Entanglement verification with realistic measurement devices via squashing operations

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Many protocols and experiments in quantum information science are described in terms of simple measurements on qubits. However in a real implementation the exact description is more difficult and more complicated observables are used. The question arises whether a claim of entanglement in the simplified description still holds, if the difference between the realistic and simplified models is taken into account. We show that a positive entanglement statement remains valid if a certain positive linear map connecting the two descriptions—a so-called squashing operation—exists; then lower bounds on the amount of entanglement are also possible. We apply our results to polarization measurements of photons using only threshold detectors, and derive procedures under which multiphoton events can be neglected.

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I. INTRODUCTION

According to Asher Peres, entanglement is "a trick that quantum magicians use to produce phenomena that cannot be imitated by classical magicians" [1]. Because of the key role of entanglement in applications lots of effort is put into realizing this fragile resource in the laboratory, for example via parametric down-conversion (PDC) sources or with ion traps, to only name a few. In a real experiment it is of course desirable to unambiguously verify the creation of entanglement, and in fact many different operational tools have been developed over the past years to achieve this task, cf. Ref. [2] for a review. A reliable entanglement verification has to satisfy a few crucial criteria [3]; most importantly the verification method should not rely on assumptions from the entanglement generation process, but instead on the information acquired about the system via measurements. Moreover the obtained data should be considered under a worst case scenario. That is, in the spirit of Ref. [4], the test is only considered to be affirmative if, in the limiting case of an infinite number of experimental runs, the data exclude compatibility with all separable states. This viewpoint is even essential for certain tasks like quantum cryptography [5].

In any case, it is typical to allow one basic ingredient: since one usually believes in quantum mechanics, it is common to assume that one is equipped with an accurate quantum mechanical description of the employed measurement devices; the actual testing or the (experimental) characterization of a measurement device is anyway often combined with other assumptions concerning the generated state [6–8]. Note that if one does not restrict oneself to this model then one can still use Bell inequalities for the verification. This leads to the known drawback that the entanglement of certain states can never be verified [9] and there is even the conjecture that complete classes of interesting entangled states may be undetectable [10]. However this will not concern us here, and we always assume to have an operator set associated with the observed data, which allows us connect the data to quantum mechanical quantities.

An example of a straightforward and hence quite often applied entanglement verification method, e.g., Ref. [11], is the procedure which we call the *tomography entanglement test* in the following: Since the useful entanglement might be confined to a low-dimensional subspace, e.g., the single photon-pair subspace of a PDC source or two very long-lived energy levels of two ions in a trap, one just performs a few different measurements to obtain tomography on this subspace. After several runs of the experiment one has collected enough data to reconstruct the underlying density operator on this subspace via some reconstruction technique. Note that here one employs the knowledge of the measurement description. In order to check for entanglement one just investigates whether this reconstructed density operator describes an entangled state or not.

However, does one really verify entanglement via this method? The problem lies within the measurement description, because such ideal measurements, as the ones used in the reconstruction mechanism, might not have actually been performed in the experiment. A good example is represented by the polarization measurement with two threshold detectors (see also Fig. 2), which is typically employed in photonic experiments. Apart from usually acting onto several input modes at once, this device does not even respond solely to the single photon subspace, since such detectors cannot resolve the number of photons. Hence the question arises whether one still verifies the entanglement if a more realistic measurement description is employed. It is the main purpose of this paper to study this question. Note that the aforementioned scenario often occurs, not because one is not aware of the more realistic model, but because an oversimplified measurement description is employed in order to ease the task of entanglement verification.

Specific instances of the problems considered here have been investigated in several works in the literature. In Ref. [12] inequalities for the detection of entanglement for two qubits have been proposed, where the measurement's devices can be misaligned to a certain degree. Bell-type inequalities which are independent of the spectrum of the measured observables have been recently introduced in Ref. [13]. Moreover, for an experiment with photons from atomic ensembles, an entanglement verification scheme which takes multiphoton events into account has been introduced [14] and implemented [15].

In this paper we proceed along the following lines. In Sec. II we provide an example of a tomography entanglement test which indeed leads to the wrong conclusion about the presence of entanglement under a small, physical change of the employed measurement description.

In Sec. III, we start to investigate under which conditions such mistakes can safely be excluded. In short, the entanglement verification process remains stable as soon as the considered set of operators are connected by a positive but not necessarily completely positive map, the so-called squashing operation. Similar relations between different measurement schemes have recently been introduced in the context of quantum key distribution (QKD), cf. Refs. [16,17], and even other known verification methods can be cast into this framework. However, complete positivity of the connecting map was required there.

In Sec. IV we reformulate the existence of such a positive map into a necessary and sufficient condition which provides a particular intuitive solution for the tomography entanglement test: The map exists if and only if each classical outcome pattern from the refined set of observables remains compatible with the oversimplified set of observables.

Then, in Sec. V we prove that the aforementioned polarization measurement with threshold detectors along all three different polarization axes represents an example which indeed can only be linked to its single photon realization by a positive but not completely positive map. This analysis concludes that the tomography entanglement test which is typically employed for a PDC source [18] or even in multipartite photonic experiments [19] (using the single photon assumption) can indeed be made error-free if the (local) double click events are taken into account.

In Sec. VI we consider the issue of entanglement quantification, proving that one can in principle get lower bounds on the entanglement of the physical state in terms of the entanglement of the operator that results from the local mapping between the observables.

Finally, we conclude and provide an outlook on possible further directions.

II. AN EXAMPLE FOR ION TRAP EXPERIMENTS

Let us first mention a simple, yet practically relevant example, which shows that the tomography entanglement test indeed can lead to a false conclusion about the presence of entanglement if the structure of the observables is not properly taken into account.

For a single ^{40}Ca ion in a trap one typically considers only the lowest two energy levels given by a lower level $|S\rangle = |1\rangle$ and the upper level $|D\rangle = |0\rangle$ and treats them as the qubit [20]. Resonance fluorescence provides a mechanism to read out the occupation number of the energy levels. An electron in the $|S\rangle$ state is coupled to a higher energy level $|P\rangle$, and observing photons from the $|S\rangle \leftrightarrow |P\rangle$ transition signals that the qubit was in the state $|S\rangle$. This overall process corresponds to a projection onto the lower energy state and consequently allows to measure the σ_z Pauli, while the measurement along different directions is achieved by a local basis rotations prior to the σ_z measurement, cf. Ref. [20].

In order to avoid too many measurements it is common to measure the occupation probability only for the state $|S\rangle$, simply because for qubits the other probability equals $p(D) = 1 - p(S)$ due to the normalization, and similar for the remaining basis settings. Suppose that one uses this measurement procedure to obtain tomography in order to verify the creation of entanglement between two separated ions in the trap. This means that one measures the expectation values $E_{ij}(\rho) = \mathrm{tr}(\rho F_i^A \otimes F_j^B)$ with $F_i^A, F_j^B \in \{|1\rangle\langle 1|, |x^-\rangle\langle x^-|, |y^-\rangle\langle y^-|, \mathbb{1}\}$, where $|x^\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ and $|y^\pm\rangle = (|0\rangle \pm i|1\rangle)/\sqrt{2}$. The inclusion of the two-dimensional identity operator, which is not explicitly measured, formalizes the knowledge that one deals with qubits. Consider now the example where a family of expectation values $E_{ij}(p)$ related to this measurement strategy and depending on a noise parameter p (a probability), are such that for any $0 \leqslant p \leqslant 1$ they allow the reconstruction of the state

$$\rho(p) = (1-p)|\psi^+\rangle\langle\psi^+| + p\frac{\mathbb{1}}{4}, \tag{1}$$

that is, $E_{ij}(p) = E_{ij}[\rho(p)]$. We remark that, by virtue of the partial-transposition criterion [21,22], the state $\rho(p)$ is entangled if and only if $p < 2/3$.

However, in practice the situation is more complicated since the ion is not a simple two-level system. To model this, one can add another energy level to only one of the ions, thereby enlarging the two-qubit system to a qutrit-qubit one. Without any additional information about the occupation number of this extra level, it is clear that (i) the assignment $p(D) = 1 - p(S)$ is incorrect and (ii) one does not obtain tomography. Indeed, working with the qutrit-qubit model, the observed data $E_{ij}(p)$ (i) must be reinterpreted and (ii) do not provide enough information to identify a specific qutrit-qubit state. In particular, reinterpretation means that the identity operator in the set of measured observables must be taken to be three-dimensional for the qutrit, while all other observables remain the same, although embedded in the larger qutrit space. Moreover, as data are compatible with a whole set of states, one verifies entanglement only if the data turn out to be incompatible with all separable qutrit-qubit states. This is only the case if $p < 0.63$, as can be checked using the method from Ref. [23]. Hence in the interval $p \in [0.63, 2/3)$ the performed tomography entanglement test indicates the presence of entanglement in the simplified model, although with the more realistic model it does not. Though this region might be small this error can become important in the multipartite scenario, where current experiments just operate at the border of genuine multipartite entanglement [24–26]. Concerning the experimental consequences, however, two facts are important:

1. For experiments with ion traps it is known that the occupation probability for levels apart from the two logical

states is very small, typically it is around 10^{-3}.[1] Given this additional measurement data, it is possible to provide a quantitative estimate of the resulting error in the used entanglement verification scheme, e.g., the mean value of an entanglement witness. For typical entanglement witnesses employed in those scenarios this error is far below the unavoidable statistical uncertainty, which is caused by the finite number of copies of a state available in any experiment.

2. Note that the probabilities $p(S)$ and $p(D)$ of each energy level can be measured independently by additional local rotations, hence at the expense of more measurements. Then the resulting probabilities correspond to the unnormalized two-level state $\rho_{\rm red}$ that is obtained from our modeled three-level system $\rho_{\rm tot}$ by a local projection, i.e., $\rho_{\rm red} = \Pi\rho_{\rm tot}\Pi$, with $\Pi = |S\rangle\langle S| + |D\rangle\langle D|$. As long as we prove entanglement of the two-qubit system $\rho_{\rm red}^{\rm AB} = \Pi \otimes \mathbb{1}\, \rho_{\rm tot}^{\rm AB}\, \Pi \otimes \mathbb{1}$, this also implies entanglement for the total state $\rho_{\rm tot}^{\rm AB}$, since the projection is local.

For instance, if one measures a witness like $\mathcal{W} = |00\rangle\langle 00| + |11\rangle\langle 11| - |x^+x^+\rangle\langle x^+x^+| - |x^-x^-\rangle\langle x^-x^-| + |y^+y^+\rangle\langle y^+y^+| + |y^-y^-\rangle\langle y^-y^-|$ [2], the mean value of $\mathcal{W}$ is nothing more than a linear combination of certain probabilities on the qubit space, and if the mean value is negative, the state $\rho_{\rm red}^{\rm AB}$ and hence $\rho_{\rm tot}^{\rm AB}$ is entangled. This shows that additional dimensions of the Hilbert space alone do not invalidate the conclusion that the state is entangled when the measurement devices are characterized properly.

III. POSITIVE SQUASHING OPERATIONS

We are ready to formulate the problem that we solve throughout the subsequent sections. For each local measurement setup one has two different sets of ordered observables; a set of simple *target* observables labeled as T_i with $i = 1, \ldots, n$ acting on the Hilbert space $\mathcal{H}_T$ which are used for the entanglement verification process or in the reconstruction mechanism, and a different set of so-called *full* operators denoted as F_i with $i = 1, \ldots, n$ onto the Hilbert space $\mathcal{H}_F$ which represent the more realistic model of the actual observables in the experiment. In the above ion-trap example we considered the case of qubit target observables, while our full operators were acting on a qutrit system.

Consider the case where in an experiment one measures the expectation values of the full operators F_i but instead one interprets them as the expectation values of the target observables T_i. The question arises, whether this may lead to a false entanglement verification. In the following we provide a simple condition on the two operator sets alone that excludes such a possibility, and hence guarantees the presence of entanglement.

Suppose that both sets of observables are connected by a positive (but not necessarily completely positive) linear map $\Lambda : \mathcal{L}(\mathcal{H}_F) \to \mathcal{L}(\mathcal{H}_T)$ which satisfies the following: the expectation value of any observable F_i with respect to an arbitrary input state ρ_F is the same as the expectation value of the corresponding target operator T_i with respect to the output

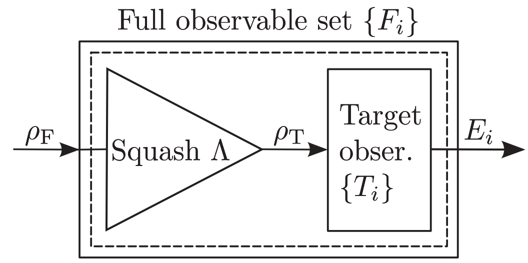

FIG. 1. Idea of the positive squashing operation: The given full observable set $\{F_i\}$ can be decomposed into a *positive* squashing operator Λ followed by a particular target observable set $\{T_i\}$ such that the same expectation values E_i are obtained for all possible input states ρ_F.

state of the corresponding map $\rho_T = \Lambda(\rho_F)$ (see Fig. 1). That is,

$$\mathrm{tr}(\rho_F F_i) = \mathrm{tr}[\Lambda(\rho_F) T_i] \quad (2)$$

holds for any input state ρ_F and for all $i = 1, \ldots, n$. Using the adjoint map $\Lambda^\dagger : \mathcal{L}(\mathcal{H}_T) \to \mathcal{L}(\mathcal{H}_F)$ this condition can be rephrased into

$$\Lambda^\dagger(T_i) = F_i \quad (3)$$

for all $i = 1, \ldots, n$, while the positivity requirement transfers also to the adjoint map $\Lambda^\dagger$.

Such a described connection between two different observables sets is an extension of the notion of a squashing operation in the QKD context [16,17] which differs from the present definition only by the extra condition of being completely positive and trace preserving. Because of these similarities we use the term *positive squashing operation* in order to refer to map Λ (or its adjoint $\Lambda^\dagger$). Note that typically we consider the case of a trace-preserving map Λ (or unital map $\Lambda^\dagger$) such that density operators are mapped to properly normalized density operator; however this requirement is not mandatory. An example of a nontrace preserving, but positive map between operator sets is given by the matrix of moments[2] [27]; the only difference is that one must be careful with entanglement criteria on the target space that explicitly employ the normalization of the density operators (e.g., the computable cross norm or realignment criterion), but one can also deal with this [27].

The advantage of such a positive squashing operation is that the structure of separable states (from the full to the target Hilbert space) remains invariant, and hence any successful entanglement verification on the target space directly translates to a positive verification statement on the full Hilbert space:

[1] Christian Roos (private communication).

[2] Though there are different applications of the matrix of moments, or equivalently the expectation value matrix, only the one from Ref. [27] exploits it in the same spirit as for the present purpose: Rather than trying to reconstruct the matrix of moments of the partially transposed state [50,51], the verification method from Ref. [27] applies the separability criteria *directly* onto the matrix of moments, since it can be considered as an unnormalized physical state. Moreover let us note that the matrix of moments is the composition of a completely positive map followed by the transposition, hence only positive but not completely positive.

Proposition III.1. (Entanglement verification.) Let us assume that each *local* set of observables is connected by a positive squashing operation, i.e., a positive, not necessarily completely positive linear map $\Lambda_A^\dagger$ relates the target and full observables on Alice's side, $\Lambda_A^\dagger(T_i^A) = F_i^A$, and a similar relation holds for Bob's observables. If the observed data verify entanglement with respect to the target observables $T_i^A \otimes T_j^B$, then these data also prove the presence of entanglement for the full operator set $F_i^A \otimes F_j^B$. An analogous statement holds for more than two parties.

Proof. For the observed data E_{ij} one has the identity $E_{ij} = \mathrm{tr}(\rho_{\mathrm{AB}} F_i^A \otimes F_j^B) = \mathrm{tr}[\Lambda_{\mathrm{AB}}(\rho_{\mathrm{AB}}) T_i^A \otimes T_j^B]$ due to the property of the squashing operation. For any separable state on the full bipartite Hilbert space $\rho_{\mathrm{AB}}^{\mathrm{sep}} = \sum_k p_k \rho_A^k \otimes \rho_B^k$, one obtains

$$\sigma_{\mathrm{AB}}^{\mathrm{sep}} := \Lambda_{\mathrm{AB}}\left(\rho_{\mathrm{AB}}^{\mathrm{sep}}\right) = \sum_k p_k \Lambda_A\left(\rho_A^k\right) \otimes \Lambda_B\left(\rho_B^k\right), \quad (4)$$

which represents a valid (normalized) separable density operator on the bipartite target Hilbert space because of positivity of the corresponding (unital) maps, and is compatible with the observed data. Consequently, if one proves the incompatibility of the mean values with all separable states on the target space, the density matrix on the full space must be entangled. Note that here one just needs positivity of Λ_A and Λ_B. It is not required that each map be complete positive, or that the combined map $\Lambda_A \otimes \Lambda_B$ be positive. ■

Note that a local squashing operation between operator sets does not represent the most general map between bipartite observable sets that preserve the structure of separable states; however we neglect other options on behalf of the "locality" of this connection. Furthermore note that since we demand only positivity of each local map, it can happen that the combined bipartite map $\Lambda_A \otimes \Lambda_B$ is not positive. Hence there are cases where the observed data can only originate from an unphysical (not positive semidefinite) density matrix on the target space. However such an operator is incompatible with a separable initial state, as this situation can only occur for an entangled state on the full bipartite Hilbert space. Hence the conclusion of the entanglement verification process remains unaffected.

Finally, let us add that the precise state reconstruction technique needed for the tomography entanglement test, either direct inversion of Born's rule or maximum likelihood estimation [28] (although there are even problems associated with them [29]), does not conflict with a positive but not completely positive squashing operation. If the corresponding operator on the target space is positive semidefinite both reconstruction techniques deliver the same operator (in the limit where one really obtains exact knowledge of the expectation value). As any separable state on the full space corresponds to the reconstruction of a valid separable target state, the possibility that a separable state is mapped to an entangled state by the reconstruction process is excluded. In the case of an unphysical "entangled" target operator a direct inversion of Born's rule would directly "witness" the entanglement.[3] In contrast the maximum likelihood method produces the closest positive semidefinite operator [29] (with respect to the likelihood "distance"), hence an unphysical, entangled target state can be mapped to a separable state via this reconstruction technique and thus escapes the tomography entanglement test. But this does not bother us here, because some entangled states are missed anyway due to the simplified operator set, and we principally aim at avoiding false claims that entanglement is present when it is not.

IV. CRITERIA FOR THE EXISTENCE OF A POSITIVE SQUASHING OPERATION

In this section we investigate which requirements need to be fulfilled by the two different operator sets in order to be connected by a positive squashing operation. There are, of course, different ways how one can tackle this problem: One method, in close analogy to that of Ref. [17], is to employ the Choi-Jamiołkowski isomorphism [30–32] between linear maps and linear operators. This isomorphism transforms positive maps into entanglement witnesses, or more precisely into linear operators that are positive on product states, while the requirements from Eq. (3) change into a set of linear equations that constrain the allowed form of the entanglement witness. For an explicit solution to this reformulated problem one first solves these linear equations and afterwards tries to choose the remaining, undetermined parameters of the operator in such a way that it meets the entanglement witness condition.

However, we take a different path that provides a clear interpretation for the existence of such a positive linear map and which is also employed in the later part to prove the positive squashing property for the polarization measurements.

Equation (2) directly allows us to read off a necessary condition: it states that for each physical density operator ρ_F in the full Hilbert space there exists a valid density operator $\Lambda(\rho_F)$ (if Λ is trace-preserving) in the target space such that both operators assign the same expectation values for the considered observables. Hence all possible expectation values E_i that can in principle be observed on the full Hilbert space must remain physical with respect to the target observables. As we will see, this condition becomes also sufficient if the target operators T_i with $i = 1, \ldots, n$ provide a complete tomographic set. Thus, in combination with Proposition III.1, we have the following solution for the question posed in the Introduction: *The tomography entanglement test is error free as long as the full local observables on Alice and Bob's side can only produce measurement results which are also consistent with the local target, or reconstruction observables.*

For the following proposition we need to define the set of possible physical expectation values associated with a given set of observables, defined as

$$\mathcal{S}_F = \{\vec{E} \in \mathbb{R}^n | \text{there is a } \rho \in \mathcal{D}(\mathcal{H}_F) \text{ such that } E_i = \mathrm{tr}(\rho F_i) \text{ for all } i = 1, \ldots n\}, \quad (5)$$

and a similar definition for the operator set on the target system $\mathcal{S}_T$. Concluding we have the following characterization:

[3] In this case one should be convinced that the actual operator description $T_i^A \otimes T_j^B$ cannot be the correct one for the experiment. However via the matrix of moments of the partially transposed state [50,51] one effectively performs such a detection.

Proposition IV.1 (Existence.) The set of full observables $\{F_i\}$ and the tomographically complete set of target observables $\{T_i\}$ are related by a unital squashing operation $\Lambda^\dagger$ if and only if it holds that $\mathcal{S}_F \subseteq \mathcal{S}_T$.

Proof. One direction of the proof is clear: Suppose that there exists a positive trace-preserving squashing operation Λ. For any $\vec{E} \in \mathcal{S}_F$ we must have a density operator ρ_F such that one obtains $E_i = \mathrm{tr}(\rho_F F_i) = \mathrm{tr}[\Lambda(\rho_F)T_i]$. Because of the properties of the corresponding map we receive a valid target density operator $\rho_T := \Lambda(\rho_F)$ which provides the same expectation values $\vec{E}$, hence $\vec{E} \in \mathcal{S}_T$. This concludes the first direction of $\mathcal{S}_F \subseteq \mathcal{S}_T$.

For the other direction, we employ the fact that the set of target operators are tomographically complete and the set inclusion $\mathcal{S}_F \subseteq \mathcal{S}_T$ to explicitly write down the positive squashing operation. First note that for a given set of physical expectation values $\vec{E} \in \mathcal{S}_T$, the corresponding target density operator is uniquely determined by a direct inversion of Born's rule, $\mathcal{R}_T : \vec{E} \mapsto \rho_T(\vec{E})$, i.e., by a linear reconstruction mechanism that maps the expectation values to its explicit density operator. Moreover for a given full density operator ρ_F the corresponding expectation values are already determined, which is described by the linear map $\mathcal{M}_F : \rho_F \mapsto \vec{E}$. Combining these two maps according to

$$\Lambda = \mathcal{R}_T \circ \mathcal{M}_F \tag{6}$$

provides the squashing operation: That is, for a given input state ρ_F one first computes the expectation values E_i of the full operator set and then uses these values in the reconstruction algorithm (that depends on the target operators) to obtain the corresponding target output state. The set inclusion guarantees that any valid full density operator is mapped to a valid target state, hence the described map is already positive. Since both maps in the decomposition are linear the overall map is linear as well. ■

In a concrete example the proposition of course only helps if one obtains knowledge on the sets $\mathcal{S}$; although this is by far not a trivial task, one can employ approximation techniques for a special set of observables or even a hyperplane characterization for the exact determination, see Ref. [33] for more details.

If the set of considered observables on the target space is not tomographically complete, then, in order to establish the existence of a positive mapping between the two sets of operators, one can still invoke Proposition IV.1 with some caution. Indeed, a positive squashing map exists if and only if it is possible to extend both observable sets by additional target and full operators, so that the extended target set is finally tomographically complete and that the two sets of physical expectation values—which depend on the choice of the extensions—satisfy the condition of Proposition IV.1.

Finally, let us note that one can also characterize a completely positive map via such a set inclusion requirement if one adds an additional reference system R of dimension equal to that of the full space (or of the target space, in the case the dual map) on each side, because complete positivity of Λ just means that $\mathrm{id}_R \otimes \Lambda$ is positive for such a reference R. For an actual investigation of such a completely positive map, however, the formalism of Ref. [17] seems more appropriate to us.

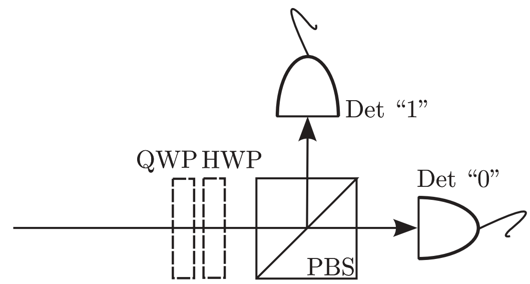

FIG. 2. Schematic setup of the considered polarization measurement: Via quarter (QWP) and half-wave plate (HWP) one can effectively adjust the polarization basis β of the corresponding polarizing beam splitter (PBS) according to the basis $\{\pm 45^\circ, \circlearrowright/\circlearrowleft, H/V\}$ that we label as $\{x,y,z\}$.

V. EXAMPLE: POLARIZATION MEASUREMENTS

In this section, we apply the developed formalism to a physical relevant measurement setup. We draw our attention to polarization measurements onto a two-mode system by using only threshold detectors, i.e., such detectors cannot resolve the number of photons. More precisely, as shown in Fig. 2, the incoming light field is separated according to a chosen polarization basis $\beta \in \{x,y,z\}$ via a polarizing beam splitter, followed by a photon number measurement on each of those modes by a simple threshold detector. Hence in total four different outcomes can be distinguished: no click at all, only one of the detector clicks or both of them trigger a signal and produce a double click. Because of its great simplicity this measurement device appears quite frequently in quantum optical experiments which employ the polarization degree of freedom (for an overview see Ref. [34]). It turns out that this measurement device provides, if measured along all three different basis settings, a nontrivial example for the difference between a positive and a completely positive squasher. This means that the corresponding map Λ can only chosen to be positive but not completely positive.

Next let us specify which observable sets should be connected by the squashing operation; see also Ref. [17]. For each chosen polarization basis β the different mode measurements are denoted as $M_{i,\beta}$ with the label $i \in \{\mathrm{vac},0,1,\mathrm{d}\}$ for all classical outcome possibilities: vacuum, click in "0", click in "1", and a double click. The target measurements are chosen such that they justify a single photon description: each click event is interpreted as a single photon state, hence as target measurements one selects the same measurement device, only that it just acts on the single photon subspace and the vacuum component. In order to achieve the squashing property, the double click events must be taken into account, since such events are clearly incompatible in a (perfect) single photon interpretation, but they nevertheless contribute to the normalization. One can incorporate this effect by a particular postprocessing scheme that represents a sort of penalty for double click events. A common scheme, originally introduced for the QKD context in Refs. [35,36], consists of randomly assigning each double click event to one of the single click outcomes. This particular set of processed measurement operators becomes the exact set of full operators $\{F_{i,\beta}\}$ with $i \in \{\mathrm{vac},0,1\}$ and $\beta \in \{x,y,z\}$ and with the relation $F_{i,\beta} = M_{i,\beta} + 1/2 M_{d,\beta}$ with $i = 0,1$ for all β.

Let us start with a perfect single-polarization mode description of the full operators; imperfections like finite efficiency or dark counts are considered later. The "no click" outcome is independent of the chosen polarization basis and becomes $F_{\mathrm{vac},\beta} = |0,0\rangle\langle 0,0|$. All other observables are block-diagonal with respect to the photon number subspace, i.e., $F_{i,\beta} = \sum_{n=1}^{\infty} F_{i,\beta}^n$ and for a fixed number of photons we have

$$F_{i,\beta}^n = \tfrac{1}{2}[\mathbb{1}_n + (-1)^i(|n,0\rangle_\beta\langle n,0| - |0,n\rangle_\beta\langle 0,n|)], \quad (7)$$

with $i = 0,1$. Here $|k,l\rangle_\beta$ denotes the corresponding two-mode Fock state in the chosen polarization basis β (e.g., for $n = 3$ the state $|2,1\rangle_z = |2H,1V\rangle$ describes a system with two horizontally and one vertically polarized photon) and $\mathbb{1}_n$ represents the identity operator onto the n-photon subspace. Note, that the states $|n,0\rangle_\beta$ and $|0,n\rangle_\beta$ are the only n-photon states that exclusively trigger one of the "single click" outcomes in the basis β; this is taken into account by the second term on the right-hand side of Eq. (7). For more details on this description we refer to Ref. [35]. This perfect single-polarization mode description is also employed for the target operators, however only acting on the vacuum $T_{\mathrm{vac},\beta} = |0,0\rangle\langle 0,0|$ or on the single photon subspace

$$T_{i,\beta} = F_{i,\beta}^1, \quad (8)$$

with $i = 0,1$.

Let us further comment on these observable sets: Note that if one selects the following standard basis for the single photon subspace $|1,0\rangle_z = |0\rangle$ and $|0,1\rangle_z = |1\rangle$, then each difference of the single click outcomes equals to a familiar Pauli operator, i.e., $\sigma_\beta = T_{0,\beta} - T_{1,\beta}$ for all β. Hence each of the single click operators $T_{i,\beta}$ with $i = 0,1$ corresponds to a projection onto one of the two different eigenstates of the related Pauli operator σ_β. Furthermore the corresponding difference between the full observables $F_\beta = F_{0,\beta} - F_{1,\beta}$ is again block diagonal and each n-photon part is given by

$$F_\beta^n = F_{0,\beta}^n - F_{1,\beta}^n = |n,0\rangle_\beta\langle n,0| - |0,n\rangle_\beta\langle 0,n|, \quad (9)$$

according to Eq. (7). Note that these observables are also accessible with a different polarization measurement that only uses a single threshold detector and which has alternatively been employed for polarization experiments, cf. Ref. [11]. Let us point out that these observables are also accessible with a polarization measurement that only uses one of the two threshold detectors[4] and which has been used, for example, in Ref. [11].

The following theorem proves the positive squashing property between the two given sets of observables; however it also applies to the aforementioned polarization measurement with only a single threshold detector.

Theorem V.1. There exists a positive, but not completely positive unital squashing operation $\Lambda^\dagger$ for the polarization measurements with operator sets $\{F_{i,\beta}\}$ and $\{T_{i,\beta}\}$ defined in Eqs. (7) and (8), respectively, i.e., $\Lambda^\dagger(T_{i,\beta}) = F_{i,\beta}$. Therefore, the interpretation of the $\{F_{i,\beta}\}$ as single photon measurements $\{T_{i,\beta}\}$ does not invalidate the entanglement verification scheme.

Proof. First let us point out that the existence of a *completely* positive squashing operation has already been ruled out in Ref. [17], to which we refer the reader for more detail.

In order to prove the existence of a positive squashing operation we only need to focus on the "click" events, since the vacuum part can be directly removed by a projection discriminating between the vacuum and all other Fock states. Note that it is sufficient to prove the squashing operation for a complete set of linear independent target operators only, because other linear dependencies are implicitly present in the linear map. In short, it is equivalent to prove a unital squashing operation $\Lambda^\dagger(\sigma_\beta) = F_\beta$ for all $\beta \in \{x,y,z\}$, where F_β is the described difference between the click outcomes of the full observables.

Since we only concentrate on the single photon subspace we are equipped with a full tomographic set and hence can readily apply Proposition IV.1, such that it remains to prove $\mathcal{S}_F \subseteq \mathcal{S}_T$. Since each full observable is photon number diagonal one obtains that $\mathcal{S}_F$ is given by the convex hull of all n-photon sets $\mathcal{S}_F^n$, i.e., the set of physical expectation values on an n-photon state. Hence we need to verify that each n-photon state can only produce expectation values which are also compatible with a single photon state, i.e., $\mathcal{S}_F^n \subseteq \mathcal{S}_F^1 = \mathcal{S}_T$ for all $n \geqslant 1$. The set $\mathcal{S}_F^1$ directly equals to the familiar Bloch sphere. Hence we prove existence of a positive squashing operation if we can show that

$$\sum_{\beta\in\{x,y,z\}} \left[\mathrm{tr}\left(\rho F_\beta^n\right)\right]^2 \leqslant 1 \quad (10)$$

holds for all n photon density operators ρ, and for all photon numbers $n \geqslant 1$.

In order to simplify the analysis in the following, each operator F_β^n can be regarded as an operator acting onto an n-qubit space. Indeed, the n-photon Hilbert space $\mathcal{H}_F^n = \mathbb{C}^{n+1}$ is isomorphic to the symmetric subspace $\mathrm{Sym}(\mathcal{H}_n)$ of an n-qubit system $\mathcal{H}_n = (\mathbb{C}^2)^{\otimes n}$. Using the given standard basis definition one obtains, for example,

$$F_z^n = |0\rangle\langle 0|^{\otimes n} - |1\rangle\langle 1|^{\otimes n}, \quad (11)$$

while for any other operator F_β^n one just replaces the states $|0\rangle,|1\rangle$ with the eigenvectors of the corresponding Pauli matrix σ_β.

Expanding these operators in a multi-qubit basis delivers

$$\begin{aligned} F_\beta^n &= \left(\frac{\mathbb{1}+\sigma_\beta}{2}\right)^{\otimes n} - \left(\frac{\mathbb{1}-\sigma_\beta}{2}\right)^{\otimes n} \\ &= \frac{1}{2^{n-1}} \sum_{j\ \mathrm{odd}} \sum_{\pi} \pi\left(\sigma_\beta^{\otimes j} \otimes \mathbb{1}^{\otimes(n-j)}\right), \end{aligned} \quad (12)$$

[4]The measurement setup is similar to the one from Fig. 2, however one only measures behind one of the outputs of the polarizing beam splitter. It is straightforward to check that the operators F_β can be obtained by using the difference on the two outputs (or alternatively with appropriate adjusted wave plates). However in order to obtain the normalization, respectively the identity $\mathbb{1}$, one has to measure the overall input via a threshold detector, i.e., with no polarizing beam splitter. It is not, as typically employed, given by the sum of both clicks events on both different outcomes.

in which $\sum_\pi$ denotes the sum over all possible permutations $\pi(\cdot)$ of the subsystems that yield different terms.

Next, we exploit the result from Ref. [37] that for odd j every quantum state ρ, hence also each state on the symmetric space, satisfies

$$\sum_{\beta=x,y,z} \langle \pi(\sigma_\beta^{\otimes j}) \rangle_\rho^2 \leqslant 1, \tag{13}$$

with the abbreviation

$$\langle \pi(\sigma_\beta^{\otimes j}) \rangle_\rho = \mathrm{tr}\big[\rho \pi(\sigma_\beta^{\otimes j} \otimes \mathbb{1}^{\otimes(n-j)})\big]. \tag{14}$$

This inequality is based on the property that the observables $\pi(\sigma_\beta^{\otimes j} \otimes \mathbb{1}^{\otimes(n-j)})$ with $\beta \in \{x,y,z\}$ have all eigenvalues equal to ± 1 and anticommute pairwise.[5] Note that this identity holds for all occurring j and for all possible permutations π. Consequently one obtains

$$\sum_\beta \big[\mathrm{tr}(\rho F_\beta^n)\big]^2 = \frac{1}{2^{2n-2}} \sum_{j,j'\mathrm{odd}}^{n} \sum_{\pi,\pi'} \left[\sum_\beta \langle \pi(\sigma_\beta^{\otimes j}) \rangle_\rho \langle \pi'(\sigma_\beta^{\otimes j'}) \rangle_\rho \right] \leqslant 1, \tag{15}$$

where the inequality (together with the Cauchy-Schwarz inequality) was used to upper bound each term in the squared bracket by 1. For the final result one needs to count the numbers of distinct permutations π, which is given by a corresponding binomial coefficient. ■

How can one use this result in the tomography entanglement test of a PDC source? First each party measures along all three different polarization axes. Next one either actively postprocesses the double click events or just passively computes the corresponding rates and/or probabilities of the full operators. Afterwards both parties can safely use the single photon assumption, or more precisely, the set of perfect single photon target operators $\{T_{i,\beta}\}$ to compute the corresponding two-qubit state ρ_{AB} (single photon subspace on each side) via their preferred reconstruction technique. In case that this reconstructed state is entangled one can be assured that the observed data still verify entanglement if both parties believe in the more realistic measurement description $\{F_{i,\beta}\}$.

Next let us focus on the imperfections of the photodetectors. Real photodetectors register only some portion of the incoming photons, a significant part is not detected. If both detectors in the setup of Fig. 2 have the same inefficiency, this inefficiency can be modeled by an additional beam splitter in front of the perfect measurement device [38], hence if one combines the beam splitter map (completely positive) with the already proven positive squashing map from the perfect case then one directly extends the validity of the positive squashing property to an inefficient measurement description. The same idea applies to dark counts, which can be modeled as a particular postprocessing scheme on the classical outcomes [35], and to misalignment errors, that are described by a fixed depolarizing channel onto each photon separately. Even the extension to a multimode description is possible if one employs the model from Ref. [39].

Concerning real experiments, one should note that double clicks in a spatial mode can arise from different physical mechanisms. First, it can happen that due to the higher orders in the PDC process more than the desired number of photons are generated and injected into the setup. Second, dark counts may lead to double click events. Finally, double-click events may occur in special setups for the generation of certain multipartite states, this case is, however, not important for our discussion.[6]

Then, it is worth mentioning that the post processing used in the above scheme is usually not applied in real experiments: double click events are typically just thrown away. In practice, however, the amount of these undesired events is quite small: For instance, in the four-photon experiment of Ref. [19] the number the coincidences where a double click occurs in one mode while in the other three modes there is one click, is around 0.77% of the (desired) events, where in each mode there is exactly one click.[7] It should be noted, however, that in experiments with more and more photons, these rates can be higher [40], so that the penalty effect of the postprocessing scheme becomes higher.

Additionally we comment on two points: As one can prove, the corresponding squashing map is *completely positive* on the single and two photon subspace [17]. Hence one only observes a violation of positivity of the corresponding target density operator if the local multiphoton contributions are very large in comparison to the single and double photon part (and even then only for very particular entangled states); consequently, to observe such a nonpositive target operator in a real PDC experiment is very unlikely.

As a last point we should make it clear that one cannot always apply Theorem V.1. Especially in multipartite experiments, it happens that one does not even want to obtain full tomography onto the multipartite target space but instead tries to measure an entanglement witness with the least number of

[5] For completeness, let us recall the proof: Let M_i be anticommuting observables (i.e., $M_iM_j + M_jM_i = 0$ for all $i \neq j$) with $M_i^2 = \mathbb{1}$ for all i and let λ_i be real coefficients with $\sum_i \lambda_i^2 = 1$. Then $(\sum_i \lambda_i M_i)^2 = \mathbb{1}$. Therefore, $(\sum_i \lambda_i \langle M_i \rangle)^2 \leqslant \langle (\sum_i \lambda_i M_i)^2 \rangle = 1$, hence $\sum_i \lambda_i \langle M_i \rangle \leqslant 1$, and, since the λ_i are arbitrary, $\sum_i \langle M_i \rangle^2 \leqslant 1$.

[6] In some setups, double click events arise from the statistical nature of the state preparation: For instance, in Ref. [41] entangled multiphoton states are generated by producing several entangled photon pairs first, and then letting them interact via beam splitters. The desired state is only produced if all the photons are distributed uniformly over all the spatial modes, that is, if each mode contains one photon. Due to the statistical properties of the beam splitters, this is not always the case, and often one of the spatial modes contains more than just one photon (and a different mode contains no photon), so that the double click rate at this outcome side drastically increases. However, neglecting these double-clicks is justified: Since in this case some spatial mode does not contain any photon, disregarding these events is equivalent to projecting the total multi-photon state onto the space where each mode contains at least one photon. Since this is a local projection, it cannot produce fake entanglement.

[7] Witlef Wieczorek (private communication).

different global measurement settings. This may require more than three different settings on each photon. For instance, in the six-photon experiment of Ref. [41] an entanglement witness was measured which required seven measurements settings of the type $M_i \otimes M_i \otimes \cdots \otimes M_i$, which is a significant advantage compared with the $3^6 = 729$ settings required for state tomography. However, on each photon seven polarization measurements have been made and the target observables are tomographically overcomplete. In such cases this theorem does not apply, because the linear dependencies imposed by the target operators are not satisfied by the full observable set, cf. Eq. (3), hence the local squashing operation does not exist—in fact the map cannot even be linear. Here one might attempt to proceed with a global, separable squashing operation, cf. comment after Proposition III.1.

VI. POSITIVE SQUASHING AND ENTANGLEMENT QUANTIFICATION

In this section we argue that a local squashing operation, even if it is not completely positive, can in principle not only provide qualitative indications about the presence of entanglement, as was proved in Proposition III.1, but also *quantitative* ones.

Let us start by recalling the notion of entanglement measure. An entanglement measure is a function from density operators to (positive) real numbers, that captures quantitatively some property of entangled states. There are many entanglement measures in the literature [42]; some of them have an operational character, while some others focus on structural properties of entangled states, for example, the fact that, by definition, they do not admit a separable decomposition. Even if some entanglement measures do not have a direct operational interpretation, they are nonetheless useful because they may provide upper and lower bounds to operational measures or other interesting quantitative properties of entanglement. Furthermore, any entanglement measure can be considered as a benchmark for the quality of an experiment designed to create "highly entangled" states and to display a good global control on more than one subsystem at a time. This is due to two facts. The first is that, although different entanglement measures do not correspond to the same ordering of states from "unentangled" to "maximally entangled", there is typically a correlation: a state that is highly entangled with respect to one measure is, in most cases of interest, highly entangled with respect to another one. The second fact is that in an axiomatic approach to entanglement measures, it is typically asked that entanglement, as quantified by some entanglement measure, does not increase under the restricted class of local operations and classical operations (LOCC). Indeed, entanglement cannot be created by LOCC operations alone, and it is natural to require that any entanglement measure does not increase under this set of operations. In this way, on one hand, entanglement is elevated to a resource that by LOCC can only be manipulated and not augmented, and on the other hand, entanglement measures satisfying such a LOCC monotonicity are a fair benchmark for the ability of the experimenters to jointly manipulate many subsystems.

Let us be more precise about the LOCC monotonicity of entanglement measures, focusing on the bipartite case. We say that E is a LOCC monotone if

$$E(\rho_{\mathrm{AB}}) \geqslant E(\Lambda_{\mathrm{LOCC}}[\rho_{\mathrm{AB}}]), \tag{16}$$

where Λ_{LOCC} is a LOCC transformation. In particular, local completely positive trace-preserving (CPTP) maps belong to the class of LOCC operations, so that E is monotone with respect to CPTP local operations:

$$E(\rho_{\mathrm{AB}}) \geqslant E\big((\Lambda_A \otimes \Lambda_B)[\rho_{\mathrm{AB}}]\big). \tag{17}$$

Thus, if the squashing is realized by local CPTP maps, the entanglement of the reconstructed squashed state $(\Lambda_A \otimes \Lambda_B)[\rho_{\mathrm{AB}}]$ is a lower bound for the entanglement of the physical state actually prepared. The point here is that one can generalize Eq. (17) to the case of positive but not completely positive maps, at least for the entanglement measure called negativity [43,44], which is one of the few entanglement measures that can be easily computed.

The negativity of a bipartite state ρ_{AB} is defined as

$$N(\rho) = \frac{\left\|\rho_{\mathrm{AB}}^{\Gamma}\right\|_1 - 1}{2}, \tag{18}$$

where $\rho_{\mathrm{AB}}^{\Gamma} = (T \otimes \mathrm{id})[\rho_{\mathrm{AB}}]$ denotes the partial transpose of the original density operator. Here, T is the transposition, which is a positive but not completely positive trace-preserving map, while "id" stands for the identity map, and $\|X\|_1 = \mathrm{tr}(\sqrt{X^\dagger X})$ is the trace norm on operators. The value of the negativity is independent of the party we choose to apply transposition to, and quantifies the degree of violation of the partial transposition separability criterion [21,22]. Indeed, it corresponds to the sum of the absolute values of the negative eigenvalues of $\rho_{\mathrm{AB}}^{\Gamma}$.

In the Appendix we prove the following inequality:

$$N(\rho_{\mathrm{AB}}) \geqslant \frac{1}{\|\tilde{\Lambda}_A \otimes \Lambda_B\|_1^H} \left(N\big((\Lambda_A \otimes \Lambda_B)[\rho_{\mathrm{AB}}]\big) - \frac{\|\tilde{\Lambda}_A \otimes \Lambda_B\|_1^H - 1}{2} \right), \tag{19}$$

with $\tilde{\Lambda}_A = T \circ \Lambda_A \circ T$ being (completely) positive if and only if Λ_A is (completely) positive, and with a norm on linear maps defined as $\|\Omega\|_1^H \equiv \max_{|\psi\rangle:\langle\psi|\psi\rangle=1} \|\Omega[|\psi\rangle\langle\psi|]\|_1$ (cf. Ref. [45]). We stress that $\|\tilde{\Lambda}_A \otimes \Lambda_B\|_1^H$ is a measure of the joint violation of complete positivity of $\tilde{\Lambda}_A$ and Λ_B. Indeed, if both maps $\tilde{\Lambda}_A$ and Λ_B are trace-preserving and completely positive then one obtains $\|\tilde{\Lambda}_A \otimes \Lambda_B\|_1^H = 1$ and one recovers the inequality given by Eq. (17).

Let us remark that $N\big((\Lambda_A \otimes \Lambda_B)[\rho_{\mathrm{AB}}]\big)$ is the negativity, as defined by Eq. (18), of the Hermitian operator $(\Lambda_A \otimes \Lambda_B)[\rho_{\mathrm{AB}}]$. If the local squashing operations Λ_A and Λ_B are not completely positive, then the latter need not be a physical state because of negative eigenvalues even before the partial transposition. The correcting terms in Eq. (19), with respect to Eq. (17), in particular the presence of $\|\tilde{\Lambda}_A \otimes \Lambda_B\|_1^H$, take care of this possibility, making inequality (19) hold.

Note that in the event that a reconstructed state on the target space results in a nonzero lower bound in Eq. (19), then the state in the full space ρ_{AB} is entangled. This is similar to Proposition III.1, where if entanglement is verified on the target space, then entanglement is verified on the full space. However, it is possible that we verify entanglement on the target space, (and so by Proposition III.1 the state in the full space is entangled, and the negativity is positive) but we have no positive lower bound on the negativity from Eq. (19).

As shown in the Appendix, a different and possibly weaker lower bound on the negativity is given by

$$N(\rho_{\rm AB}) \geqslant \frac{1}{\|\Lambda_A\|_\diamond \|\Lambda_B\|_\diamond}\left(N\big((\Lambda_A \otimes \Lambda_B)[\rho_{\rm AB}]\big) - \frac{\|\Lambda_A\|_\diamond \|\Lambda_B\|_\diamond - 1}{2}\right). \quad (20)$$

Here $\|\Omega\|_\diamond \equiv \|\Omega \otimes {\rm id}\|_1$ is the diamond norm [46], with the identity map that can be considered as acting on the same input space as Ω, and $\|\Omega\|_1 \equiv \sup_{\|X\|_1 \leqslant 1} \|\Omega[X]\|_1$ the trace norm for maps.

We further remark that, in the case of a positive but not completely positive squashing operation, it might be possible to obtain lower bounds for the entanglement of the original state also for other entanglement measures. Although we are unable to provide further explicit examples at this time, we observe that this might be true for the relative entropy of entanglement [47,48]. The latter is defined for a state $\rho_{\rm AB}$ as

$$E_R(\rho_{\rm AB}) = \min_{\sigma_{\rm AB}^{\rm sep}} S(\rho_{\rm AB} \| \sigma_{\rm AB}), \quad (21)$$

where the minimum runs over all separable states and $S(\rho_{\rm AB}\|\sigma_{\rm AB}) = {\rm tr}[\rho_{\rm AB}(\log_2 \rho_{\rm AB} - \log_2 \sigma_{\rm AB})]$ is the relative entropy. Monotonicity of this measure under CPTP LOCC operations can be easily checked as follows:

$$\begin{aligned} E_R(\rho_{\rm AB}) &= \min_{\sigma_{\rm AB}^{\rm sep}} S\big(\rho_{\rm AB} \big\| \sigma_{\rm AB}^{\rm sep}\big) \\ &\geqslant \min_{\sigma_{\rm AB}^{\rm sep}} S\big(\Lambda_{\rm LOCC}[\rho_{\rm AB}] \| \Lambda_{\rm LOCC}\big[\sigma_{\rm AB}^{\rm sep}\big]\big) \\ &\geqslant \min_{\tau_{\rm AB}^{\rm sep}} S(\Lambda_{\rm LOCC}[\rho_{\rm AB}] \| \tau_{\rm AB}) \\ &= E_R(\Lambda_{\rm LOCC}[\rho_{\rm AB}]). \end{aligned} \quad (22)$$

In the first inequality one uses monotonicity of the relative entropy under CPTP maps; for the second inequality one employs the fact that a CPTP LOCC map transforms a separable state into another separable state. Now, a local map $\Lambda_A \otimes \Lambda_B$ also transforms a separable state into a separable state as soon as the maps Λ_A and Λ_A are positive and trace-preserving—this was the key fact used in Proposition III.1. If monotonicity of the relative entropy holds under some local map $\Lambda_A \otimes \Lambda_B$, even if Λ_A and Λ_B are just positive but not completely positive, then the inequality $E_R(\rho_{\rm AB}) \geqslant E_R\big((\Lambda_A \otimes \Lambda_B)[\rho_{\rm AB}]\big)$ still remains true. This possibility is left open for example by the fact that the requirement often used to prove monotonicity of the relative entropy is not complete positivity, but the weaker request of two-positivity [49] (together with a trace preservation condition). A given map Ω is two-positive if

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \geqslant 0 \Rightarrow \begin{pmatrix} \Omega[A] & \Omega[B] \\ \Omega[C] & \Omega[D] \end{pmatrix} \geqslant 0, \quad (23)$$

where A, B, C, and D are matrices themselves. Hence, if both maps Λ_A and Λ_B are positive and trace-preserving, and the combined local map $\Lambda_A \otimes \Lambda_B$ is two-positive, then the inequality $E_R(\rho_{\rm AB}) \geqslant E_R\big((\Lambda_A \otimes \Lambda_B)[\rho_{\rm AB}]\big)$ still holds.

In conclusion, a positive squashing operation does not only provide qualitative statements about entanglement, but potentially also quantitative ones. Open problems regard the application of the derived bounds on the negativity to specific cases, and the analysis of other entanglement measures. Let us remark that in case of the negativity a detailed analysis of lower bounds on the entanglement essentially deals with the issue of separating the negativity due to the application of the local squashing maps from the negativity due to partial transposition. As the bounds are conservative, only states that are sufficiently entangled may result in a nontrivial lower bound. Indeed, it is clear that if $N\big((\Lambda_A \otimes \Lambda_B)[\rho_{\rm AB}]\big) = 0$—this is the case for a separable $\rho_{\rm AB}$ and positive Λ_A and Λ_B—and $\|\tilde{\Lambda}_A \otimes \Lambda_B\|_1^H > 1$ or $\|\Lambda_A\|_\diamond \|\Lambda_B\|_\diamond > 1$, respectively, then the right-hand sides of Eqs. (19) and (20) are actually negative. It is worth remarking that in the derivation of the bounds for the negativity we have not made use of the positivity of the squashing operations. This indicates that if one considers local squashing operations with the aim of entanglement verification and quantification, then one may hope to be able to further relax the requirements on the corresponding maps, not only going beyond complete positivity, but also beyond positivity if adequate care is taken.

VII. CONCLUSIONS

Entanglement verification typically assumes that one knows the underlying measurement operators so that each classical outcome gets an accurate quantum mechanical interpretation. We have addressed the question under what conditions an affirmative entanglement statement remains valid if a simplified description of the measurement apparatus is used. This situation can occur if the actual measurement observables are different from the ones used in the verification analysis, simply because of imperfections or wrong calibration. However it can even happen on purpose: Indeed one can try, despite being aware of certain differences, to explain the data via an oversimplified model, e.g., a very low-dimensional description, that eases then the task of applying an entanglement criteria. Such a case occurs for example for an active polarization measurement with threshold detectors to analyze the entanglement from a PDC source. Here one may choose a single photon description only, although one knows that certain multiphoton states can also trigger events that are indistinguishable from a single photon case, because then one easily obtains "tomography" by using three different measurement settings and directly checks for entanglement on the reconstructed two-qubit state.

Summarizing, a positive entanglement statement remains valid if the two operator sets can be related by a positive (not necessarily completely positive) map. In case that the reconstruction operators provide complete tomography such a positive maps exists if and only if all measurement results from the refined, actual measurement devices are compatible with the assumed measurement description of the device. We have shown that the aforementioned polarization measurement, measured along all three different polarization axes, constitutes a physical relevant example of such a connection that is positive but not completely positive. This result shows that most of the performed tomography entanglement tests for a PDC source are indeed error-free if one incorporates a penalty for double clicks during the state reconstruction. This verifies entanglement for a more realistic model, with imperfections and multiphoton contributions, of the measurement used. Finally, we argued that it might be possible to obtain not only a positive qualitative statement about the presence of entanglement, but also a quantitative one, even in cases where the squashing map is not completely positive and standard results about monotonicity of entanglement measures cannot directly be used.

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APPENDIX

For a Hermitian operator $\rho_{\rm AB}$ normalized to satisfy $\text{tr}(\rho_{\rm AB}) = 1$, we define negativity as

$$N(\rho_{\rm AB}) = \frac{\left\|\rho_{\rm AB}^{\Gamma}\right\|_1 - 1}{2}, \quad \text{(A1)}$$

where $\rho_{\rm AB}^{\Gamma} = (T \otimes \text{id})[\rho_{\rm AB}]$, and T is the transposition. Negativity corresponds to the sum of the negative eigenvalues of $\rho_{\rm AB}^{\Gamma}$.

Any Hermiticity preserving map Λ can be written as $\Lambda[X] = \sum_i c_i K_i X K_i^{\dagger}$, $c_i \in \mathbb{R}$. Then $T \circ \Lambda = \tilde{\Lambda} \circ T$, with $\tilde{\Lambda} : X \mapsto \sum_i c_i K_i^{*} X K_i^{T}$, i.e., $\tilde{\Lambda} = T \circ \Lambda \circ T$. If Λ is (completely) positive, that is if $c_i \geqslant 0$ for all i, then $\tilde{\Lambda}$ is (completely) positive. It also holds that Λ is trace-preserving if and only if $\tilde{\Lambda}$ is trace-preserving.

For any map Ω we define the norm $\|\Omega\|_1^H \equiv \max_{|\psi\rangle:\langle\psi|\psi\rangle=1} \|\Omega[|\psi\rangle\langle\psi|]\|_1$ [45]. Moreover, we observe that the trace norm of a Hermitian operator X can be expressed as $\|X\|_1 = \max_{-\mathbb{1}\leqslant M\leqslant\mathbb{1}} \text{tr}(MX)$. For any $-\mathbb{1} \leqslant M \leqslant \mathbb{1}$,

$$\begin{aligned}|\langle\psi|\Omega^{\dagger}[M]|\psi\rangle &= |\text{tr}\big(\Omega^{\dagger}[M]|\psi\rangle\langle\psi|\big)| \\ &= |\text{tr}\big(M\Omega[|\psi\rangle\langle\psi|]\big)| \leqslant \|\Omega\|_1^H. \quad \text{(A2)}\end{aligned}$$

Therefore, if $-\mathbb{1} \leqslant M \leqslant \mathbb{1}$, then $-\mathbb{1} \leqslant \frac{\Omega^{\dagger}[M]}{\|\Omega\|_1^H} \leqslant \mathbb{1}$.

Thus, assuming that Λ_A and Λ_B are trace-preserving—so that $\text{tr}\big((\Lambda_A \otimes \Lambda_B)[\rho_{\rm AB}]\big) = 1$:

$$\begin{aligned}
&N\big((\Lambda_A \otimes \Lambda_B)[\rho_{\rm AB}]\big) \\
&= \frac{\|\big((T\circ\Lambda_A)\otimes\Lambda_B\big)[\rho_{\rm AB}]\|_1 - 1}{2} \\
&= \frac{\left\|(\tilde{\Lambda}_A\otimes\Lambda_B)\big[\rho_{\rm AB}^{\Gamma}\big]\right\|_1 - 1}{2} \\
&= \frac{1}{2}\left\{\max_{-\mathbb{1}\leqslant M\leqslant\mathbb{1}} \text{tr}\big(M(\tilde{\Lambda}_A\otimes\Lambda_B)\big[\rho_{\rm AB}^{\Gamma}\big]\big) - 1\right\} \\
&= \frac{1}{2}\left\{\max_{-\mathbb{1}\leqslant M\leqslant\mathbb{1}} \text{tr}\big((\tilde{\Lambda}_A\otimes\Lambda_B)^{\dagger}[M]\rho_{\rm AB}^{\Gamma}\big) - 1\right\} \\
&= \frac{1}{2}\left\{\|\tilde{\Lambda}_A\otimes\Lambda_B\|_1^H \max_{-\mathbb{1}\leqslant M\leqslant\mathbb{1}} \text{tr}\left(\frac{(\tilde{\Lambda}_A\otimes\Lambda_B)^{\dagger}[M]}{\|\tilde{\Lambda}_A\otimes\Lambda_B\|_1^H}\rho_{\rm AB}^{\Gamma}\right) - 1\right\} \\
&\leqslant \frac{1}{2}\left\{\|\tilde{\Lambda}_A\otimes\Lambda_B\|_1^H \max_{-\mathbb{1}\leqslant M'\leqslant\mathbb{1}} \text{tr}\big(M'\rho_{\rm AB}^{\Gamma}\big) - 1\right\} \\
&= \frac{1}{2}\left\{\|\tilde{\Lambda}_A\otimes\Lambda_B\|_1^H \|\rho_{\rm AB}^{\Gamma}\| - 1\right\} \\
&= \|\tilde{\Lambda}_A\otimes\Lambda_B\|_1^H N(\rho_{\rm AB}) + \frac{\|\tilde{\Lambda}_A\otimes\Lambda_B\|_1^H - 1}{2}. \quad \text{(A3)}
\end{aligned}$$

Solving for $N(\rho_{\rm AB})$ we finally find

$$\begin{aligned}N(\rho_{\rm AB}) \geqslant \frac{1}{\|\tilde{\Lambda}_A\otimes\Lambda_B\|_1^H}\bigg(&N\big((\Lambda_A\otimes\Lambda_B)[\rho_{\rm AB}]\big) \\ &- \frac{\|\tilde{\Lambda}_A\otimes\Lambda_B\|_1^H - 1}{2}\bigg). \quad \text{(A4)}\end{aligned}$$

For a Hermiticity preserving map Θ one easily checks that $\|\Omega\circ\Theta\|_1^H \leqslant \|\Omega\|_1^H\|\Theta\|_1^H$. In our case

$$\begin{aligned}\|\tilde{\Lambda}_A\otimes\Lambda_B\|_1^H &\leqslant \left\|\tilde{\Lambda}_A\otimes\text{id}_{B_{\rm out}}\right\|_1^H \left\|\text{id}_{A_{\rm in}}\otimes\Lambda_B\right\|_1^H \\ &= \left\|\Lambda_A\otimes\text{id}_{B_{\rm out}}\right\|_1^H \left\|\text{id}_{A_{\rm in}}\otimes\Lambda_B\right\|_1^H \\ &\leqslant \|\Lambda_A\|_{\diamond}\|\Lambda_B\|_{\diamond}. \quad \text{(A5)}\end{aligned}$$

By $\text{id}_{B_{\rm out}}$ and $\text{id}_{A_{\rm in}}$ we have denoted the identity map on the output space of Λ_B and on the input space of Λ_A, respectively. The equality in the second line is due to the fact that $\|T\circ\Omega\circ T\|_1^H = \|\Omega\|_1^H$. The diamond norm [46] is defined as $\|\Omega\|_{\diamond} \equiv \|\Omega\otimes\text{id}\|_1$, with the identity map that can be taken as acting on the same input space as Ω, and with $\|\Omega\|_1 \equiv \sup_{\|X\|_1\leqslant 1}\|\Omega[X]\|_1$ the trace norm for maps. The last inequality is due to the fact that $\|\Omega\otimes\text{id}\|_1^H \leqslant \|\Omega\otimes\text{id}\|_1 \leqslant \|\Omega\|_{\diamond}$, for id acting on an arbitrary dimension [46]. Thus, we finally obtain the lower bound

$$\begin{aligned}N(\rho_{\rm AB}) \geqslant \frac{1}{\|\Lambda_A\|_{\diamond}\|\Lambda_B\|_{\diamond}}\bigg(&N\big((\Lambda_A\otimes\Lambda_B)[\rho_{\rm AB}]\big) \\ &- \frac{\|\Lambda_A\|_{\diamond}\|\Lambda_B\|_{\diamond} - 1}{2}\bigg). \quad \text{(A6)}\end{aligned}$$

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