limit of Eq. (5) has to show properties of an integrable model [3]. In general, for finite R, integrability is not present and therefore solitons may become unstable and will occasionally not survive collisions. In these cases they are better termed solitary waves. Before we analyze the stationary solutions of Eq. (3), we briefly discuss the symmetry transformations of spaces of constant curvature.

IV. SYMMETRIES

Spaces of constant curvatures are spaces of maximal symmetry and therefore translational invariance is present [10,11]. Consider the metric (2) or, equivalently, the nonlinear Helmholtz equation (3) to be invariant under the coordinate transformation $\eta = \eta(\eta',\xi')$ and $\xi = \xi(\eta',\xi')$. An immediate consequence of this symmetry is that, if $\psi(\eta,\xi)$ is a solution of (3), so is $\psi'(\eta',\xi') = \psi[\eta(\eta',\xi'),\xi(\eta',\xi')]$. In particular, it can be shown that the coordinate transformation

$$\begin{aligned} \sin_\sigma \eta' &= \frac{\sin_\sigma \eta + v_0 \cos_\sigma \eta \sin_\sigma \xi}{\sqrt{1+v_0^2}}, \\ \sin_\sigma \xi' &= \frac{\cos_\sigma \eta \sin_\sigma \xi - v_0 \sin_\sigma \eta}{\sqrt{1+v_0^2 - \sigma(\sin_\sigma \eta + v_0 \cos_\sigma \eta \sin_\sigma \xi)^2}}, \end{aligned} \tag{6}$$

is a symmetry of the metric in (2). The parameter v_0 is associated with a shift in the velocity of a given solution, since we have

$$\left.\frac{d\eta'}{d\xi'}\right|_{\eta,\xi=0} = \frac{\left.\frac{d\eta}{d\xi}\right|_{\eta,\xi=0} + v_0}{1 - v_0 \left.\frac{d\eta}{d\xi}\right|_{\eta,\xi=0}}, \tag{7}$$

the addition theorem for velocities. We obtain the usual Galilean addition of velocities for sufficiently small values of v_0. This reflects the fact that, within the paraxial approximation, only small velocities or angles with respect to the propagation direction are allowed [12,13]. A translation is associated with the transformation

$$\begin{aligned} \sin_\sigma \eta' &= \cos_\sigma \eta_0 \sin_\sigma \eta - \sin_\sigma \eta_0 \cos_\sigma \eta \cos_\sigma \xi, \\ \cos_\sigma \xi' &= \frac{\cos_\sigma \eta_0 \cos_\sigma \eta \cos_\sigma \xi + \sigma \sin_\sigma \eta_0 \sin_\sigma \eta}{\sqrt{1 - \sigma(\cos_\sigma \eta_0 \sin_\sigma \eta - \sin_\sigma \eta_0 \cos_\sigma \eta \cos_\sigma \xi)^2}}, \end{aligned} \tag{8}$$

where η_0 is the initial shift or translation, since we have $\eta' = \eta - \eta_0$ and $\xi' = 0$ for $\xi = 0$. The third symmetry transformation corresponds to the simple translation $\xi' = \xi + \xi_0$. Therefore for every solution we find a three- parametric family of solutions. This is in total analogy with the usual flat case. In the following we only need the infinitesimal version of the preceding symmetries. These are given by

$$\begin{aligned} \eta' &= \eta + v_0 \sin_\sigma \xi, \\ \xi' &= \xi - v_0 \tan_\sigma \eta \cos_\sigma \xi, \end{aligned} \tag{9}$$

for an infinitesimal velocity shift $v_0 \ll 1$ and

$$\begin{aligned} \eta' &= \eta - \eta_0 \cos_\sigma \xi, \\ \xi' &= \xi - \sigma\eta_0 \tan_\sigma \eta \sin_\sigma \xi, \end{aligned} \tag{10}$$

for a translation with $\eta_0 \ll 1$. From these transformations we are also able to identify the meaning of the parameters v_0 and η_0 as a shift in the velocity or position of a given solution.

V. NONPARAXIAL SOLITONS

In flat space the slowly varying envelope approximation of (3) describes the evolution of the system well. However, in spaces of nonzero constant curvature a paraxial approximation breaks the translational symmetry, which can have serious effects on the stability properties. Therefore, here we do not make use of the slowly varying envelope approximation. In contrast, we are looking for soliton solutions of the nonlinear Helmholtz equation (3). Nevertheless, as in the flat case, we are able to interpret the ξ direction as the propagation direction [12,13]. Similarly to the flat case, we now assume propagation in the ξ direction. To simplify Eq. (3) we transform the field, as

$$\psi(\eta,\xi) = (\beta^2 \cos_\sigma \eta)^{-1/2} u(\eta,\xi),$$

and look for stationary solutions of the newly introduced field $u(\eta,\xi)$ such as $u(\eta,\xi) = u_0(\eta)\exp(im\xi)$, where m is the propagation constant of the soliton and $\beta = kR$. With this ansatz we find

$$-\partial_\eta^2 u_0 + V(\eta)u_0 - \gamma(\eta)|u_0|^2 u_0 = \left(\beta^2 + \frac{\sigma}{4}\right) u_0, \qquad (11)$$

where $\gamma(\eta) = \gamma_0/\cos_\sigma \eta$. Although translational invariance is not broken, Eq. (11) is similar to one describing propagation in flat space, but in the presence of an attracting ($\sigma = 1$) or repulsive ($\sigma = -1$) potential

$$V(\eta) = \frac{m^2 - \frac{\sigma}{4}}{\cos_\sigma^2 \eta}. \qquad (12)$$

Therefore the exact potential is not harmonic as found in (5).

A. Positively curved space

We start with the linear or low-power case ($\gamma_0 \to 0$), where the stationary solutions of (3) are rescaled spherical functions as $u(\eta,\xi) = (\cos\eta)^{1/2} Y_l^m(\eta + \pi/2,\xi)$ [14], which we denote $|l,m\rangle$. Hence, for a given frequency we find $2l + 1$ localized solutions with $l \in \mathbb{N}$, $m \in \mathbb{Z}$, $|m| \leqslant l$, and $\beta^2 = l(l+1)$. As shown in Figs. 2(a) and 2(b) these linear modes branch off, as a nonlinear continuation, in $2l + 1$ nonlinear modes. In the focusing case each maximum of the linear mode evolves into a separate hump, which all together form a lattice of $l - m + 1$ bright solitons. Bright solitons of opposite phase, which would repel each other in flat space, are bound together by the focusing effect of positive curvature. As discussed, using the symmetries of these spaces we are also able to introduce a velocity and a position of the solitonic solution and can study collisions (see Fig. 3).

In contrast, in the defocusing case the excitation tends to spread. Phase jumps of the linear solutions are transformed into $l - m$ dark solitons which are embedded in a Thomas-Fermi sea. This background, as shown in Fig. 2(b), is well explained by the Thomas-Fermi approximation

$$\gamma_0|u_0(\eta)|^2 \sim \left(\beta^2 + \frac{1}{4}\right)\cos\eta - \frac{m^2 - \frac{1}{4}}{\cos\eta}, \qquad (13)$$

for $|\eta| \leqslant \arccos[(m^2 - 1/4)/(\beta^2 + 1/4)]$.

To analyze the linear stability of the multihump solitons we perturb a given stationary solution $u_0(\eta)$ with propagation constant m by a small ($\epsilon \ll 1$) linear fluctuation, that is, $u(\eta,\xi) = [u_0(\eta) - \epsilon\chi(\eta,\xi)]\exp(im\xi)$. By linearizing (3), in the case of positive curvature, we obtain the Hamilton system $\partial_\xi \mathbf{\Psi} = J\mathcal{H}\mathbf{\Psi}$, with

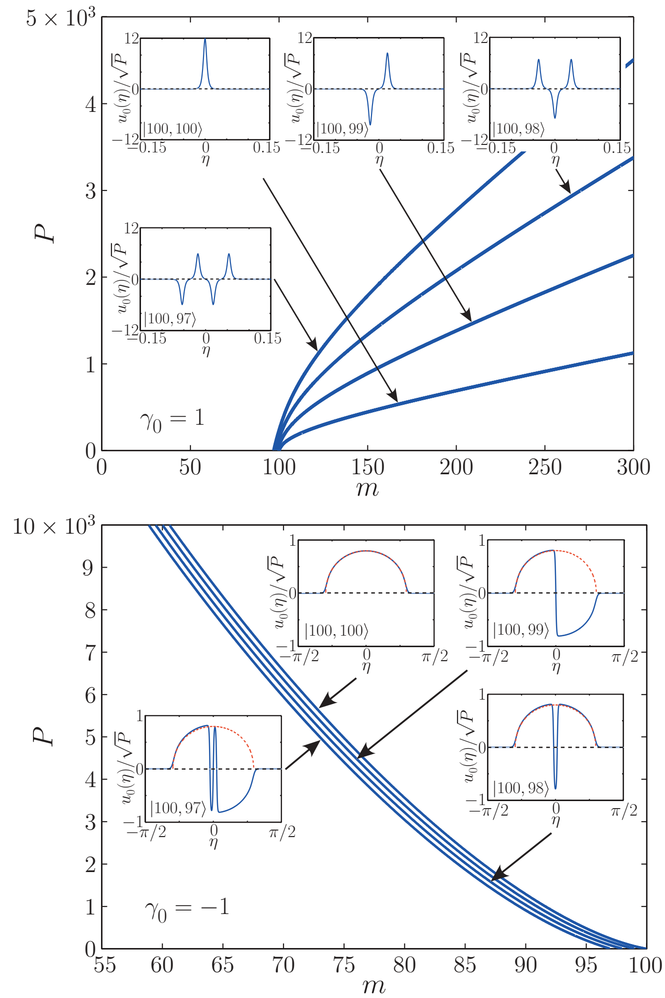

FIG. 2. (Color online) Branching of the first linear modes $|100,m\rangle$ (a) for the focusing and (b) for the defocusing case. Numerically calculated solitons are displayed in the insets for $m = 300$ (a) and $m = 60$ (b). The dashed (red) curve in (b) shows the Thomas-Fermi approximation of $|u_0(\eta)|$. The power $P = 1/m \int \mathrm{Im}(u^*\partial_\xi u)/(\cos^2\eta) d\eta$ is conserved, that is, $dP/d\xi = 0$.

$$\mathcal{H} = \begin{pmatrix} L & -\gamma(\eta)u_0^2 & im & 0 \\ -\gamma(\eta)u_0^{*2} & L & 0 & -im \\ -im & 0 & -\cos^2\eta & 0 \\ 0 & im & 0 & -\cos^2\eta \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix},$$

$$L = -\partial_\eta^2 - \beta^2 - \frac{1}{4} - \frac{1}{4\cos^2\eta} - 2\gamma(\eta)|u_0|^2,$$

and $\mathbf{\Psi} = (\chi, \chi^*, \Pi, \Pi^*)^T$. The canonical momentum is found to be $\Pi = -\cos^{-2}\eta(\partial_\xi\chi + im\chi)$. Now, setting $\mathbf{\Psi}(\eta,\xi) = \mathbf{\Phi}(\eta)\exp(i\Omega\xi)$, we find the linear eigenvalue problem

$$i\Omega\mathbf{\Phi} = J\mathcal{H}\mathbf{\Phi}. \qquad (14)$$

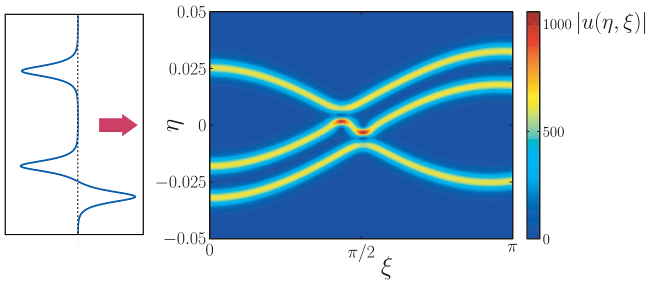

FIG. 3. (Color online) Collision between translated one-hump ($\eta_0 = 0.025, v_0 = 0$) and two-hump ($\eta_0 = -0.025, v_0 = 0$) solitons [see Fig. 2(a)], with $l = 10^4$ and $m = l + 10$, for focusing nonlinearity (paraxial propagation). Stationary solutions were translated with the use of the symmetry (8) and, therefore, are solutions as well. Hence the solitons are robust with respect to collisions. The color illustrates $|u(\eta,\xi)|$.

Therefore the soliton is said to be stable if all eigenvalues Ω of (14) are real. It is interesting to look at the zero modes, which are $\Omega = 0$ solutions of (14). In flat space there are two zero modes corresponding to the breaking of translation and phase invariance [15]. In positively curved space, simple flat translation invariance is not present. However, there is still a kind of translational symmetry realized. We consider an infinitesimal symmetry transformation (10) with $\eta_0 = \epsilon \ll 1$. This is given by $\eta \rightarrow \eta - \cos\xi\epsilon$ and $\xi \rightarrow \xi - \sin\xi\tan\eta\epsilon$. Here ϵ is the analog to an infinitesimal shift of the transverse position of the soliton in flat space. Hence, if $u_0(\eta)\exp(im\xi)$ is a solution, this is also the case for $u(\eta,\xi) = [u_0(\eta) - \epsilon\chi_\pm(\eta,\xi)]\exp(im\xi)$, with

$$\chi_\pm(\eta,\xi) = \frac{1}{2}\left[\frac{du_0}{d\eta} \pm \left(m \mp \frac{1}{2}\right)\tan(\eta)u_0(\eta)\right]e^{\pm i\xi}. \quad (15)$$

From this we deduce that in the case of positively curved space, the translation mode $du_0/d\eta$ known from flat space appears as a stable linear mode with an eigenvalue $\Omega = \pm 1$. This corresponds, when expressed as the real lengths x and z, to $\Omega = \pm 1/R$. For $R \rightarrow \infty$ we obtain $\Omega = 0$ the flat case. Note that the symmetry (9) gives no further linear independent modes.

To analyze the stability behavior further we discuss the linear or low-power case ($\gamma_0 = 0$), where the soliton just corresponds to a linear mode with the propagation constant m. For this case we find the eigenfunctions

$$\begin{aligned} \boldsymbol{\Psi}_+ &= \mathbf{(}1,0,-i(m+\Omega)\cos^{-2}\eta,0\mathbf{)}^T|l,m+\Omega\rangle, \\ \boldsymbol{\Psi}_- &= \mathbf{(}0,1,0,i(m-\Omega)\cos^{-2}\eta\mathbf{)}^T|l,m-\Omega\rangle, \end{aligned} \quad (16)$$

with the total spectrum $-(m+l) \leqslant \Omega \leqslant m+l$ and $\Omega \in \mathbb{Z}$. The key aspect is, now, that we always have twofold degenerated eigenvalues in the spectrum (see Fig. 4).

By continuation to nonlinear modes, these real degenerated eigenvalues could interact with each other and become complex, even for defocusing nonlinearity (see Figs. 5 and 6). Therefore, due to this degeneracy multiple-hump solitons can become unstable. This picture is confirmed by the Krein criteria [15], since the Krein signature of the two corresponding eigenfunctions $\boldsymbol{\Psi}_+$ and $\boldsymbol{\Psi}_-$ is different for all values of m and l.

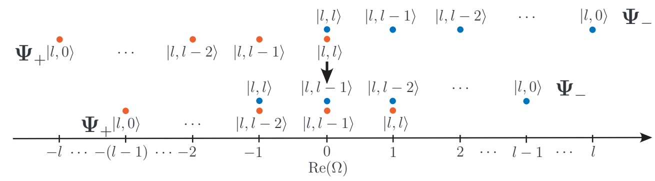

FIG. 4. (Color online) Degenerated eigenvalues $\Omega = m - l, \ldots, l - m$. For $m = l$ only $\Omega = 0$ is twofold degenerated. For $m = l - 1$ the two spectra are shifted against each other by the modulus of 1. Therefore we find three degenerated eigenvalues for $m = l - 1$, and so on. We are only interested in fluctuations, which are propagating with the soliton, thus $-m < \mathrm{Re}(\Omega) < m$.

Physically the Krein signature is the sign of the energy of the corresponding mode. Usually the interaction of positive and negative energy modes leads to instabilities, that is, complex eigenvalues [16].

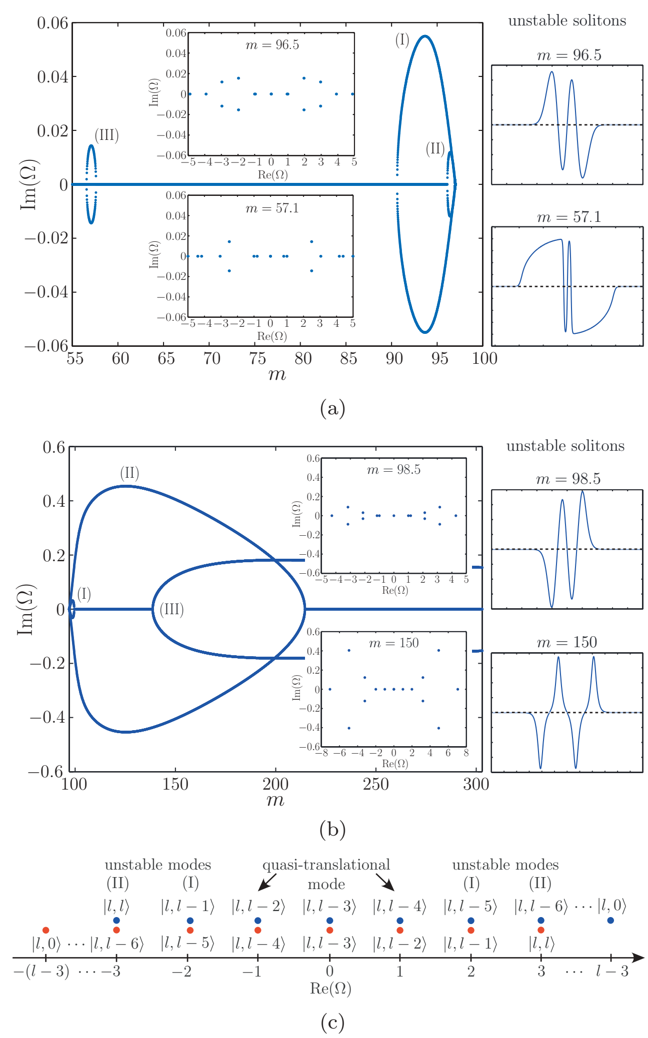

FIG. 5. (Color online) Instabilities for the branch $|100,97\rangle$. In both cases, defocusing (a) and focusing (b), (I) arises from the linear modes $|100,97 \pm 2\rangle$, whereas (II) arises from $|100,97 \pm 3\rangle$ [see (c) for $l = 100$]. Therefore (II) breaks the parity symmetry of the underlying soliton. (III) Second collision of the two eigenvalues of (I). For $|100,98\rangle$ only (II) is not present. Branches $|100,99\rangle$ and $|100,100\rangle$ are stable.

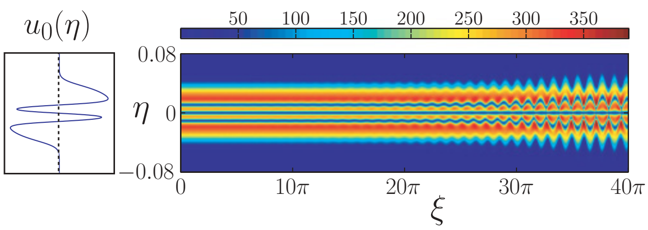

FIG. 6. (Color online) Oscillatory instability in the defocusing case, for the branch $|l,l-3\rangle$ with $l = 10^4$ and $m = l - 7$ (paraxial propagation). Initially random noise of the order $\sim 10^{-2}$ was added. The color illustrates $|u(\eta,\xi)|$.

However, additional symmetries can prevent such a transition. This is indeed the case for the $m = l - 1$ double-hump soliton. Here, the degeneracy of $\Omega = \pm 1$ cannot lead to an instability of the soliton, since the quasitranslation mode (15) is always present as a mode with a real eigenvalue. Therefore the rotational symmetry forbids the annihilation of the two linear modes present at $\Omega = 1$ or $\Omega = -1$. This explains the stability of the double-hump soliton, where solitons with more than two humps are usually unstable for both signs of the nonlinearity. Interestingly, a similar stability scenario was also observed for Bose-Einstein condensates which are confined in a harmonic trap [17].

B. Negatively curved space

The case of negative curvature is quite different. Here, there are no localized solutions at all, since in the limit $\eta \to \pm\infty$ we find the asymptotic $u_0(\eta) \sim \exp(\pm i\sqrt{\beta^2 - 1/4}\eta)$. Now we launch a bright Schrödinger soliton, $u_0(\eta) = \sqrt{2}\kappa/\cosh(\kappa\eta)$ with $\kappa = \sqrt{m^2 - \beta^2}$, which would be stable in flat space. In negatively curved space we observe strong radiation emanating from the beam. Analogously to [18] the ambient radiation arises from the self-induced nonlinear tunneling of the bright

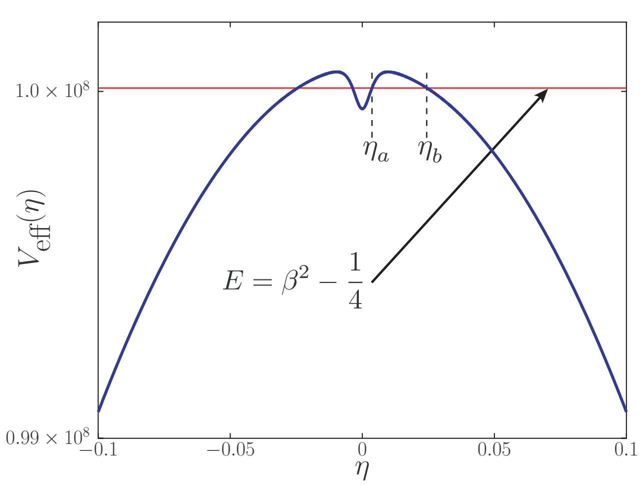

FIG. 7. (Color online) Effective potential for the propagation of a bright soliton in negatively curved space [(blue) curve]. The horizontal (red) line shows the associated energy. The parameters are $l = 10^4$ and $m = l + 3.5$.

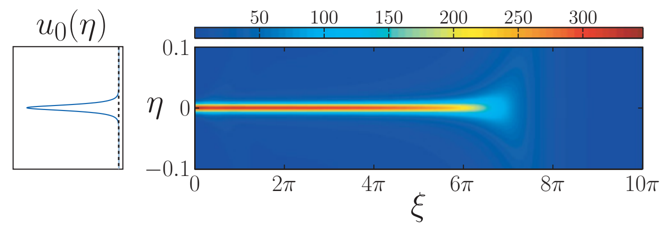

FIG. 8. (Color online) Evaporation of a bright soliton in negatively curved space (paraxial propagation). Parameters are $l = 10^4$ and $m = l + 3.5$. The color illustrates $|u(\eta,\xi)|$.

soliton through a potential barrier. This effective potential is given by

$$V_{\mathrm{eff}}(\eta) = \frac{m^2 + \frac{1}{4}}{\cosh^2\eta} - \frac{2\kappa^2}{\cosh\eta\cosh^2\kappa\eta}, \tag{17}$$

if we interpret the nonlinear interaction $\gamma(\eta)|u_0(\eta)|^2$ to be a self-induced potential. Now we consider (11) as the motion of a quantum mechanical particle in the potential (17) with the energy $E = \beta^2 - 1/4$.

Following standard WKB considerations [19], we can find the tunneling probability of the bright soliton, as $\ln\Gamma = -2\mathrm{Im}S$, where $\mathrm{Im}S = \int_{\eta_a}^{\eta_b}\sqrt{V_{\mathrm{eff}}(\eta) - E}\,d\eta$ is the imaginary part of the classical action S and η_a,η_b are the classical turning points, satisfying $V_{\mathrm{eff}}(\eta_{a,b}) = E$ (see Fig. 7). In the limit m and β large, we can neglect the second term in (17), since it mainly contributes a constant shift to $\mathrm{Im}S$, and find $\ln\Gamma \sim -\pi(m-\beta)$. This shows that the soliton tunneling rate is very low for large propagation constants. Therefore the soliton can stabilize itself for some time. However, tunneling then, finally, leads to evaporation or even explosion of the beam (see Fig. 8).

Regarding experimental observation of soliton evaporation, for example, in a repulsive harmonic trap, one faces the difficulty that the center of mass of the soliton is unstable [18]. We use the symmetries of the negatively curved space and consider an infinitesimal translation (10) of the bright soliton, that is, $\eta \to \eta - \cosh\xi\epsilon$ and $\xi \to \xi + \sinh\xi\tanh\eta\epsilon$. Hence, compared to (15) the translation mode is shifted to $\Omega = \pm i/R$ and seems to induce an instability as well. However, this interpretation is wrong in a translationally invariant space, since this exponential growth just reflects the exponential increase in the separation between neighboring geodesics in negatively curved space [8]. This means that the center of mass is stable and the beam always remains on top of the potential. Therefore soliton evaporation in negatively curved space will always occur, for any initial position and velocity of the bright soliton.

VI. CONCLUSION

In conclusion, we have analyzed, numerically and analytically, the existence and stability of solitons in spaces of constant curvature. Employing symmetry arguments we could naturally generalize the notion of solitons known from flat space and define respective soliton parameters, such as velocity and position. In positively curved space we found localized solutions even in the defocusing case and stable double-hump solitons for focusing and defocusing nonlinearity. Both are not present in flat space. In the case of a negatively curved space, no localized solution exists. Even nonlinearity cannot withstand the spreading of negatively curved space, and a Schrödinger bright soliton which is quite robust in flat space decays after a finite length due to a nonlinear tunneling process. Furthermore, we have shown that translation modes known as neutral or zero modes in flat space occur as an oscillation or exponential divergency in curved space, where they describe the motion along geodesics.

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