Confinement limit of the Dirac particle in one dimension

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For a particle of mass m that obeys the time-independent Dirac equation in one dimension with a symmetric potential, Unanyan *et al.* [Phys. Rev. A **79**, 044101 (2009)] recently pointed out that the inequality $\Delta x = \sqrt{\langle x^2\rangle} > \lambda/2$ can be derived simply from the Dirac equation. Here $\lambda = \hbar/(mc)$ is the Compton wavelength, x is the particle coordinate, and $\langle x^2\rangle$ is the expectation value of x^2. We conjecture that a new, more stringent limit $\Delta x \geqslant \lambda/\sqrt{2}$ holds for any symmetric potential. We present a model analysis on which the conjecture is based.

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Consider a particle in a stationary state that is described by the one-dimensional Dirac equation,

$$H\psi(x) = E\psi(x), \quad H = c\alpha p + \beta mc^2 + V(x), \tag{1}$$

where $\psi(x)$ is a two-component wave function, c is the speed of light, $p = -i\hbar\, d/dx$, and m is the rest mass of the particle. Here α and β are 2×2 Dirac matrices, which we will specify shortly. The potential $V(x)$ is real and time-independent. Wave function $\psi(x)$ is normalized as

$$\int_{-\infty}^{\infty} \rho(x)dx = 1, \quad \rho(x) = \psi^\dagger(x)\psi(x). \tag{2}$$

By assuming that $V(x)$ is symmetric, that is, $V(x) = V(-x)$, Unanyan *et al.* [1] recently pointed out that the inequality

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} > \lambda/2, \quad \lambda = \hbar/(mc), \tag{3}$$

follows simply from the Dirac equation (1). Here λ is the Compton wavelength and $\langle x^n\rangle = \int_{-\infty}^{\infty}\rho(x)x^n dx$. Since $V(x)$ is symmetric, the density $\rho(x)$ is also symmetric and hence $\langle x\rangle = 0$ and $\Delta x = \sqrt{\langle x^2\rangle}$. It is noteworthy that Unanyan *et al.* did not use the uncertainty principle $\Delta x\Delta p \geqslant \hbar/2$ and that the validity of Eq. (3) does not depend on the explicit form of $V(x)$.

The purpose of this Brief Report is to conjecture a more stringent confinement limit, $\Delta x \geqslant \lambda/\sqrt{2}$. We do this in two steps. First, by way of reexamining Unanyan *et al.*'s derivation of Eq. (3), we find that $\Delta x = \lambda/\sqrt{2}$ holds when $V(x)$ represents a point interaction. Then, we present a model analysis that leads to the conjecture that $\Delta x \geqslant \lambda/\sqrt{2}$ holds for any symmetric potential. The equality applies if and only if the range of the potential is zero.

Let us reexamine Unanyan *et al.*'s derivation of $\Delta x > \lambda/2$. For the Dirac matrices α and β, any two of the 2×2 Pauli matrices σ_x, σ_y, and σ_z can be used. We use $\alpha = \sigma_y$ and $\beta = \sigma_z$ as Unanyan *et al.* did. Then $\alpha p = -i\sigma_y\hbar d/dx$ becomes real and hence $\psi(x)$ can be chosen to be real. Equation (1) can be written as

$$\begin{aligned} -c\hbar v_x + mc^2 u + Vu &= Eu, \\ c\hbar u_x - mc^2 v + Vv &= Ev, \end{aligned} \tag{4}$$

where $\psi = \psi^* = \binom{u}{v}$, $u_x = du/dx$, and $v_x = dv/dx$. From Eq. (4) we can obtain $uv = (\hbar/2mc)(uu_x + vv_x) = (\hbar/4mc)(d\rho/dx)$, and then

$$\int_{-\infty}^{\infty} uvx dx = \frac{\hbar}{4mc}\int_{-\infty}^{\infty}\frac{d\rho}{dx}x dx = -\frac{\lambda}{4}. \tag{5}$$

Note that $\rho = u^2 + v^2 \geqslant 2|uv|$, which leads to

$$\langle |x|\rangle = \int_{-\infty}^{\infty}\rho|x|dx \geqslant 2\left|\int_{-\infty}^{\infty} uvx dx\right| = \frac{\lambda}{2}. \tag{6}$$

Using the Cauchy-Schwarz inequality $\langle x^2\rangle = \langle |x|^2\rangle \geqslant \langle |x|\rangle^2$ and Eq. (6), we obtain

$$\Delta x \geqslant \lambda/2. \tag{7}$$

This result does not depend on $V(x)$. This is how Unanyan *et al.* proceeded.

A question that arises here is whether the *equality* $\Delta x = \lambda/2$ can hold. In order for the equality to hold, we require that $u^2 + v^2 = 2|uv|$. Unanyan *et al.* stated that this equality holds only if $u(x) = v(x)$ and that $u(x) = v(x)$ is incompatible with the Dirac equation; hence $\Delta x = \lambda/2$ does not hold. This statement of Unanyan *et al.*, however, is not quite correct. The equality $u^2 + v^2 = 2|uv|$ only requires $|u| = |v|$. Since $V(x)$ is symmetric, we can assume that $u(x) = u(-x)$ and $v(x) = -v(-x)$. Then $|u| = |v|$ leads to $u(x) = (x/|x|)v(x)$ or $u(x) = -(x/|x|)v(x)$. As we show in the following, such a solution is allowed if (and only if) $E = 0$ and $V(x)$ is a point interaction acting at the origin.

The simplest point interaction is in the form of the Dirac delta function. The point interaction in its most general form can be characterized by means of a boundary condition that relates $u(x)$ and $v(x)$ for $x = 0^+$ to those for $x = 0^-$. The boundary condition can be characterized in terms of three real parameters [2,3]. Such details about the general point interaction, however, are actually unimportant in the following. The Dirac equation with a point interaction can be easily solved. There can be one bound state. We focus on the situation in which the interaction is symmetric, that is, invariant with respect to $x \leftrightarrow -x$. The $u(x)$ and $v(x)$ of the bound state can be chosen as

$$u(x) = N\, e^{-|x|/\lambda_E}, \quad (8)$$

$$v(x) = -N\sqrt{\frac{mc^2 - E}{mc^2 + E}}\, \frac{x}{|x|}\, e^{-|x|/\lambda_E}, \quad (9)$$

where N is the normalization factor, E is the energy eigenvalue, and λ_E is defined by

$$\lambda_E = c\hbar/\sqrt{(mc^2)^2 - E^2}. \quad (10)$$

Note that $-mc^2 < E < mc^2$ and $\lambda_E \geqslant \lambda$. If $V(x)$ is not a point interaction, that is, if $V(x) \neq 0$ for $x \neq 0$, Eq. (4) does not allow a solution such that $u(x) \propto v(v)$.

The normalized density distribution in the bound state is given by

$$\rho(x) = (1/\lambda_E)e^{-2|x|/\lambda_E}. \quad (11)$$

The maximum (strongest) confinement is accomplished when $E = 0$, that is, when the binding energy is mc^2 and $\lambda_E = \lambda$. In that case we obtain

$$(\Delta x)^2 = \langle x^2\rangle = \int_{-\infty}^{\infty} \rho(x)x^2 dx = \lambda^2/2, \quad (12)$$

that is, $\Delta x = \lambda/\sqrt{2}$. When (and only when) $E = 0$, we find that $u(x) = -(x/|x|)v(x)$ and the equality $u^2 + v^2 = 2|uv|$ holds. We agree with Unanyan *et al.* in that the equality $\Delta x = \lambda/2$ cannot hold but the reasoning we have given above is different from that of Unanyan *et al.*

We examine Δx for two models, I and II. In model I, $V(x)$ is a square-well (rectangular) potential, namely,

$$V(x) = -(g/2a)\,\theta(a - |x|), \quad (13)$$

where g and $a > 0$ are constants and $\theta(x) = 1$ (0) if $x > 0$ ($x < 0$). In numerical illustrations, we assume that $g > 0$ such that $V(x) < 0$. Regarding the sign of g, the following observation would be in order. Let H' be the H of Eq. (1) in which V is replaced by $-V$. Then we can obtain $H'\psi' = -E\psi'$, where ψ' is related to ψ by $\psi' = \sigma_x\psi$. Note that $\psi'^\dagger\psi' = \psi^\dagger\psi = \rho$. This means that, when the sign of V is reversed, the density distribution and hence Δx remain the same. The sign of g is unimportant as far as ρ and Δx are concerned.

Functions $u(x)$ and $v(x)$ can, respectively, be chosen as even and odd functions of x. Outside the potential, that is, if $|x| > a$, $u(x)$ and $v(x)$ are both proportional to $\exp[-(|x| - a)/\lambda_E]$. The maximum confinement results when $E = 0$ and $\lambda_E = \lambda$. We focus on such a situation in the following. The Dirac equation with $E = 0$ and the V of Eq. (13) is satisfied by the following $u(x)$ and $v(x)$:

$$u(x) = \begin{cases} N\cos\kappa a\; e^{-(|x|-a)/\lambda} & \text{if } |x| > a, \\ N\cos\kappa x & \text{if } |x| \leqslant a, \end{cases} \quad (14)$$

$$v(x) = \begin{cases} -N\cos\kappa a\;(x/|x|)\,e^{-(|x|-a)/\lambda} & \text{if } |x| > a, \\ -Nc\hbar\kappa\;(x/|x|)\,\sin\kappa x/[mc^2 + (g/2a)] & \text{if } |x| < a, \end{cases} \quad (15)$$

where

$$(c\hbar\kappa)^2 = (g/2a)^2 - (mc^2)^2 \quad (16)$$

and N is the normalization factor that will be given shortly. It is understood that $(g/2a)^2 > (mc^2)^2$ so that κ is real. If this inequality does not hold, there is no solution with $E = 0$. We can assume $\kappa > 0$ without losing generality. Because $V(x)$ is finite, $u(x)$ and $v(x)$ are continuous functions. The function $u(x)$ of Eq. (14) is continuous at $x = \pm a$. On the other hand the continuity of $v(x)$ of Eq. (15) at $x = \pm a$ requires that

$$c\hbar\kappa\tan\kappa a = mc^2 + (g/2a). \quad (17)$$

With the κ that is subject to Eq. (17), we find that $v(x) = -N\cot\kappa a\sin\kappa x$ for $|x| < a$. Note also that $u(x) = -(x/|x|)v(x)$ for $|x| > a$.

The density $\rho(x)$ is given by

$$\rho(x) = N^2\{2\cos^2\kappa a\; e^{-2(|x|-a)/\lambda}\,\theta(|x| - a) + (\cos^2\kappa x + \cot^2\kappa a\sin^2\kappa x)\,\theta(a - |x|)\}, \quad (18)$$

where

$$\frac{1}{N^2} = 2\lambda\cos^2\kappa a + \left(a - \frac{\sin 4\kappa a}{4\kappa}\right)\mathrm{cosec}^2\kappa a. \quad (19)$$

We then obtain

$$\frac{\langle x^2\rangle}{N^2} = \lambda(2a^2 + 2a\lambda + \lambda^2)\cos^2\kappa a + \frac{a^3}{3}\mathrm{cosec}^2\kappa a - \frac{1}{8\kappa^3}\{[2(\kappa a)^2 - 1]\sin 4\kappa a + 4\kappa a\cos^2 2\kappa a\}\mathrm{cosec}^2\kappa a. \quad (20)$$

In the limit of $a \to 0$, we obtain Eqs. (8)–(12). In taking this limit, care has to be exercised as discussed in Refs. [4,5].

For all numerical illustrations we take units such that $c = \hbar = m = \lambda = 1$. For a, we have tried many values between 0 and 1. For an assumed value of a, g is determined by Eq. (17). For example, $g = 1.699118$ when $a = 0.1$. Figure 1 shows that Δx increases as a increases. This is natural. Consequently, $\Delta x > \lambda/\sqrt{2}$ holds for any value of $a > 0$.

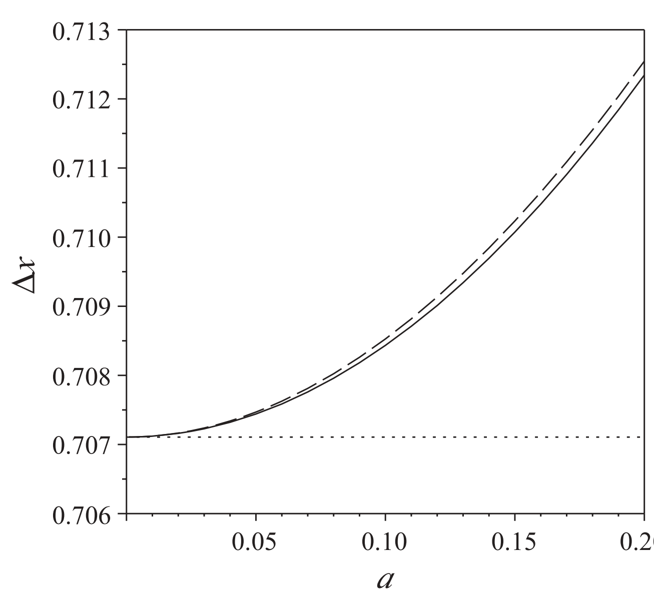

FIG. 1. Δx of model I plotted as a function of a. The solid and dashed lines are, respectively, for the cases with potentials of Eqs. (13) and (21). The dotted line indicates $\Delta x = \lambda/\sqrt{2}$.

In order to examine how Δx depends on the shape of the potential, we examined cases in which the square (rectangular) form is replaced by a smooth form such as Gaussian, Woods-Saxon potential, etc. We found that the relation between Δx and a found in the square-well case remains essentially the same. Δx becomes slightly larger when the potential is made smooth. In Fig. 1 we also show the results obtained with a potential of Gaussian form,

$$V(x) = -(g/\sqrt{\pi}\,a)e^{-(x/a)^2}. \tag{21}$$

Note that $\int_{-\infty}^{\infty} V(x)dx = -g$ for both of the potentials defined by Eqs. (13) and (21).

The particle can be more strongly confined if something like barriers are added outside the square-well potential assumed in model I. In order to explore such a possibility we consider model II with

$$V(x) = -(g/2a)\,\theta(a-|x|) + (h/b)\,\theta(|x|-a) \times \theta(a+b-|x|), \tag{22}$$

where h and $b > 0$ are constants. For the sign of h, we assume that $h > 0$ so that the two terms of $V(x)$ have opposite signs. If we assume that the two terms are of the same sign, then we are essentially making the range of $V(x)$ of model II greater than that of model I. It is obvious that Δx of such a case is greater than its counterpart of model I. We are not interested in such a situation. The Dirac equation with $V(x)$ of Eq. (22) can be handled in a way similar to what we did for model I with the square-well potential of Eq. (13). In choosing the parameters of the potential, we start by assuming some values for a, b, and h. Then we determine g such that there is a bound state with $E = 0$.

If we start with a set of fixed values of a and b and increase h starting with $h = 0$, Δx slightly decreases in the beginning. Beyond a certain value of h, Δx becomes an increasing function of h. The value of Δx always stays above $\lambda/\sqrt{2}$. This feature is illustrated in Fig. 2 in which $a = 0.1$ is combined with $b = 0.005$, 0.01, 0.05, 0.1, and 0.15. We have also examined cases with smaller values of b. Lines for $a = 0.1$ and $b < 0.005$, however, are virtually indistinguishable from the one for $a = 0.1$ and $b = 0.005$. On the other hand, if we start with a set of fixed values of a and h and increase b starting with $b = 0$, Δx slightly decreases in the beginning. Beyond a certain value of b, Δx becomes an increasing function of b. The value of Δx always stays above $\lambda/\sqrt{2}$. Close scrutiny reveals that, for $a = 0.1$, Δx becomes minimum at approximately $b = 0.002$ and $h = 0.25$. The value of g in this case is $g = 2.15709$. This feature remains essentially the same when the value of a is changed.

Figure 3 compares the density $\rho(x)$ of the (square version of) model I and that of model II. In both models we assume $a = 0.1$. In model II b and h are chosen such that Δx becomes approximately minimum. In addition the density given by Eq. (11) with $E = 0$ is shown with a dotted line. We have also examined smoothed versions of model II such as the one in which the rectangular potential is replaced by a potential with two Gaussian functions. We found that the density remains almost the same.

In summary, when $V(x)$ is a symmetric point interaction acting at the origin, we obtain $\Delta x = \lambda/\sqrt{2}$. We examined Δx for a large variety of symmetric, finite-ranged potentials and found that $\Delta x > \lambda/\sqrt{2}$. On the basis of this extensive model analysis we conjecture that $\Delta x \geqslant \lambda/\sqrt{2}$ holds for any symmetric potential. The equality applies if and only if the range of the potential is zero.

Before ending let us mention the confinement problem with a more general form of the Dirac Hamiltonian, that is,

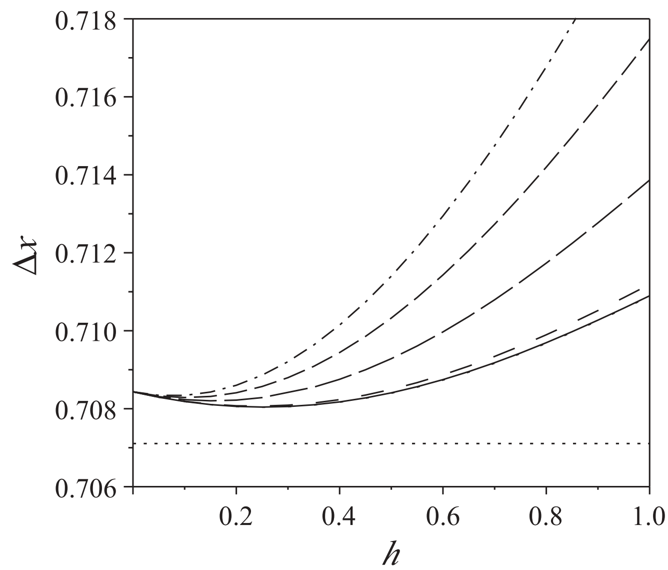

FIG. 2. Δx of model II plotted for a fixed value of $a = 0.1$ and $b = 0.005$ (solid line), 0.01 (spaced-dashed line), 0.05 (long dashed line), 0.1 (dashed line), and 0.15 (dashed-dotted line). Lines for $b < 0.005$ are virtually indistinguishable from that of $b = 0.005$. Δx becomes minimum around $b = 0.002$ and $h = 0.25$. The dotted line indicates $\Delta x = \lambda/\sqrt{2}$.

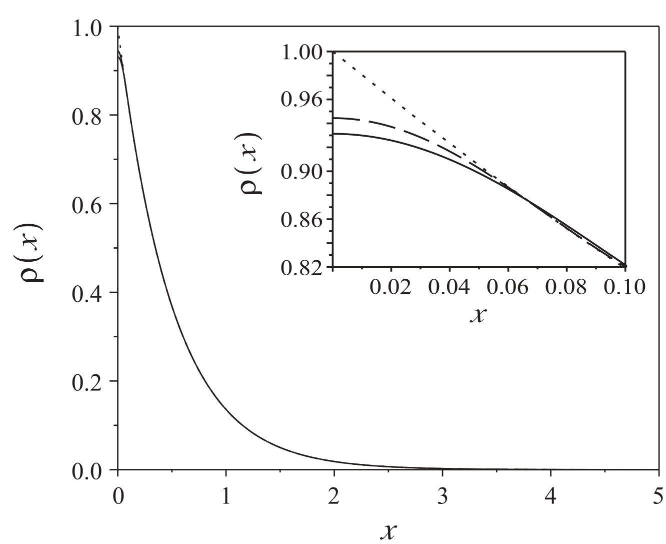

FIG. 3. Comparison of the densities of models I and II. The solid and dashed lines show, respectively, $\rho(x)$ for models I and II. In both models, $a = 0.1$ is assumed. In model II, we assume $b = 0.002$ and $h = 0.25$. For these values of b and h, Δx becomes approximately minimum. In addition, $\rho(x)$ of Eq. (11) with $E = 0$ is shown with a dotted line.

$H = \alpha p + \beta[mc^2 + S(x)] + V(x)$, where $S(x)$ is a real function of x [6–8]. Assume for example that $S(x) = (g_s/2a)\,\theta(|x| - a)$. If we let $g_s \to \infty$, the particle is completely confined in the region of $|x| < a$. The value of a can be chosen arbitrarily small. This is not surprising because we are essentially assuming that the mass of the particle is infinitely large. Such a scalar potential is used for confining quarks within a "bag" (see, e.g., Ref. [9]). There is another type of $S(x)$ with which we can confine a particle to an arbitrarily small region. Assume that $S(x) \propto x$ and $V(x) = 0$. This leads to a solution of the Dirac equation such that density $\rho(x)$ is Gaussian. Its width is related to $S(x)/x$, which is a constant. This constant can be chosen such that the density can be confined to an arbitrarily small region [10].

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[6] Unanyan *et al.*, called $V(x)$ the "scalar potential." We, however, prefer the more commonly used terminology in which $V(x)$ and $S(x)$ are, respectively, referred to as vector and scalar potentials [7,8]. In the three-dimensional case, the V can be the Coulomb potential. In electromagnetic theory, the Coulomb potential is referred to as the "scalar potential" in distinction from the "vector potential," which is related to the magnetic field. In a relativistic theory, however, the Coulomb potential is the time component of a Lorentz vector. Potential S appears together with the rest mass m, which is a Lorentz scalar.

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