Orthopositronium decay form factors and two-photon correlations

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We give results for the orthopositronium decay form factors through one-loop order. We use the form factors to calculate momentum correlations of the final-state photons $\langle k_{Ai} k_{Bj} \rangle$ and $\langle \hat{k}_{Ai} \hat{k}_{Bj} \rangle$, including one-loop corrections, for ensembles of initial orthopositronium atoms having arbitrary polarization.

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I. INTRODUCTION

Positronium physics has been a rich source of tests of QED and the discrete symmetries C, P, and T, singly and in various combinations. (For reviews see [1–4].) Positronium is described almost completely by pure QED—standard model strong and weak interaction effects are small [5–8]. Any deviations from QED predictions for positronium properties at present and proposed levels of experimental sensitivity would indicate new physics in the leptonic sector. Positronium is relatively easy to prepare and study and so forms an attractive system to use in the search for new physics.

In this work we will focus on the three-photon decay of the spin-1, $C = -1$ state orthopositronium (ortho-Ps). The lowest order ortho-Ps decay rate contribution was calculated by Ore and Powell [9] in 1949 as

$$\Gamma_0 = \frac{2}{9\pi}(\pi^2 - 9)m\alpha^6. \tag{1}$$

The $O(\alpha)$ correction was computed with increasing precision over the years [10–13], culminating with the analytic evaluation in 2008 [14]. The analytic expression is too long to quote here, but the numerical value to seventeen digits is

$$\gamma_1 = -10.286\,614\,808\,628\,262 \tag{2}$$

in units of $(\alpha/\pi)\Gamma_0$. The corrections to Γ_0 of relative orders $\alpha^2 \ln \alpha$ [11], α^2 [15,16], $\alpha^3 \ln^2 \alpha$ [17], and $\alpha^3 \ln \alpha$ [18–20] have all been worked out. The calculated ortho-Ps decay rate is in excellent agreement with the latest experimental results [21–23].

Our goals for this paper are twofold. First, we will give explicit forms for the three $O(\alpha)$ ortho-Ps $\to 3\gamma$ form factors. These form factors are in analytic form, and with their use all one-loop ortho-Ps decay calculations are greatly simplified. For example, the calculation of γ_1 in [14] was set up as a two-dimensional integration of a function found by use of these form factors. The ortho-Ps decay form factors were employed earlier in high-precision numerical calculations of γ_1 and in obtaining some contributions to the $O(\alpha^2)$ decay-rate correction [13,24], but explicit forms were not given then due to their length. Now the form factors are available on the EPAPS electronic depository [25].

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Our second goal for this paper is to give results for a number of two-photon momentum correlations of the form $\langle k_{Ai} k_{Bj} \rangle$ and $\langle \hat{k}_{Ai} \hat{k}_{Bj} \rangle$ among photons produced in ortho-Ps $\to 3\gamma$ decay for ortho-Ps ensembles of arbitrary polarization. Our results include the effects of one-loop radiation corrections. These correlations include but are more general than the cross-product average $\langle (\vec{k}_A \times \vec{k}_B)_a \rangle = \epsilon_{aij} \langle k_{Ai} k_{Bj} \rangle$ that has been used for tests of CPT [5,26–32].

This paper is organized as follows. In Sec. II we give a general discussion of the ortho-Ps decay calculation for ortho-Ps ensembles of arbitrary polarization. In Sec. III we define the ortho-Ps $\to 3\gamma$ form factors and give an EPAPS link where they may be found. In Sec. IV we show how the form factors can be used to calculate one-loop corrections, specifically, the one-loop decay rate correction γ_1. In Sec. V we present the calculations and results for the two-photon momentum correlations at lowest and one-loop orders. Finally, in Sec. VI we give a discussion of our results.

II. ORTHOPOSITRONIUM DECAY

The expression for the decay rate of ortho-Ps to three photons is

$$\Gamma = \frac{1}{3!}\frac{1}{2(2W)} \int \frac{d^3k_1}{(2\pi)^3 2\omega_1}\frac{d^3k_2}{(2\pi)^3 2\omega_2}\frac{d^3k_3}{(2\pi)^3 2\omega_3} \times (2\pi)^4 \delta(P - k_1 - k_2 - k_3)\overline{|M|^2}, \tag{3}$$

where $P = (2W, \vec{0}\,)$ in the ortho-Ps rest frame, W is half the ortho-Ps mass, $\omega_i = k_i^0 = |\vec{k}_i|$ is the ith photon energy, and $\overline{|M|^2}$ is the decay amplitude squared and summed over final-state photon polarizations. The initial ortho-Ps ensemble is described by a density matrix [26]

$$\rho = \sum_{ij} |\text{Ps},i\rangle \rho_{ij} \langle \text{Ps},j|, \tag{4}$$

where $|\text{Ps},i\rangle$ is the ortho-Ps state with polarization given by the ith Cartesian basis vector $\hat{e}_i$. The density matrix elements can be expressed as

$$\rho_{ij} = \frac{1}{3}\delta_{ij} - \frac{i}{2}\epsilon_{ijk} s_k - t_{ij}, \tag{5}$$

where $t_{ij} = t_{ji}$, $t_{ii} = 0$. (We make use of the summation convention to sum over repeated indices.) The quantities s_k and t_{ij} measure the vector and tensor polarizations present in the initial ensemble of ortho-Ps atoms. For example, the $m = 0$ pure state has

$$\vec{s} = 0, \quad t = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -2/3 \end{pmatrix}, \tag{6}$$

and the $m = \pm 1$ pure states have

$$\vec{s} = \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}, \quad t = \begin{pmatrix} -1/6 & 0 & 0 \\ 0 & -1/6 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}. \tag{7}$$

The spin-averaged mixed state has $\vec{s} = \vec{0}$ and $t_{ij} = 0$.

We will use the energy-momentum conserving delta function to perform four of the nine phase-space integrals. The five remaining integrals will be separated into three over Euler angles and two describing the relative orientation of the three photons in the decay plane. To start with, we evaluate the $\vec{k}_3$ integral using the three-momentum δ function to find

$$\Gamma = \frac{1}{3!4W} \int \frac{d^3k_1\, d^3k_2}{(2\pi)^6 8\omega_1\omega_2\omega_3}(2\pi) \times \delta(2W - \omega_1 - \omega_2 - \omega_3)\overline{|M|^2}. \tag{8}$$

The remaining integration variables will be parametrized as follows. We form the momentum vectors $\vec{k}_1, \vec{k}_2$ by first placing the decay plane in the $x'y'$ plane with photon 1 along the x' axis and photon 2 at angle β:

$$\hat{k}_1' = (1,0,0), \quad \hat{k}_2' = (\cos\beta, \sin\beta, 0). \tag{9}$$

The xyz frame is obtained from this $x'y'z'$ frame by an Euler angle rotation $R_E = R_{z'}(\chi)R_{x'}(\theta)R_{z'}(\phi)$. The momentum unit vectors in the xyz frame are

$$\hat{k}_1 = R_E\hat{k}_1', \quad \hat{k}_2 = R_E\hat{k}_2', \quad \hat{k}_3 = \frac{-(\omega_1\hat{k}_1 + \omega_2\hat{k}_2)}{|\omega_1\hat{k}_1 + \omega_2\hat{k}_2|}. \tag{10}$$

The volume elements are related by

$$d^3k_1\, d^3k_2 = \omega_1^2\omega_2^2 d\omega_1 d\omega_2 \sin\beta d\beta d\omega, \tag{11}$$

where

$$d\omega = d\chi\, \sin\theta d\theta\, d\phi \tag{12}$$

is the volume element in the space of Euler angles. The β integral can be done by using the energy-conserving delta function. We write $x_i = \omega_i/W$, in terms of which energy conservation states

$$x_1 + x_2 + x_3 = 2. \tag{13}$$

Moreover, momentum conservation

$$\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = \vec{0} \tag{14}$$

allows us to express the angle α_{ij} between any two photons, say i and j, in the decay plane in terms of the x_i as

$$\cos(\alpha_{ij}) = \hat{k}_i \cdot \hat{k}_j = 1 - \frac{2\bar{x}_k}{x_i x_j} \tag{15}$$

among many equivalent forms, where $(ijk) = (123)$ or a permutation thereof, and $\bar{x}_k \equiv 1 - x_k$. The decay rate is

$$\Gamma = \frac{W}{768\pi^3} \int \frac{d\omega}{8\pi^2} \int d\Delta\overline{|M|^2}, \tag{16}$$

where $\int d\Delta$ is the integral over a triangular region in the x_1x_2 plane:

$$0 \leqslant x_1 \leqslant 1, \quad 1 - x_1 \leqslant x_2 \leqslant 1. \tag{17}$$

We note that W, half of the ortho-Ps mass, is nearly equal to m: $W = m\left[1 + O(\alpha^2)\right]$. In our work here, which is only through $O(\alpha)$ corrections, we can always take $W \to m$.

Correlations such as $\langle k_{Ai}k_{Bj}\rangle$ can be calculated according to

$$\langle O \rangle = \frac{\int d\Delta \int \frac{d\omega}{8\pi^2} Op}{\int d\Delta \int \frac{d\omega}{8\pi^2} p}, \tag{18}$$

where p is a probability density proportional to $\overline{|M|^2}$. We label the photons according to the magnitudes of their energies, so that A and B represent one of I, II, or III with $\omega_{\rm I} > \omega_{\rm II} > \omega_{\rm III}$. We show that the denominator of (18) is proportional to the spin-averaged decay rate. This is so because p is proportional to the photon-polarization-averaged decay amplitude squared $\overline{|M|^2}$, and $\overline{|M|^2}$ contains a factor of ρ_{ij}. It follows that p can be written as a sum of terms representing the spin-averaged contribution, the vector polarization contribution, and the tensor polarization contribution:

$$p = p_R + p_S + p_T. \tag{19}$$

The spin-averaged part, which we call p_R, depends only on the dot products of the $\vec{k}_i$, so p_R can be expressed as a function of the x_i only with no Euler angle dependence. The vector polarization part p_S is proportional to $\epsilon_{ijk}s_k$ with two free indices, so it involves terms like $k_{1i}k_{2j}$ that do depend on Euler angles. Likewise, the tensor polarization part p_T is proportional to t_{ij} and involves terms like $k_{1i}k_{1j}$ and $k_{1i}k_{2j}$, etc., and depends on the Euler angles. The denominator of (18) is integrated over Euler angles, and $\int d\omega\, k_{ai}k_{bj} \propto \delta_{ij}$, so the $\epsilon_{ijk}s_k$ and t_{ij} terms do not contribute. The denominator of (18) is just proportional to the spin-averaged decay rate, which we obtain in Sec. IV.

III. ORTHOPOSITRONIUM DECAY FORM FACTORS

In this section we describe the calculation of the ortho-Ps decay form factors and present the results. These form factors were obtained and used earlier [13,24] but not explicitly written there due to their length. We now make them available through the EPAPS service [25].

Orthopositronium decay is governed by the decay matrix element, which can be written as

$$M = \epsilon^*_{1\mu_1}\, \epsilon^*_{2\mu_2}\, \epsilon^*_{3\mu_3}\, \epsilon_\alpha\, M^{\mu_1\mu_2\mu_3\alpha}(k_1,k_2,k_3) \tag{20}$$

for the decay of a state with polarization ϵ to three photons having polarizations ϵ_1, ϵ_2, ϵ_3. It can be shown, based on Lorentz symmetry, gauge invariance, and Bose statistics, that the decay tensor can be expressed as

$$M^{\mu_1\mu_2\mu_3\alpha}(k_1,k_2,k_3) = \sum_{S_3} \mathcal{M}^{\mu_1\mu_2\mu_3\alpha}(k_1,k_2,k_3), \tag{21}$$

where the sum is over the six photon permutations [24,33]. The tensor $\mathcal{M}^{\mu_1\mu_2\mu_3\alpha}$ can be written in terms of three scalar functions A_1, A_2, A_3 as

$$\begin{aligned}
&\mathcal{M}^{\mu_1\mu_2\mu_3\alpha}(k_1,k_2,k_3)\\
&= A_1(k_1,k_2,k_3)\frac{1}{k_1k_3}\left(\frac{k_3^{\mu_1}k_1^{\mu_3}}{k_1k_3}-g^{\mu_1\mu_3}\right)\\
&\times k_1^{\alpha}\left(\frac{k_3^{\mu_2}}{k_2k_3}-\frac{k_1^{\mu_2}}{k_1k_2}\right)+A_2(k_1,k_2,k_3)\\
&\times\left\{\frac{1}{k_2k_3}\left(\frac{k_1^{\alpha}k_3^{\mu_1}}{k_1k_3}-g^{\alpha\mu_1}\right)\left(\frac{k_1^{\mu_2}k_2^{\mu_3}}{k_1k_2}-g^{\mu_2\mu_3}\right)\right.\\
&\left.+\frac{1}{k_1k_3}\left(\frac{k_1^{\mu_2}}{k_1k_2}-\frac{k_3^{\mu_2}}{k_2k_3}\right)\left(k_1^{\mu_3}g^{\alpha\mu_1}-k_1^{\alpha}g^{\mu_1\mu_3}\right)\right\}\\
&+A_3(k_1,k_2,k_3)\frac{1}{k_1k_3}\left(\frac{k_1^{\alpha}k_3^{\mu_1}}{k_1k_3}-g^{\alpha\mu_1}\right)\\
&\times\left(\frac{k_3^{\mu_2}k_2^{\mu_3}}{k_2k_3}-g^{\mu_2\mu_3}\right).
\end{aligned}\tag{22}$$

The scalar functions A_i depend only on scalar products of the k_i and so are really functions of x_1, x_2, x_3 (of which only two are independent). It is convenient to define form factors by removing a common factor from the A_i according to

$$A_i(k_1,k_2,k_3) = A_0F_i(x_1,x_2,x_3),\tag{23}$$

where

$$A_0 = 16i\pi m^2\alpha^3\frac{\bar{x}_1\bar{x}_2\bar{x}_3}{x_1x_2x_3}.\tag{24}$$

Explicit calculation shows that the lowest order (in α) contributions to F_1, F_2, and F_3 are 0, 1, and 0, respectively [24]. The expansions in α of the form factors F_i have the form

$$F_1 = 0+\frac{\alpha}{\pi}F_1^{(1)}+O(\alpha^2),\tag{25a}$$

$$F_2 = 1+\frac{\alpha}{\pi}F_2^{(1)}+O(\alpha^2),\tag{25b}$$

$$F_3 = 0+\frac{\alpha}{\pi}F_3^{(1)}+O(\alpha^2).\tag{25c}$$

The one-loop form factors are linear combinations of the functions $h_1(x_i)$, ..., $h_5(x_i)$, and $h_6(x_i,x_j)$ times rational functions of the x_i [34], where

$$h_1(x_1) = \ln(2x_1),\tag{26a}$$

$$h_2(x_1) = \sqrt{\frac{x_1}{\bar{x}_1}}\theta_1,\tag{26b}$$

$$h_3(x_1) = \frac{1}{2x_1}\{\zeta(2)-\mathrm{Li}_2(1-2x_1)\},\tag{26c}$$

$$h_4(x_1) = \frac{1}{2x_1}\left\{\left(\frac{\pi}{2}\right)^2-\theta_1^2\right\},\tag{26d}$$

$$h_5(x_1) = \frac{1}{2\bar{x}_1}\theta_1^2,\tag{26e}$$

$$h_6(x_1,x_3) = \frac{1}{\sqrt{x_1\bar{x}_1x_3\bar{x}_3}}\{\mathrm{Li}_2(r^+,\bar{\theta}_1)-\mathrm{Li}_2(r^-,\bar{\theta}_1)\},\tag{26f}$$

with

$$\theta_1 = \arctan(\sqrt{\bar{x}_1/x_1}),\tag{27a}$$

$$\bar{\theta}_1 = \arctan(\sqrt{x_1/\bar{x}_1}),\tag{27b}$$

$$r^{\pm} = \sqrt{\bar{x}_1}\left(1\pm\sqrt{\frac{x_1\bar{x}_3}{\bar{x}_1x_3}}\right).\tag{27c}$$

The dilogarithm functions used here are defined by Lewin [35]. Explicit results for the $O(\alpha)$ form factors are given in EPAPS as text files in a form readable by MATHEMATICA, in FORTRAN and C formats, and as a MATHEMATICA notebook [25].

IV. ORTHOPOSITRONIUM DECAY RATE THROUGH ONE-LOOP ORDER

In this section we describe the calculation of the ortho-Ps decay rate. Up to a factor, this decay rate gives us the denominator of expression (18) for the two-photon correlations. As described at the end of Sec. II, the S and T parts of the density matrix do not affect the decay rate, so

$$\Gamma = \frac{W}{768\pi^3}\int d\Delta\overline{|M|^2_R},\tag{28}$$

where R indicates the spin-averaged part of $\overline{|M|^2}$. The Euler angle integral is trivial because the integrand is independent of Euler angles. We obtain $\overline{|M|^2}$ by taking the complex square of M [in (20)]. The result, for physical (spatial) photon and ortho-Ps polarization vectors, is

$$\overline{|M|^2} = \sum_{\epsilon_1}\epsilon_{1i_1}\epsilon^*_{1j_1}\sum_{\epsilon_2}\epsilon_{2i_2}\epsilon^*_{2j_2}\sum_{\epsilon_3}\epsilon_{3i_3}\epsilon^*_{3j_3}\rho_{ij}M^{*i_1i_2i_3i}M^{j_1j_2j_3j}.\tag{29}$$

The photon polarization sums are done by using the transverse delta function

$$\sum_{\epsilon}\epsilon_i\epsilon_j^* = \delta^T_{ij}(\hat{k}) = \delta_{ij}-\hat{k}_i\hat{k}_j,\tag{30}$$

so

$$\overline{|M|^2} = \delta^T_{i_1j_1}(\hat{k}_1)\delta^T_{i_2j_2}(\hat{k}_2)\delta^T_{i_3j_3}(\hat{k}_3)\rho_{ij}M^{*i_1i_2i_3i}M^{j_1j_2j_3j}.\tag{31}$$

Here $M^{i_1i_2i_3i}$ is given in terms of the form factors by (21)–(24). For the lowest order spin-averaged contribution, we find

$$\overline{|M|^2_{R0}} = \frac{2^9\pi^2\alpha^6}{3}\left\{\left(\frac{\bar{x}_1}{x_2x_3}\right)^2+\left(\frac{\bar{x}_2}{x_3x_1}\right)^2+\left(\frac{\bar{x}_3}{x_1x_2}\right)^2\right\}.\tag{32}$$

The lowest order decay rate comes by integration:

$$\Gamma_0 = \frac{m}{768\pi^3}\int_0^1dx_1\int_{1-x_1}^1dx_2\overline{|M|^2_{R0}} = \frac{2}{9\pi}(\pi^2-9)m\alpha^6,\tag{33}$$

which is the result of Ore and Powell [9]. The spin-averaged contribution to $\overline{|M|^2}$ at one-loop order can be expressed in terms of the one-loop form factors as

$$\begin{aligned}
\overline{|M|^2_{R1}} = \frac{2}{3m^4}|A_0|^2\sum_{S_3}&\left\{\frac{-1}{\bar{x}_1\bar{x}_3}F_1^{(1)}(x_1,x_2,x_3)\right.\\
&+\frac{(x_2\bar{x}_2+x_3\bar{x}_3)}{\bar{x}_1^2\bar{x}_2\bar{x}_3}F_2^{(1)}(x_1,x_2,x_3)\\
&\left.+\frac{x_1}{\bar{x}_2^2\bar{x}_3}F_3^{(1)}(x_1,x_2,x_3)\right\}.
\end{aligned}\tag{34}$$

We set up a numerical integration of $\overline{|M|^2_{R1}}$ using MATHEMATICA and easily obtained high precision. We found it convenient to break the integration region into the six parts connected by the S_3 symmetry—and then just integrate over one of them. We used the region specified by $\omega_1 \geqslant \omega_2 \geqslant \omega_3$. We then broke the integration region into parts inside of which the various denominators present in the one-loop form factors (such as $1-2x_3$) are not singular. (We note, despite the dangerous denominator factors, that the one-loop form factors are continuous functions.) Specifically, we evaluated the integration according to

$$\Gamma_1 = \frac{m}{768\pi^3}\int d\Delta \overline{|M|^2_{R1}} = \frac{m}{768\pi^3} 3! \left\{ \int_0^{1/2} dx_3 \int_{1-x_3}^{1-x_3/2} dx_2 + \int_{1/2}^{2/3} dx_3 \int_{x_3}^{1-x_3/2} dx_2 \right\} \overline{|M|^2_{R1}}. \quad (35)$$

The result, to 17 digits, is $\Gamma_1 = \gamma_1 \Gamma_0$, where γ_1 is given in (2). This result is consistent with earlier, less precise, evaluations [10,12,13] and with the recent analytic result of Kniehl, Kotikov, and Veretin [14].

V. TWO-PHOTON MOMENTUM CORRELATIONS

In this section we describe the calculation of two-photon momentum correlations and give our results. According to (18), the correlations $\langle k_{Ai} k_{Bj} \rangle$ can be written as

$$\langle k_{Ai} k_{Bj} \rangle = \frac{1}{D} \langle k_{Ai} k_{Bj} \rangle^N, \quad (36)$$

where the numerator is

$$\langle k_{Ai} k_{Bj} \rangle^N = \int d\Delta \int \frac{d\omega}{8\pi^2}\, p\, k_{Ai} k_{Bj} \quad (37)$$

and the denominator is

$$D = \int d\Delta \int \frac{d\omega}{8\pi^2} p = \int d\Delta \int \frac{d\omega}{8\pi^2} p_R = \int d\Delta p_R. \quad (38)$$

[The S and T contributions to the denominator D vanish as discussed after Eq. (18).] We have chosen to normalize p so that the lowest order term is exactly one:

$$D = \int d\Delta p_R = \frac{\Gamma}{\Gamma_0} = 1 + \frac{\alpha}{\pi}\gamma_1 + O(\alpha^2). \quad (39)$$

The numerator factor in the correlation is

$$\langle k_{Ai} k_{Bj} \rangle^N = \langle k_{Ai} k_{Bj} \rangle^N_0 + \frac{\alpha}{\pi} \langle k_{Ai} k_{Bj} \rangle^N_1 + O(\alpha^2) \quad (40)$$

so that

$$\langle k_{Ai} k_{Bj} \rangle = \langle k_{Ai} k_{Bj} \rangle^N_0 + \frac{\alpha}{\pi}\left\{ \langle k_{Ai} k_{Bj} \rangle^N_1 - \gamma_1 \langle k_{Ai} k_{Bj} \rangle^N_0 \right\} + O(\alpha^2). \quad (41)$$

There are two nonvanishing contributions to the correlations through one-loop order, the spin-averaged part with $\rho_{ij} \to \delta_{ij}/3$ (labeled R) and the tensor polarization part with $\rho_{ij} \to -t_{ij}$ (labeled T):

$$\langle k_{Ai} k_{Bj} \rangle = \langle k_{Ai} k_{Bj} \rangle_R + \langle k_{Ai} k_{Bj} \rangle_T. \quad (42)$$

We calculate the two contributions in turn. For the spin-averaged contribution, the relevant probability distribution is

$$p_R = \frac{1}{\Gamma_0}\frac{m}{768\pi^3}\overline{|M|^2_R} = p_{R0} + \frac{\alpha}{\pi} p_{R1} + O(\alpha^2) \quad (43)$$

with

$$p_{R0} = \frac{1}{\pi^2-9}\left\{ \left(\frac{\bar{x}_1}{x_2 x_3}\right)^2 + \left(\frac{\bar{x}_2}{x_3 x_1}\right)^2 + \left(\frac{\bar{x}_3}{x_1 x_2}\right)^2 \right\} \quad (44)$$

and

$$p_{R1} = \frac{1}{\pi^2-9}\left(\frac{\bar{x}_1\bar{x}_2\bar{x}_3}{x_1 x_2 x_3}\right)^2 \sum_{S_3} \left\{ \frac{-1}{\bar{x}_1\bar{x}_3} F_1^{(1)}(x_1,x_2,x_3) + \frac{(x_2\bar{x}_2 + x_3\bar{x}_3)}{\bar{x}_1^2\bar{x}_2\bar{x}_3} F_2^{(1)}(x_1,x_2,x_3) + \frac{x_1}{\bar{x}_2^2\bar{x}_3} F_3^{(1)}(x_1,x_2,x_3) \right\}. \quad (45)$$

The Euler angle integrals are immediate since p_R does not depend on them:

$$\langle k_{Ai} k_{Bj} \rangle^N_R = \frac{m^2}{3}\delta_{ij} \int d\Delta\, x_A x_B \cos(\alpha_{AB}) p_R \equiv m^2 \delta_{ij} R^N_{AB}. \quad (46)$$

Phase space consists of six regions corresponding to the various orderings of x_1, x_2, x_3 (see Fig. 1). In region I, for instance, we have $x_1 \geqslant x_2 \geqslant x_3$, while in region IV $x_3 \geqslant x_2 \geqslant x_1$. It follows that the identity of $\vec{k}_A$ varies by region. For example, for the correlation of the most energetic and second most energetic photons one has

$$\langle k_{A=1,i} k_{B=2,j} \rangle^N_R = \langle k_{1i} k_{2j} \rangle^N_{\rm RI} + \langle k_{1i} k_{3j} \rangle^N_{\rm RII} + \langle k_{3i} k_{1j} \rangle^N_{\rm RIII} + \langle k_{3i} k_{2j} \rangle^N_{\rm RIV} + \langle k_{2i} k_{3j} \rangle^N_{\rm RV} + \langle k_{2i} k_{1j} \rangle^N_{\rm RVI} = 6\langle k_{1i} k_{2j} \rangle^N_{\rm RI}, \quad (47)$$

the last equality following by symmetry. The lowest order contribution is

$$R^N_{AB0} = 2\int_I d\Delta\, x_A x_B \cos(\alpha_{AB}) p_{R0}, \quad (48)$$

where region I of phase space is parametrized as in (35). The results for the various R^N_{AB0} were obtained analytically and are

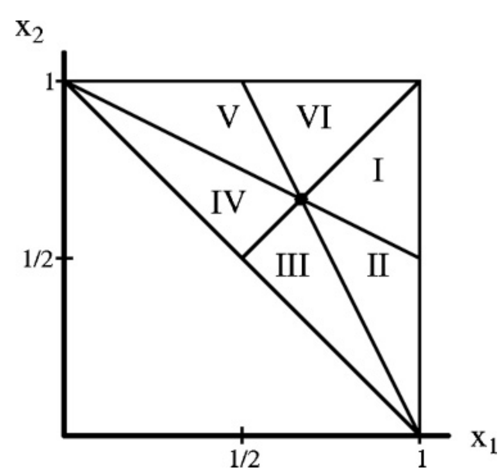

FIG. 1. Phase space for the three-photon decay of a massive particle. The physical region is the triangle in the upper right half of the diagram. Phase space can be decomposed into six regions: region I where the order of (x_1,x_2,x_3) is (123), II with order (132), III with order (312), IV with order (321), V with order (231), and VI with order (213).

TABLE I. Coefficients of the various terms appearing in the lowest order spin-averaged correlation factors R^N_{AB0}. The results still must be divided by $(\pi^2 - 9)$.

AB	$\mathrm{Li}_2(1/3)$	$\ln^2 3$	$\ln 3 \ln 2$	$\ln^2 2$	$\ln 3$	$\ln 2$	π^2	1
11	−52	−26	52	−26	−40	40	13/3	−3
12	−52	−26	52	−52	−40	80	13/6	11/4
13	104	52	−104	78	80	−120	−13/2	1/4
22	104	52	−104	78	80	−120	−13/3	−251/12
23	−52	−26	52	−26	−40	40	13/6	109/6
33	−52	−26	52	−52	−40	80	13/3	−221/12

displayed in Table I. For example,

$$R^N_{110} = \frac{1}{\pi^2 - 9}\left\{-52\mathrm{Li}_2\left(\frac{1}{3}\right) - 26\ln^2\left(\frac{3}{2}\right) - 40\ln\left(\frac{3}{2}\right) + \frac{13}{3}\pi^2 - 3\right\}, \qquad (49)$$

where $\mathrm{Li}_2(x)$ is a dilogarithm (see Lewin [35]). Because of momentum conservation $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = \vec{0}$, the results satisfy

$$R^N_{110} + R^N_{120} + R^N_{130} = 0, \qquad (50a)$$

$$R^N_{120} + R^N_{220} + R^N_{230} = 0, \qquad (50b)$$

$$R^N_{130} + R^N_{230} + R^N_{330} = 0. \qquad (50c)$$

Numerical results for the lowest order contributions are shown in the first column of Table II. The one-loop contributions

$$R^N_{AB1} = 2\int_I d\Delta\, x_A x_B \cos(\alpha_{AB}) p_{R1} \qquad (51)$$

were obtained numerically and are given in the second column of Table II. Total results for

$$R_{AB} = R^N_{AB0} + \frac{\alpha}{\pi}\left\{R^N_{AB1} - \gamma_1 R^N_{AB0}\right\} \qquad (52)$$

including both lowest and first orders are shown in the third column of Table II.

TABLE II. Numerical values for the spin-averaged correlations of photon momentum vectors. The three columns give the numerical values for the lowest order contribution R^N_{AB0}, the one-loop contribution R^N_{AB1}, and the total result through one-loop order R_{AB}. The numerical uncertainty of each entry is less than one in the least significant digit shown.

AB	R^N_{AB0}	R^N_{AB1}	R_{AB}
11	0.2669520	−2.775663	0.266883
12	−0.1930664	2.036270	−0.192950
13	−0.0738857	0.739392	−0.073934
22	0.1762007	−1.856327	0.176099
23	0.0168657	−0.179943	0.016851
33	0.0570200	−0.559449	0.057083

TABLE III. Coefficients of the various terms appearing in the lowest order spin-averaged correlation factors $\hat{R}^N_{AB0}$. The results still must be divided by $24(\pi^2 - 9)$.

AB	π^2	1
12	7	−301/4
13	7	−294/4
23	7	−269/4

We also calculated the correlations of unit vectors in the directions of the photon momenta:

$$\langle \hat{k}_{Ai}\hat{k}_{Bj}\rangle^N_R = \frac{1}{3}\delta_{ij}\int d\Delta \cos(\alpha_{AB}) p_R \equiv \delta_{ij}\hat{R}^N_{AB}. \qquad (53)$$

Our definitions regarding $\hat{R}^N_{AB0}$, $\hat{R}^N_{AB1}$, and $\hat{R}_{AB}$ parallel those of R^N_{AB0}, R^N_{AB1}, and R_{AB}. Clearly $\hat{R}^N_{AA0} = 1/3$ and $\hat{R}^N_{AA1} = \gamma_1/3$ (no sum over A), but there are no longer consistency conditions coming from momentum conservation. The results from the lowest order spin-averaged quantities $\hat{R}^N_{AB0}$ appear in Table III, and numerical results for the one-loop and total results are shown in Table IV.

We turn now to the tensor polarization part of $\langle k_{Ai}k_{Bj}\rangle_T$. The tensor probability distribution has contributions at $O(\alpha^0)$ and $O(\alpha^1)$:

$$p_T = \frac{1}{\Gamma_0}\frac{m}{768\pi^3}\overline{|M|^2_T} = p_{T0} + \frac{\alpha}{\pi}p_{T1} + O(\alpha^2). \qquad (54)$$

The distribution contains t_{mn} linearly and is a Cartesian scalar, so it also contains factors of $\hat{k}_{am}\hat{k}_{bn}$:

$$p_T = \{P_{11}\hat{k}_{1m}\hat{k}_{1n} + P_{12}\hat{k}_{1m}\hat{k}_{2n} + P_{13}\hat{k}_{1m}\hat{k}_{3n} + P_{22}\hat{k}_{2m}\hat{k}_{2n} + P_{23}\hat{k}_{2m}\hat{k}_{3n} + P_{33}\hat{k}_{3m}\hat{k}_{3n}\}t_{mn}. \qquad (55)$$

There is some ambiguity in the definition of P_{ab} because of relations among the $\hat{k}_{am}\hat{k}_{bn}$ due to momentum conservation. At lowest order we can choose

$$P^{(0)}_{aa} = \frac{3}{2(\pi^2 - 9)}\left(\frac{\bar{x}_a}{x_b x_c}\right)^2, \qquad (56)$$

$$P^{(0)}_{ab} = 0 \qquad (57)$$

TABLE IV. Numerical values for the spin-averaged correlations of photon momentum unit vectors. The three columns give the numerical values for the lowest order contribution $\hat{R}^N_{AB0}$, the one-loop contribution $\hat{R}^N_{AB1}$, and the total result through one-loop order $\hat{R}_{AB}$. The numerical uncertainty of each entry is less than one in the least significant digit shown.

AB	$\hat{R}^N_{AB0}$	$\hat{R}^N_{AB1}$	$\hat{R}_{AB}$
11	0.3333333	−3.428872	0.333333
12	−0.2952861	3.059727	−0.295234
13	−0.2114357	2.176862	−0.211431
22	0.3333333	−3.428872	0.333333
23	0.0880300	−0.964953	0.087892
33	0.3333333	−3.428872	0.333333

for (abc) a permutation of (123). This leads to

$$p_{T0} = \frac{3}{2(\pi^2-9)}\left\{\left(\frac{\bar{x}_1}{x_2x_3}\right)^2 \hat{k}_{1m}\hat{k}_{1n} + \left(\frac{\bar{x}_2}{x_3x_1}\right)^2 \hat{k}_{2m}\hat{k}_{2n} + \left(\frac{\bar{x}_3}{x_1x_2}\right)^2 \hat{k}_{3m}\hat{k}_{3n}\right\} t_{mn}. \tag{58}$$

The one-loop distribution can be written as

$$p_{T1} = \frac{1}{2}\sum_{S_3}\left(P_{11}^{(1)}\hat{k}_{1m}\hat{k}_{1n} + P_{12}^{(1)}\hat{k}_{1m}\hat{k}_{2n}\right)t_{mn}, \tag{59}$$

where

$$(\pi^2-9)P_{11}^{(1)} = \frac{3\bar{x}_1\bar{x}_2\bar{x}_3}{4x_2^2x_3^2}\left\{\frac{x_2}{\bar{x}_3}F_1^{(1)}(x_1,x_2,x_3) + \frac{x_3}{\bar{x}_2}F_1^{(1)}(x_1,x_3,x_2)\right\} + \frac{3\bar{x}_1^3}{4x_2^2x_3^2}\left\{F_2^{(1)}(x_2,x_1,x_3) + F_2^{(1)}(x_2,x_3,x_1) + F_2^{(1)}(x_3,x_1,x_2) + F_2^{(1)}(x_3,x_2,x_1)\right\} + \frac{3\bar{x}_1^3}{4x_2^2\bar{x}_2x_3^2\bar{x}_3}\left\{\bar{x}_3^2F_3^{(1)}(x_1,x_2,x_3) + \bar{x}_2^2F_3^{(1)}(x_1,x_3,x_2)\right\} \tag{60}$$

and

$$\begin{aligned}(\pi^2-9)P_{12}^{(1)} = {}& \frac{3}{4x_1x_2x_3^2}\Big\{(x_1-x_3)\bar{x}_2^2F_1^{(1)}(x_1,x_2,x_3) + (x_2-x_3)\bar{x}_1^2F_1^{(1)}(x_2,x_1,x_3) - x_3\bar{x}_2\bar{x}_3F_1^{(1)}(x_1,x_3,x_2)\\ &- \bar{x}_1x_3\bar{x}_3F_1^{(1)}(x_2,x_3,x_1) - (x_2+\bar{x}_3)\bar{x}_2^2\big[F_2^{(1)}(x_1,x_2,x_3) + F_2^{(1)}(x_1,x_3,x_2)\big]\\ &-(x_1+\bar{x}_3)\bar{x}_1^2\big[F_2^{(1)}(x_2,x_1,x_3) + F_2^{(1)}(x_2,x_3,x_1)\big] - \bar{x}_1\bar{x}_2x_3\big[F_2^{(1)}(x_3,x_1,x_2) + F_2^{(1)}(x_3,x_2,x_1)\big]\Big\}\\ &- \frac{3\bar{x}_3}{4x_1x_2x_3^2}\left\{\bar{x}_1^2F_3^{(1)}(x_1,x_2,x_3) + \bar{x}_2^2F_3^{(1)}(x_2,x_1,x_3)\right\}\\ &- \frac{3\bar{x}_1\bar{x}_2}{4x_1x_2x_3^2\bar{x}_3}\left\{(x_1+\bar{x}_3)\bar{x}_1F_3^{(1)}(x_1,x_3,x_2) + (x_2+\bar{x}_3)\bar{x}_2F_3^{(1)}(x_2,x_3,x_1)\right\}.\end{aligned} \tag{61}$$

The Euler averages involved in the calculation of the correlations are straightforward. The Euler average of any product of an odd number of momenta vanishes. The required even Euler averages are

$$\int\frac{d\omega}{8\pi^2}\hat{k}_{Ai}\hat{k}_{Bj} = \frac{1}{3}\cos(\alpha_{AB})\delta_{ij}, \tag{62a}$$

$$\int\frac{d\omega}{8\pi^2}\hat{k}_{Ai}\hat{k}_{Aj}\hat{k}_{Bk}\hat{k}_{C\ell} = \left\{\frac{2}{15}\cos(\alpha_{BC}) - \frac{1}{15}\cos(\alpha_{AB})\cos(\alpha_{AC})\right\}\delta_{ij}\delta_{k\ell} + \left\{\frac{-1}{30}\cos(\alpha_{BC}) + \frac{1}{10}\cos(\alpha_{AB})\cos(\alpha_{AC})\right\}(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}), \tag{62b}$$

which hold even when A, B, and C are not distinct. The tensor correlation numerator is

$$\langle k_{Ai}k_{Bj}\rangle_T^N = \int d\Delta\int\frac{d\omega}{8\pi^2}k_{Ai}k_{Bj}\left\{p_{T0} + \frac{\alpha}{\pi}p_{T1} + O(\alpha^2)\right\} \equiv \left[T_{AB0}^N + \frac{\alpha}{\pi}T_{AB1}^N + O(\alpha^2)\right]t_{ij}. \tag{63}$$

TABLE V. Coefficients of the various terms appearing in the lowest order tensor correlation factors T_{AB0}^N. The results still must be divided by (π^2-9).

AB	$\mathrm{Li}_2(1/3)$	$\ln^2 3$	$\ln 3\ln 2$	$\ln^2 2$	$\ln 3$	$\ln 2$	π^2	1
11	−24	−12	24	−12	−6	6	67/20	−99/5
12	−24	−12	24	−24	−6	12	13/40	231/20
13	48	24	−48	36	12	−18	−147/40	33/4
22	48	24	−48	36	12	−18	−13/20	−431/20
23	−24	−12	24	−12	−6	6	13/40	10
33	−24	−12	24	−24	−6	12	67/20	−73/4

Analytic results for the various T_{AB0}^N are shown in Table V. These expressions satisfy momentum conservation consistency conditions analogous to (50). Numerical results for these

TABLE VI. Numerical values for the tensor correlations of photon momentum vectors. The three columns give the numerical values for the lowest order contribution T_{AB0}^N, the one-loop contribution T_{AB1}^N, and the total result through one-loop order T_{AB}. The numerical uncertainty of each entry is less than one in the least significant digit shown.

AB	T_{AB0}^N	T_{AB1}^N	T_{AB}
11	0.0787061	−0.921559	0.078446
12	−0.0502105	0.632466	−0.049941
13	−0.0284956	0.289093	−0.028505
22	0.0353879	−0.471884	0.035137
23	0.0148226	−0.160582	0.014804
33	0.0136729	−0.128511	0.013701

TABLE VII. Coefficients of the various terms appearing in the lowest order tensor correlation factors $\hat{T}^N_{AB0}$. The results still must be divided by $160(\pi^2 - 9)$.

AB	π^2	1
11	32	−303
12	19	−793/4
13	19	−399/2
22	32	−306
23	19	−713/4
33	32	−303

lowest order contributions are shown in the first column of Table VI. The one-loop numerator factors T^N_{AB1} are given numerically in the second column of Table VI. Total results for

$$T_{AB} = T^N_{AB0} + \frac{\alpha}{\pi}\left\{T^N_{AB1} - \gamma_1 T^N_{AB0}\right\} \tag{64}$$

are given in the third column of Table VI.

Finally, we obtained the tensor contribution to the unit vector correlations

$$\langle \hat{k}_{Ai}\hat{k}_{Bj}\rangle_T = \hat{T}_{AB} t_{ij}. \tag{65}$$

Analytic results for the lowest order tensor quantities $\hat{T}^N_{AB0}$ are given in Table VII, and numerical results for the lowest order terms $\hat{T}^N_{AB0}$, the one-loop numerators $\hat{T}^N_{AB1}$, and total results $\hat{T}_{AB}$ are shown in Table VIII.

TABLE VIII. Numerical values for the tensor correlations of photon momentum unit vectors. The three columns give the numerical values for the lowest order contribution $\hat{T}^N_{AB0}$, the one-loop contribution $\hat{T}^N_{AB1}$, and the total result through one-loop order $\hat{T}_{AB}$. The numerical uncertainty of each entry is less than one in the least significant digit shown.

AB	$\hat{T}^N_{AB0}$	$\hat{T}^N_{AB1}$	$\hat{T}_{AB}$
11	0.0921924	−1.075395	0.091897
12	−0.0771005	0.941265	−0.076756
13	−0.0860845	0.901127	−0.086048
22	0.0706308	−0.879555	0.070275
23	0.0666430	−0.740360	0.066516
33	0.0921924	−0.875417	0.092362

VI. DISCUSSION

Form factors, such as the ones given in this work, provide an efficient encoding of the effects of radiative corrections. These one-loop form factors are given in analytic form and involve functions no more complicated than dilogarithms. Through their use, the decay matrix element, and all things that follow from it, can be obtained analytically through one-loop order.

We have obtained results for the two-momentum correlations $\langle k_{Ai}k_{Bj}\rangle$ and $\langle \hat{k}_{Ai}\hat{k}_{Bj}\rangle$ through one-loop order for polarized positronium. To this order, the only parts of the density matrix ρ_{ij} that contribute are the spin-average part $\frac{1}{3}\delta_{ij}$ and the tensor-polarization part $-t_{ij}$. Since these are both symmetric in ij, the CPT-sensitive correlation $\langle \vec{k}_A \times \vec{k}_B\rangle$ vanishes to one-loop order. It is evident from (28) that $|M|^2$ will only have an antisymmetric vector-polarization contribution when the form factors, and hence the decay matrix element M, have nonvanishing imaginary parts. For positronium decay, this occurs first at two-loop order due to final-state interactions. (See Bigi and Sanda [36] for a discussion of final-state interactions in this context.) The QED contributions to $\langle \vec{k}_A \times \vec{k}_B\rangle$ and related correlations were calculated by Bernreuther *et al.* [26] and are small compared to experimental limits [27,28,30–32].

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[1] S. Berko and H. N. Pendleton, Annu. Rev. Nucl. Part. Sci. **30**, 543 (1980).
[2] A. Rich, Rev. Mod. Phys. **53**, 127 (1981).
[3] A. Rich *et al.*, in *New Frontiers in Quantum Electrodynamics and Quantum Optics*, edited by A. O. Barut (Plenum Press, New York, 1990), pp. 257–274.
[4] S. G. Karshenboim, Int. J. Mod. Phys. A **19**, 3879 (2004).
[5] W. Bernreuther and O. Nachtmann, Z. Phys. C **11**, 235 (1981).
[6] R. Alcorta and J. A. Grifols, Ann. Phys. (NY) **229**, 109 (1994).
[7] J. Govaerts and M. Van Caillie, Phys. Lett. B **381**, 451 (1996).
[8] A. Czarnecki and S. G. Karshenboim, e-print arXiv:hep-ph/9911410.
[9] A. Ore and J. L. Powell, Phys. Rev. **75**, 1696 (1949).
[10] W. E. Caswell, G. P. Lepage, and J. Sapirstein, Phys. Rev. Lett. **38**, 488 (1977).
[11] W. E. Caswell and G. P. Lepage, Phys. Rev. A **20**, 36 (1979).
[12] G. S. Adkins, Ann. Phys. (NY) **146**, 78 (1983).
[13] G. S. Adkins, Phys. Rev. Lett. **76**, 4903 (1996).
[14] B. A. Kniehl, A. V. Kotikov, and O. L. Veretin, Phys. Rev. Lett. **101**, 193401 (2008).
[15] G. S. Adkins, R. N. Fell, and J. Sapirstein, Phys. Rev. Lett. **84**, 5086 (2000).
[16] G. S. Adkins, R. N. Fell, and J. Sapirstein, Ann. Phys. (NY) **295**, 136 (2002).
[17] S. G. Karshenboim, Sov. Phys. JETP **76**, 541 (1993) [Zh. Eksp. Teor. Fiz. **103**, 1105 (1993)].
[18] R. J. Hill and G. P. Lepage, Phys. Rev. D **62**, 111301(R) (2000).
[19] B. A. Kniehl and A. A. Penin, Phys. Rev. Lett. **85**, 1210 (2000); **85**, 3065(E) (2000).
[20] K. Melnikov and A. Yelkhovsky, Phys. Rev. D **62**, 116003 (2000).
[21] O. Jinnouchi, S. Asai, and T. Kobayashi, Phys. Lett. B **572**, 117 (2003).

[22] R. S. Vallery, P. W. Zitzewitz, and D. W. Gidley, Phys. Rev. Lett. **90**, 203402 (2003).
[23] Y. Kataoka, S. Asai, and T. Kobayashi, Phys. Lett. B **671**, 219 (2009).
[24] G. S. Adkins, Phys. Rev. A **72**, 032501 (2005).
[25] See supplementary material at http://link.aps.org/supplemental/10.1103/PhysRevA.81.042507 for electronic versions of the form factors.
[26] W. Bernreuther, U. Löw, J. P. Ma, and O. Nachtmann, Z. Phys. C **41**, 143 (1988).
[27] B. K. Arbic, S. Hatamian, M. Skalsey, J. Van House, and W. Zheng, Phys. Rev. A **37**, 3189 (1988).
[28] M. Skalsey, Mod. Phys. Lett. A **7**, 2251 (1992).
[29] S. K. Andrukhovich, N. Antovich, and A. V. Berestov, Instrum. Exp. Tech. (USSR) **43**, 453 (2000) [translated from Pribory i Tekhnika Eksperimenta].
[30] P. A. Vetter and S. J. Freedman, Phys. Rev. Lett. **91**, 263401 (2003).
[31] P. A. Vetter, Int. J. Mod. Phys. A **19**, 3865 (2004).
[32] T. Namba, K. Nishihara, T. Yamazaki, S. Asai, and T. Kobayashi, in *Cold Antimatter Plasmas and Application to Fundamental Physics*, edited by Y. Kanai and Y. Yamazaki (American Institute of Physics, Melville, NY, 2008), pp. 56–65.
[33] E. W. N. Glover and A. G. Morgan, Z. Phys. C **60**, 175 (1993).
[34] In [24] the function $h_7(x_i, x_j)$ also appeared. The identity $h_6(x_i, x_j) + h_6(x_j, x_i) + h_7(x_i, x_j) + h_7(x_j, x_i) = 0$ allows h_7 to be eliminated, resulting in more compact expressions.
[35] L. Lewin, *Polylogarithms and Associated Functions* (Elsevier North-Holland, New York, 1981).
[36] I. I. Bigi and A. I. Sanda, *CP Violation* (Cambridge University Press, Cambridge, 2000), Sec. 4.10.