Amplification of maximally-path-entangled number states

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We examine the behavior of a non-Gaussian state like the maximally path-entangled number state commonly known as a $N00N$ state under phase-insensitive amplification. We derive an analytical result for the density matrix of the $N00N$ state for arbitrary gain of the amplifier. We consider cases of both symmetric and antisymmetric amplification of the two modes of the $N00N$ state. We quantitatively evaluate the loss of entanglement by the amplifier in terms of the logarithmic negativity parameter. We find that $N00N$ states are more robust than their Gaussian counterparts.

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I. INTRODUCTION

Among the continuous variable (CV) entangled states [1], non-Gaussian states are generally believed to possess much more robust entanglement *vis a vis* the Gaussian states—states characterized by Gaussian quasiprobability distributions and hence by their first and second moments. Though mathematically not as well understood as the Gaussian states [2] in so far as their entanglement properties are concerned, non-Gaussian states, by virtue of the robustness of their entanglement, have in recent years emerged as strong contenders for potential applications in quantum information technology. The fact that the non-Gaussian states are defined by what they are not makes a general discussion of their entanglement properties impossible and one is forced to restrict oneself to subfamilies of non-Gaussian states such as states obtained by adding or subtracting a fixed number of photons to Gaussian states [3] and forming suitable superpositions thereof. One such widely discussed family of non-Gaussian states parameterized by an integer N is the family of the maximally path-entangled number state commonly known as $N00N$ states [4]

$$|N00N\rangle = \frac{1}{\sqrt{2}}[|N,0\rangle + |0,N\rangle]. \quad (1)$$

These states, similar in structure to the Einstein-Podolsky-Rosen (EPR) states, have attracted much attention in recent years and can be viewed as a two-mode state consisting of a superposition of states containing N photons in one mode and none in the other and vice versa. Schemes for reliable production of such states have been proposed [5,6] and their usefulness as a practical tool in making superprecision measurements in optical interferometry, atomic spectroscopy, and in sensing extremely small magnetic fields than hitherto possible have been highlighted [7]. This circumstance makes it imperative to investigate their behavior under attenuation and amplification. While studies on the decoherence effects on $N00N$ states under specific models for system-bath interactions already exist in the literature [8], in the present work we focus on the question of amplification of $N00N$ states and the consequent degradation of entanglement therein and to compare and contrast it with the behavior of entanglement in Gaussian states under the amplification investigated in [9,10]. It is important to understand that the amplification of $N00N$ states as quantum communication protocols would use both amplifiers and attenuators [11]. In the present work we consider only phase-insensitive amplification and model this in the standard way as a bath consisting of N two-level atoms of which N_1 are in the excited state and N_2 the ground state with $N_1 > N_2$. Under the assumptions that atomic transitions have a large width and that the bath is maintained in a steady state, the time evolution of the density operator ρ for a single mode of radiation field on resonance with the atomic transition is described, in the interaction picture, by the master equation

$$\frac{\partial\rho}{\partial t} = -\kappa N_1(aa^\dagger\rho - 2a^\dagger\rho a + \rho aa^\dagger) - \kappa N_2(a^\dagger a\rho - 2a\rho a^\dagger + \rho a^\dagger a), \quad (2)$$

where a and $a^\dagger$ are the annihilation and creation operators of the field mode.

A brief outline of this work is as follows. In Sec. II, with the $N00N$ state as the input to the amplifier, we obtain expressions for the output density operator for the case when both the modes are symmetrically amplified and for the case when only one mode is subjected to amplification and the other is not amplified at all. We investigate how the entanglement in the output state varies with the amplifier gain using logarithmic negativity as the quantifier for entanglement [12]. Section III contains our concluding remarks and further outlook. Our study complements the work done by Vitelli *et al.* [6] on amplification of a $N00N$ state by a phase-sensitive amplifier.

II. EVOLUTION OF THE $N00N$ STATE UNDER PHASE-INSENSITIVE AMPLIFICATION

In our earlier work [9], we found that for a two-mode squeezed vacuum as the input, there are limits on the gain beyond which the output of the amplifier has no entanglement between the two modes and the limiting values of the gain in the two cases considered, symmetric and asymmetric amplification, were found to be

$$G^2 = \left(\frac{2+2\eta}{1+2\eta+e^{-2r}}\right) \text{ symmetric amplification,} \quad (3)$$

$$G^2 = 1 + \frac{1}{\eta} \text{ asymmetric amplification.} \quad (4)$$

where the gain $G = \exp[(N_1 - N_2)\kappa t]$ and $\eta = N_2/(N_2 - N_1)$. Here r is the squeezing parameter for the two-mode squeezed state. In particular, when $\eta \to 0$, the limiting gain in the symmetric case remains finite and that for the asymmetric case the limiting value moves off to infinity [13].

In this section we discuss how an input $N00N$ state evolves under the action of a phase-insensitive amplifier as modeled by the master equation (2) confining ourselves for simplicity to the $\eta \to 0$ limit. The solution of master equation (2) can be written in terms of the Fock-state matrix elements. However, such a solution is rather involved [14]. It is instructive to work in terms of phase-space distributions. A distribution which is especially useful is the Q function introduced by Kano, Sudarshan, and Mehta [15,16]. This function is defined by

$$Q(\alpha) \equiv \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle. \tag{5}$$

The master equation (2) for $\eta = 0$ then leads to

$$\frac{\partial Q}{\partial t} = -G\frac{\partial(\alpha Q)}{\partial \alpha} - G\frac{\partial(\alpha^* Q)}{\partial \alpha^*}. \tag{6}$$

Note that this differential equation for the Q function involves only the first-order derivatives with respect to phase-space variables and hence its solution is simple [17]

$$Q_{\rm in}(\alpha) \equiv \frac{1}{\pi}\langle\alpha|\rho_{\rm in}|\alpha\rangle \to Q_{\rm out}(\alpha) = \frac{1}{G^2}Q_{\rm in}(\alpha/G). \tag{7}$$

Let us see what the result (7) means. Let us consider the input state to be a vacuum, then the output would be

$$\begin{aligned} Q_{\rm in}(\alpha) &= \frac{1}{\pi}\langle\alpha|0\rangle\langle 0|\alpha\rangle \\ &= \frac{1}{\pi}e^{-|\alpha|^2} \to Q_{\rm out}(\alpha) \equiv \frac{1}{\pi G^2}e^{-|\alpha|^2/G^2}. \end{aligned} \tag{8}$$

Such an output Q function is equivalent to a thermal density matrix with the mean number of photons equal to $(G^2 - 1)$. Thus the vacuum state on amplification becomes a thermal state with a mean number of photons that grows with the gain of the amplifier. We note that a result like (6) extends to the multimode case.

We now examine two cases, the symmetric case in which both the modes a and b in the input $N00N$ state are symmetrically amplified and the asymmetric case in which only one mode, say a, is amplified. Before discussing the amplification of the $N00N$ state we examine quantitatively the entanglement in the state (1). We compute the log-negativity parameter which is defined as

$$E_N = \log_2(2N+1),$$

where N is the absolute value of the sum of all the negative eigenvalues of the partial transpose of the density matrix ρ. It is clear that the partial transpose of the density matrix associated with the state (1) is

$$\begin{aligned} \rho^{\rm pt} = \tfrac{1}{2}[&|N,0\rangle\langle N,0| + |0,N\rangle\langle 0,N| \\ &+ |N,N\rangle\langle 0,0| + |0,0\rangle\langle N,N|], \end{aligned} \tag{9}$$

which can be written in the diagonal form as

$$\begin{aligned} \rho^{\rm pt} = &\tfrac{1}{2}(|N,0\rangle\langle N,0| + |0,N\rangle\langle 0,N|) \\ &+ \tfrac{1}{4}(|N,N\rangle + |0,0\rangle)(\langle N,N| + \langle 0,0|) \\ &- \tfrac{1}{4}(|N,N\rangle - |0,0\rangle)(\langle N,N| - \langle 0,0|). \end{aligned} \tag{10}$$

The partial transpose has a negative eigenvalue $-1/2$ and hence the logarithmic negativity parameter $E_N = 1$.

Symmetric case: The Q function corresponding to the density operator for the input $N00N$ state

$$\begin{aligned} \rho_{\rm in} &= \frac{1}{2}[|N,0\rangle\langle N,0| + |N,0\rangle\langle 0,N| \\ &\quad + |0,N\rangle\langle N,0| + |0,N\rangle\langle 0,N|] \\ &= \frac{1}{2N!}[a^{\dagger N}\rho_0 a^N + a^{\dagger N}\rho_0 b^N + b^{\dagger N}\rho_0 a^N + b^{\dagger N}\rho_0 b^N], \end{aligned}$$

$$\rho_0 = |0,0\rangle\langle 0,0|, \tag{11}$$

is found to be

$$\begin{aligned} Q_{\rm in}(\alpha,\beta) &\equiv \frac{1}{\pi^2}\langle\alpha,\beta|\rho_{\rm in}|\alpha,\beta\rangle \\ &= \frac{1}{2N!\pi^2}|\alpha^N + \beta^N|^2\exp[-(|\alpha|^2 + |\beta|^2)]. \end{aligned} \tag{12}$$

Following the prescription in (7), under a symmetric phase-insensitive amplification, the Q function evolves as follows:

$$\begin{aligned} &Q_{\rm in}(\alpha,\beta) \to Q_{\rm out}(\alpha,\beta) \\ &\quad = \frac{1}{G^4}Q_{\rm in}(\alpha/G,\beta/G) = \frac{1}{2N!\pi^2G^{2N+4}}|\alpha^N + \beta^N|^2 \\ &\quad \times \exp[-(|\alpha|^2 + |\beta|^2)/G^2]. \end{aligned} \tag{13}$$

The $N00N$ state is highly nonclassical. A quantitative measure for nonclassicality is obtained by examining zeros of the Q function [18]. We note that the zeros of the function $Q_{\rm out}$ are identical to the zeros of the function $Q_{\rm in}$ and hence we have the remarkable result that the nonclassical character of the input $N00N$ state is preserved.

We can now find the density matrix after amplification by using the results (8) and (13)

$$\begin{aligned} \rho_{\rm in} \to \rho_{\rm out} &= \frac{1}{2N!G^{2N}}[a^{\dagger N}\rho_G a^N + a^{\dagger N}\rho_G b^N \\ &\quad + b^{\dagger N}\rho_G a^N + b^{\dagger N}\rho_G b^N], \\ \rho_G &= \frac{1}{G^4}e^{-\beta(a^\dagger a + b^\dagger b)}, \\ \beta &= \ln\left(\frac{G^2}{G^2-1}\right). \end{aligned} \tag{14}$$

We note that the structure of (14) is such that it cannot be written in a separable form. This is seen more clearly if we write (14) as

$$\rho_{\rm out} = \frac{1}{2N!G^{2N}}\{a^{\dagger N} + b^{\dagger N}\}\rho_G\{a^N + b^N.\} \tag{15}$$

We further note that the output state has the structure of a two-mode photon-added thermal state in which either mode has added photons. The single-mode version of the photon-added thermal state was introduced by Agarwal and Tara [19]. These states have been experimentally studied recently [20].

Writing ρ_G in the number state basis as

$$\rho_G = \frac{1}{G^4}\sum_{n,m=0}^{\infty}\left(\frac{G^2-1}{G^2}\right)^{n+m}|n,m\rangle\langle n,m|, \tag{16}$$

we have

$$\rho_{\text{out}} = \frac{1}{2N!G^{2N+4}} \sum_{n,m=0}^{\infty} \left(\frac{G^2-1}{G^2}\right)^{n+m} \times [a^{\dagger N}|n,m\rangle\langle n,m|a^N + a^{\dagger N}|n,m\rangle\langle n,m|b^N + b^{\dagger N}|n,m\rangle\langle n,m|a^N + b^{\dagger N}|n,m\rangle\langle n,m|b^N]$$
$$= \frac{1}{2N!G^{2N+4}} \sum_{n,m=0}^{\infty} \left(\frac{G^2-1}{G^2}\right)^{n+m} \times \left[\frac{(n+N)!}{n!}|n+N,m\rangle\langle n+N,m| + \frac{(m+N)!}{m!}|n,m+N\rangle\langle n,m+N| + \sqrt{\frac{(n+N)!}{n!}\frac{(m+N)!}{m!}}(|n+N,m\rangle\langle n,m+N| + |n,m+N\rangle\langle n+N,m|)\right], \tag{17}$$

which immediately gives us the expression for the operator $\rho_{\text{out}}^{\text{PT}}$ obtained by partially transposing ρ_{out} (with respect to the b mode):

$$\rho_{\text{out}}^{\text{PT}} = \frac{1}{2N!G^{2N+4}} \sum_{n,m=0}^{\infty} \left(\frac{G^2-1}{G^2}\right)^{n+m} \times \left[\frac{(n+N)!}{n!}|n+N,m\rangle\langle n+N,m| + \frac{(m+N)!}{m!}|n,m+N\rangle\langle n,m+N| + \sqrt{\frac{(n+N)!}{n!}\frac{(m+N)!}{m!}}(|n+N,m+N\rangle\langle n,m| + |n,m\rangle\langle n+N,m+N|)\right]$$
$$= \frac{1}{2N!G^{2N}}[a^{\dagger N}\rho_G a^N + b^{\dagger N}\rho_G b^N + a^{\dagger N}b^{\dagger N}\rho_G + \rho_G a^N b^N]. \tag{18}$$

The object of interest now is to calculate the logarithmic negativity $E_{\mathcal{N}}$, the sum of the logarithmic negativity eigenvalues of $\rho_{\text{out}}^{\text{PT}}$ and to see how it varies as a function of G^2. We carry out this task numerically and the results are displayed in Fig. 1 where we plot $E_{\mathcal{N}}$ as a function of G^2 for $N = 2$, 4, and 6. In Fig. 1 we also show the log-negativity parameter for the two-mode squeezed vacuum state (taken from Ref. [9].) as well as the state for $N = 1$. The state (1) for $N = 1$ is especially relevant for Mach-Zehnder interferometers [21,22] based on single photon input. Clearly the $N00N$ state is much more robust under amplification.

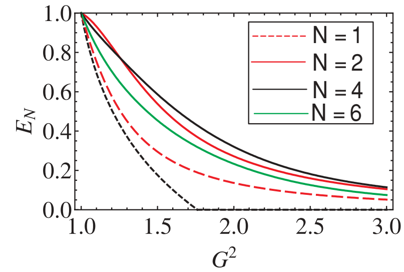

FIG. 1. (Color online) Behavior of the logarithmic negatively as a function of G^2 for the symmetric case. The dashed curve (black) gives the normalized log-negativity parameter for the two-mode squeezed vacuum state under amplification for squeezing parameter r.

Asymmetric case: Proceeding as above, one finds that

$$Q_{\text{in}}(\alpha,\beta) \to Q_{\text{out}}(\alpha,\beta) = \frac{1}{G^2}Q_{\text{in}}(\alpha/G,\beta) = \frac{1}{2N!\pi^2G^{N+2}}|(\alpha/G)^N + \beta^N|^2 \times \exp[-(|\alpha|^2/G^2 + |\beta|^2)], \tag{19}$$

and hence

$$\rho_{\text{in}} \to \rho_{\text{out}} = \frac{1}{2N!G^{2N}}[a^{\dagger N}\tilde{\rho}a^N + G^N a^{\dagger N}\tilde{\rho}b^N + G^N b^{\dagger N}\tilde{\rho}a^N + G^{2N}b^{\dagger N}\tilde{\rho}b^N]$$
$$\tilde{\rho} = \frac{1}{G^2}e^{-\beta(a^\dagger a)}|0\rangle\langle 0|;\ \beta = \ln\left(\frac{G^2}{G^2-1}\right). \tag{20}$$

Writing $\tilde{\rho}$ as

$$\tilde{\rho} = \frac{1}{G^2}\sum_{n=0}^{\infty}\left(\frac{G^2-1}{G^2}\right)^n |n,0\rangle\langle n,0|, \tag{21}$$

we can write ρ_{out} in terms of number states as

$$\rho_{\text{out}} = \frac{1}{2N!G^{2N+2}}\sum_{n=0}^{\infty}\left(\frac{G^2-1}{G^2}\right)^n \times [a^{\dagger N}|n,0\rangle\langle n,0|a^N + G^N a^{\dagger N}|n,0\rangle\langle n,0|b^N + G^N b^{\dagger N}|n,0\rangle\langle n,0|a^N + G^{2N}b^{\dagger N}|n,0\rangle\langle n,0|b^N]$$
$$= \frac{1}{2N!G^{2N+2}}\sum_{n=0}^{\infty}\left(\frac{G^2-1}{G^2}\right)^n \times \left[\frac{(n+N)!}{n!}|n+N,0\rangle\langle n+N,0| + G^{2N}N!|n,N\rangle\langle n,N| + G^N\sqrt{\frac{(n+N)!}{n!}}N!(|n+N,0\rangle\langle n,N| + |n,N\rangle\langle n+N,0|)\right], \tag{22}$$

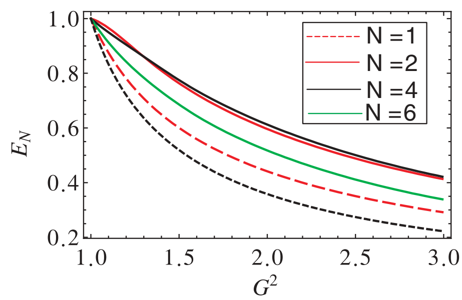

FIG. 2. (Color online) Behavior of the logarithmic negatively as a function of G^2 for the asymmetric case. The dashed curve (black) is the result for the two-mode squeezed vacuum.

and hence

$$\begin{aligned}\rho_{\rm out}^{\rm PT} &= \frac{1}{2N!G^{2N+2}}\sum_{n=0}^{\infty}\left(\frac{G^2-1}{G^2}\right)^n \\ &\quad\times\left[\frac{(n+N)!}{n!}|n+N,0\rangle\langle n+N,0| \right.\\ &\quad + G^{2N}N!|n,N\rangle\langle n,N| \\ &\quad + G^N\sqrt{\frac{(n+N)!}{n!}}N!(|n+N,N\rangle\langle n,0| \\ &\quad \left. + |n,0\rangle\langle n+N,N|)\right] \\ &= \frac{1}{2N!G^{2N}}[a^{\dagger N}\tilde{\rho}a^N + G^{2N}b^{\dagger N}\tilde{\rho}b^N \\ &\quad + G^N a^{\dagger N}b^{\dagger N}\tilde{\rho} + G^N\tilde{\rho}a^N b^N]. \end{aligned} \tag{23}$$

The logarithmic negativity of $\rho_{\rm out}^{\rm PT}$ is computed numerically and the results are shown in Fig. 2 for $N = 1$, 2, 4, and 6. Clearly in the asymmetric case the loss of entanglement is much slower. Again, entanglement in the $N00N$ state is more robust then that in the squeezed vacuum state. Finally, we compare the amplification of the $N00N$ state with that of a photon-added two-mode squeezed vacuum state (i.e., the state, for brevity the normalization factors are ignored)

$$|\Phi\rangle \propto a^\dagger b^\dagger \exp\{\zeta a^\dagger b^\dagger - \zeta^* ab\}|00\rangle, \quad \zeta = r. \tag{24}$$

This is a non-Gaussian state. The input and output Q functions are found to be

$$Q_{\rm in} \propto |\alpha|^2|\beta|^2 Q_{\rm sq}, \tag{25}$$

$$Q_{\rm out} \propto \frac{|\alpha|^2|\beta|^2}{G^4} Q_{{\rm sq},G} \tag{26}$$

where $Q_{\rm sq}$ is the Q function for the two-mode squeezed vacuum and $Q_{{\rm sq},G}$ is the Q function obtained by amplification of the squeezed vacuum. Thus the density operator after amplification of the non-Gaussian state can be written as

$$\rho_{\rm out} \propto a^\dagger b^\dagger(\rho_{\rm sq\,out})ab, \tag{27}$$

where $\rho_{\rm sq\,out}$ is the density operator for the squeezed vacuum after amplification. Now $\rho_{\rm sq\,out}$ becomes separable for G greater than that given by (3) and hence $\rho_{\rm out}$ becomes separable if $G^2 > (2+2\eta)/(1+2\eta+e^{-2r})$. Thus the non-Gaussian states obtained from Gaussian states by the addition of photons would behave under symmetric amplification in a manner similar to Gaussian states. We have therefore found that the $N00N$ states behave quite differently under amplification.

III. CONCLUSION

In conclusion, we found that the $N00N$ states are more robust under amplification than their Gaussian counterparts such as a two-mode squeezed vacuum produced by a downconverter. We presented results for the logarithmic negativity as a function of the gain of the amplifier. We presented numerical results for cases of states which have already been realized experimentally. We also found that the $N00N$ state does better than say a photon-added two-mode squeezed vacuum state.

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