Family of fish-eye-related models and their supersymmetric partners

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A large family of potentials related to the Maxwell fish-eye model is derived with the help of conformal mappings. It is shown that the whole family admits square-integrable $E = 0$ solutions of the Schrödinger equation for discrete values of the coupling constant. A corresponding supersymmetric family of partner potentials to the preceding ones is derived as well. Some applications of the considered potentials are also discussed.

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I. INTRODUCTION

The idea initiated by Maxwell in 1854 [1], nowadays known under the name "Maxwell's fish-eye problem," addresses the issue of focusing properties of optical media with continuously varying refractive index. Media of this kind have been manufactured since the early 1960s and are used, for example, in modern fiber optics and in the production of optical instruments like lenses and prisms.

Maxwell considered the index of refraction $n_1(r) = n_0/[1 + (r/R)^2]$, where R stands for the radius of a sphere, for which we assume $R = 1$, r is the radial distance from its center, and n_0 is a positively determined constant. A medium with the index possesses the remarkable property of giving rise to the circular motion of rays and represents a perfect optical focusing device in the sense that every point of the xy plane has the corresponding perfect conjugate point in the plane.

The formula for $n_1(r)$ can be derived in a number of ways. One of them is via the stereographic projection $x = X/(1 - Z)$, $y = Y/(1 - Z)$ of the points on sphere $X^2 + Y^2 + Z^2 = 1$ onto the xy plane. In this way, we can get [2] for the optical line element $ds^2 = n_1(r)(dx^2 + dy^2)$. Since the projection is a conformal one, we may map the xy plane conformally upon itself. Using $f(z) = u(x,y) + iv(x,y)$, with $z = x + iy$ and u and v being real functions, one can show that [2] $n(x,y) = 2|f'(z)|/[1 + |f(z)|^2]$, where $f'(z) = df/dz$ and $n(x,y) = n_1(r)$ for $f(z) = z$.

Now, it is not difficult to show that all light rays, in the medium described by the refractive index $n(x,y)$, trace out the curves being the solutions of the equation [3],

$$|f|^2 + af + a^* f^* = 1, \tag{1}$$

where $a = \alpha + i\beta$ is an arbitrary complex constant.

Detailed discussion in [2] shows that the light rays may be considered as trajectories of particles of mass, say μ, if the refractive index $n(x,y)$ is replaced with the potential

$$V(x,y) \to V(u,v) = \frac{-w|f'(z)|^2}{2[1 + |f(z)|^2]^2} \tag{2}$$

for the fixed total energy $E = 0$ and the strength parameter $w > 0$. Again, for $f(z) = z$ the potential gives rise to the circular motion as in the case of Maxwell's fish-eye model. For this and other $f(z)$ mappings, all trajectories can be found from Eq. (1).

It is the purpose of this article to show that all members of the family of Maxwell's fish-eye-related potentials in Eq. (2) can be quantized and for discrete values of $w > 0$ have solutions of the Schrödinger equation expressible in terms of Gegenbauer polynomials. We show this in Sec. II. In the next section, the supersymmetric partner potentials to the family (2) are also discussed. Finally, Sec. IV contains concluding remarks.

II. QUANTUM SOLUTIONS

Let us start with the stationary, $E = 0$, Schrödinger equation:

$$\left[\frac{-\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + V(x,y)\right]\psi(x,y) = 0. \tag{3}$$

Under the conformal transformation $f(z) = u(x,y) + iv(x,y)$, $u,v \in \mathbb{R}$, it takes the form

$$\left[\frac{-\hbar^2}{2\mu}|f'(z)|^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + V(u,v)\right]\psi(u,v) = 0. \tag{4}$$

With the choice of potentials,

$$V(u,v) = |f'(z)|^2 U(u,v), \tag{5}$$

Eq. (4) has exactly the form as Eq. (3). Thus, from Eqs. (5), (4), and (2), we have

$$\left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + \frac{\mu w}{\hbar^2(1 + u^2 + v^2)^2}\right]\psi(u,v) = 0. \tag{6}$$

Introducing now the polar coordinates: $u = \rho\cos\varphi$ and $v = \rho\sin\varphi$, we can write the radial part of the Schrödinger equation as

$$\frac{d^2\phi}{d\rho^2} + \frac{1}{\rho}\frac{d\phi}{d\rho} + \left[\frac{\mu w}{\hbar^2}\frac{1}{(1+\rho^2)^2} - \frac{\lambda^2}{\rho^2}\right]\phi(\rho) = 0, \tag{7}$$

where we have used $\psi(\rho,\varphi) = \phi(\rho)\exp(\pm i\lambda\varphi)$, and obviously $\rho^2 = u^2 + v^2 = |f(z)|^2$.

Solutions to Eq. (7), at least some of them, can be square integrable provided that the coupling constant w takes discrete values [4]. To find the solutions and the dependence of w on quantum numbers, it is now convenient to change the variable

$$\xi = \frac{1 - \rho^2}{1 + \rho^2} = \frac{1 - |f(z)|^2}{1 + |f(z)|^2}. \tag{8}$$

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Postulating additionally that $\phi(\xi) = (1-\xi^2)^{\lambda/2}F(\xi)$, we get

$$(1-\xi^2)\frac{d^2F}{d\xi^2} - 2(\lambda+1)\xi\frac{dF}{d\xi} + \left[\frac{\mu w}{4\hbar^2} - \lambda(\lambda+1)\right]F = 0. \tag{9}$$

This equation can be reduced to the standard equation for the Gegenbauer polynomials $C_{n_r}^{\lambda+1/2}(\xi)$ if w is quantized according to the rule

$$\frac{\mu w}{4\hbar^2} = (n_r+\lambda)(n_r+\lambda+1), \tag{10}$$

where $n_r = 0,1,2,\ldots$ and is "the radial" quantum number. Thus, for the full solution of Eq. (6), including also the angle part, we have

$$\psi_{n_r}(x,y) = N\exp(\pm i\lambda\varphi)\left[\frac{|f(z)|}{1+|f(z)|^2}\right]^{\lambda} \times C_{n_r}^{\lambda+1/2}\left[\frac{1-|f(z)|^2}{1+|f(z)|^2}\right], \tag{11}$$

where N stands for the normalization constant and λ is a positively determined parameter whose detailed values depend on the used mapping $f(z)$ and boundary conditions. We can only say at this point that for the weight function of the Gegenbauer polynomials to be real and integrable, we need $\lambda + 1/2 > -1/2$. The first three polynomials are the following: $C_0^{\lambda+1/2}(x) = 1$, $C_1^{\lambda+1/2}(x) = 2(\lambda+1/2)x$, and $C_2^{\lambda+1/2}(x) = 2(\lambda+1/2)(\lambda+3/2)x^2 - (\lambda+1/2)$. The normalization, for the cases considered in what follows, is given in [5,6].

When $f(z) \to f_1(z) = x + iy$, Eq. (11) represents an exact regular solution for Maxwell's fish-eye model at the threshold energy $E = 0$. In this case, from the normalization [5], it follows that $\lambda = l > 1$, and the potential from Eq. (2) reads as $V_1(x,y) = -w/[2(1+r^2)^2]$, $r^2 = x^2 + y^2$. This potential has found many applications. For instance, it was used in geometric optics for constructing perfectly focusing lenses [2,7], in the theory of quantum dots with smooth boundaries [8], in projecting invisibility devices [9], and, its generalization, in microwave antenna design [10]. Besides, it is interesting from the formal point of view. Let us mention only that the model and the Kepler problem are formally equivalent [11,12]. This can be proved with the help of the Kustaanheimo-Stiefel transformation [13]. One can also show that the potential $V_1(x,y)$ can be obtained as a superposition of the Yukawa potential [14].

When $f(z) \to f_k(z) = z^k$, $k \neq 0$, we obtain the large family of Lenz-Demkov-Ostrovsky potentials [15] $V_k(x,y) = -wr^{2k}/[2r^2(1+r^{2k})^2]$, with the property that $V_k(x,y) = V_{-k}(x,y)$, for which the values of λ in Eqs. (10) and (11) are $\lambda = l/|k|$. Classical orbits can be found from Eq. (1) and few examples are given in [5]. Again, the potentials $V_k(x,y)$ have found interesting applications: the case of $k = 1/2$ for describing the *Aufbau* law for building up the periodic system of elements [15–17], the case of $k = 3$ for explaining some grouping of levels observed in medium-size sodium clusters [18], and for studying the quantum-classical correspondence for a fixed k [5].

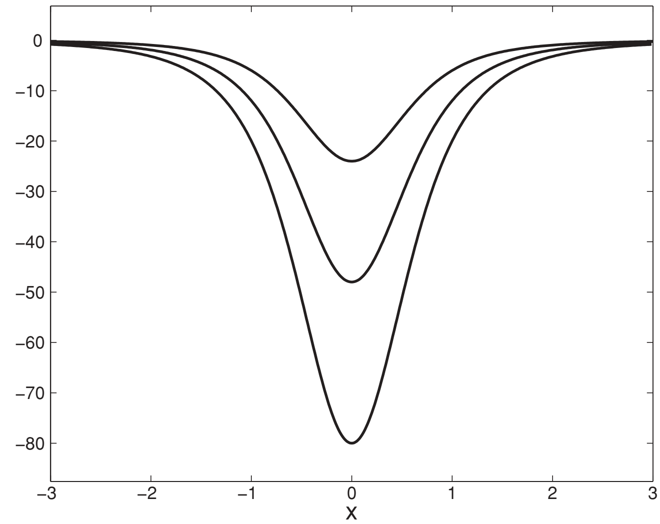

FIG. 1. The plot, in units of $\hbar^2/(2\mu)$, of the fish-eye potential in Eq. (2), in section with the plane $y = 0$, where $f(z) = z$ and the coupling constant $w = (4\hbar^2/\mu)l(l+1)$, for $l = 2$, 3, and 4. The deeper the potential well, the larger the value of l.

When $f(z) = \exp(iz/L)$, $L =$ const, and we restrict ourselves to the strip $0 < x < L$, $-\infty < y < \infty$ of the xy plane, then the potential from Eq. (2) is now $V(x,y) = -w/[8L^2\cosh^2(y/L)]$ in the y direction and has infinite walls at $x = 0$ and $x = L$. This time $\lambda = l\pi$ in Eqs. (10) and (11) and the noncentral potential is again an example of completely solvable model with the help of identical mapping in both the classical and the quantum formulations. It was recently used for studying relations between quantum probability distributions and classical orbits [6].

Some other members of the family (2), for which Eqs. (1), (10), and (11) constitute exact solutions, can be derived in the way discussed previously. Since all of them lead to Eq. (6), as for the fish-eye model, we shall call the model "the founding father" of the whole family of potentials related to the Maxwell one. The illustration in Fig. 1 shows its behavior for three values of the coupling constant w.

III. SUPERSYMMETRIC PARTNER POTENTIALS

It is now not difficult to derive another family of potentials that is related to that in Eq. (2). They are called supersymmetric (SUSY) partner potentials and the procedure of obtaining them has been described many times (see, e.g., [19]) and is based on using Witten's Hamiltonian [20]. To this end, we start the discussion from the equation

$$\begin{aligned} H_-\chi &= \frac{-\hbar^2}{2\mu}\frac{d^2\chi}{d\rho^2} + \left[\frac{\hbar^2}{2\mu}\frac{\lambda^2 - 1/4}{\rho^2} - \frac{w}{2(1+\rho^2)^2}\right]\chi \\ &= \frac{-\hbar^2}{2\mu}\frac{d^2\chi}{d\rho^2} + V_-(\lambda,\rho)\chi = 0, \end{aligned} \tag{12}$$

which can be obtained from Eq. (7) after the substitution $\phi(\rho) = \rho^{-1/2}\chi(\rho)$. The square bracket represents an effective potential $V_-(\lambda,\rho)$ for the fish eye. According to Witten, the potential and its SUSY partner $V_+(\lambda,\rho)$ are both defined with the help of a superpotential $W(\rho)$,

$$V_{\mp} = W^2 \mp \frac{\hbar}{\sqrt{2\mu}} W', \tag{13}$$

where $W' = dW/d\rho$.

This nonlinear Riccati-type equation can be formally integrated if one observes that $W_1 = (-\hbar/\sqrt{2\mu})(d/d\rho)\ln\chi_0$ is its particular solution. In our case, $\chi_0(\rho)$, according to Eq. (11), is the nodeless solution of Eq. (12) in the form

$$\chi_0(\rho) = N_0 \frac{\rho^{\lambda+1/2}}{(1+\rho^2)^{\lambda}} = N_0 \frac{|f(z)|^{\lambda+1/2}}{[1+|f(z)|^2]^{\lambda}}. \tag{14}$$

The function is obviously square integrable and for the normalization constant N_0, we get $N_0^2 = 2\Gamma(2\lambda)/[\Gamma(\lambda+3/2)\Gamma(\lambda-3/2)]$, $\lambda > 3/2$. Next, using the transformation $W(\rho) = W_1(\rho) + Q(\rho)$, we will get a Bernoulli-type equation for $Q(\rho)$, which can be further transformed into a linear equation for $S(\rho)$ if $Q(\rho) = S^{-1}(\rho)$. In this way, the general solution for the superpotential $W(\rho)$ reads as [21]

$$W(\rho) = \frac{-\hbar}{\sqrt{2\mu}}\left(\frac{\chi_0'}{\chi_0}\right) + \left[\chi_0^2\left(\alpha - \frac{\sqrt{2\mu}}{\hbar}\int^{\rho} \chi_0^{-2} d\rho\right)\right]^{-1}. \tag{15}$$

The particular value of an arbitrary parameter α is motivated by physical considerations. To have $W(\rho)$ finite in the asymptotic region $\rho \to \infty$, we take $\alpha = \infty$. Thus, we obtain

$$\begin{aligned} W(\rho) = W_1(\rho) &= \frac{-\hbar}{\sqrt{2\mu}}\left(\frac{\chi_0'}{\chi_0}\right) \\ &= \frac{-\hbar}{\sqrt{2\mu}}\left[\frac{2\lambda+1+(1-2\lambda)\rho^2}{2\rho(1+\rho^2)}\right]. \end{aligned} \tag{16}$$

One can easily check that with the superpotential, the formula (13) precisely reproduces the potential $V_-(\lambda,\rho)$ defined previously, and for $V_+(\lambda,\rho)$, we obtain

$$V_+(\lambda,\rho) = \frac{\hbar^2}{2\mu}\left[\frac{\rho^4(\lambda^2-2\lambda+3/4)+\rho^2(-2\lambda^2+4\lambda+3/2)+\lambda^2+2\lambda+3/4}{\rho^2(1+\rho^2)^2}\right], \tag{17}$$

which takes everywhere positive values for all $\lambda \geqslant 0$.

We should also add at this point that the nodeless solution in Eq. (14) is for the radial quantum number $n_r = 0$ and, from Eq. (10), we have now $\mu w = 4\hbar^2\lambda(\lambda+1)$. The eigenfunctions for $n_r \neq 0$ can be written utilizing Eq. (11) and all of them correspond to the energy $E = 0$. It plays also the role of factorization energy for the Hamiltonian H_- in Eq. (12), which may be formulated as well as

$$H_-\chi_0 = A^+A^-\chi_0 = 0, \tag{18}$$

where the operators $A^{\pm}$ are given by

$$A^{\pm} = \mp\frac{\hbar}{\sqrt{2\mu}}\frac{d}{d\rho} + W(\rho). \tag{19}$$

Since there is a normalizable function χ_0 obeying $A^-\chi_0 = 0$, then also Eq. (18) is fulfilled. Inversely, the equality $\langle\chi_0|A^+A^-|\chi_0\rangle = 0$ implies that $A^-\chi_0 = 0$. In consequence, the Hamiltonian $H_+ = A^-A^+$ has no normalizable eigenstate with $E = 0$; that is, the family of potentials $V_+(\lambda,\rho) = V_+[\lambda,|f(z)|]$ does not have it either. This is clearly visible from Fig. 2, where we have plotted $V_+(\lambda,\rho)$ and compared it with the effective fish-eye potential $V_-(\lambda,\rho)$

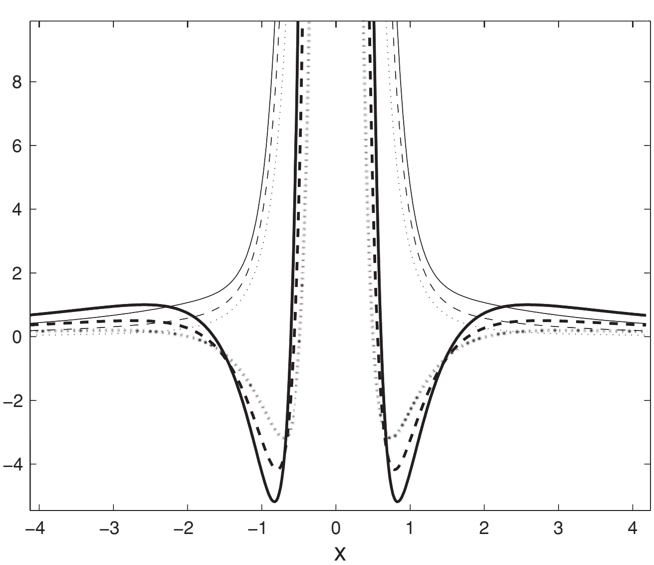

FIG. 2. The $y = 0$ sections of the effective potentials $V_-(\lambda,\rho = \sqrt{x^2+y^2})$ (thick lines) and the corresponding partner potentials $V_+(\lambda,\rho = \sqrt{x^2+y^2})$ (thin lines) for $\lambda = l = 2$ (dotted lines), $l = 3$ (dashed lines), and $l = 4$ (solid lines). The plots are in units of $\hbar^2/(2\mu)$.

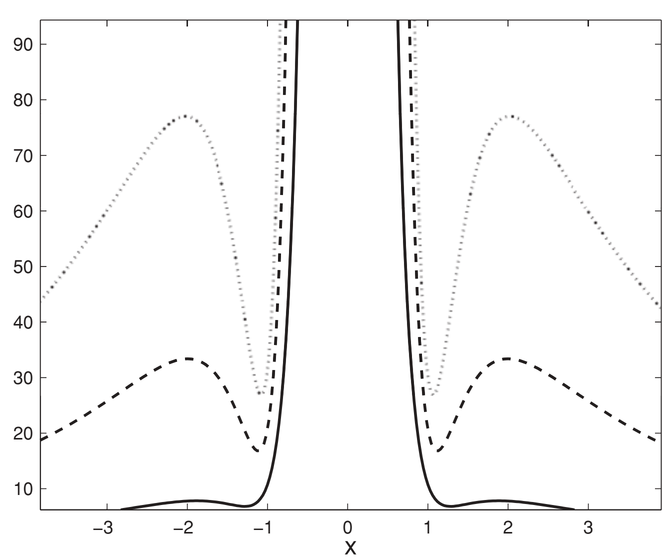

FIG. 3. The $y = 0$ sections of the potentials $V_+(\lambda,\rho = \sqrt{x^2+y^2})$ for $\lambda = l = 10$ (solid lines), $l = 20$ (dashed lines), and $l = 30$ (dotted lines). The plots are in units of $\hbar^2/(2\mu)$.

for three values of $\lambda = l = 2$, 3, and 4. For higher values of λ the curves for $V_+(\lambda,\rho)$ are no longer monotonic functions of x and some structures appear. They are visible in Fig. 3 for $\lambda = 10$, 20, and 30. A similar observation was made earlier in [22].

Commonly discussed in quantum mechanics are usually one-dimensional SUSY problems, or more precisely, formulated in one variable. This is the case, for example, for models with cylindrical or spherical symmetries. It is then a simple matter to derive the SUSY pair potentials. When $V_-[\lambda,|f(z)|]$ does not have this property, the preceding procedure has to be modified. In such a case, as, for example, for the mapping $f(z) = \exp(iz/L)$ discussed in Sec. II, the separability of the problem in the x and y variables allows us only to consider the one-dimensional SUSY partner potentials and then to combine the results. This kind of approach has been considered in [23] for the two-dimensional square well.

IV. CONCLUDING REMARKS

We have derived a large family of the fish-eye-related potentials and showed that they are all soluble in terms of the Gegenbauer polynomials. This is the case for discrete values of the coupling constant in the potentials. The procedure is based on using suitable conformal mappings, which is a known approach with many applications. In connection with the present article, we mention the discussion on fish-eye models in [24] and the article [25], where conformal transformations are utilized to decide whether, and in what way, two problems are related to each other.

Furthermore, we have obtained the SUSY partner potentials to the derived family. It is an open question as to whether they will find any applications in optics, as the fish-eye related ones did. To some extent, similar SUSY considerations, but in a different context, can be found in [22,26].

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