Overlap of quantum many-body states with a separable state and phase transitions in the Dicke model: Zero and finite temperature

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(Received 10 February 2010; published 30 April 2010)

Overlap with the separable state is introduced in this article for the purpose of characterizing the overall correlation in many-body systems. This definition has clear geometric and physical meaning and moreover can be considered as the generalization of the concept of the Anderson orthogonality catastrophe. As an exemplification, it is used to mark the phase transition in the Dicke model for zero and finite temperatures, and the discussion shows that it can faithfully reflect the phase transition properties of this model whether for zero or finite temperature. Furthermore, the overlap for the ground state also indicates the appearance of multipartite entanglement in the Dicke model.

DOI: 10.1103/PhysRevA.81.042112 PACS number(s): 03.65.Ud, 64.60.−i

I. INTRODUCTION

Correlation in condensed matter systems predominates the understanding of many-body effects. Fundamentally, one can define different correlation functions for describing the unusual connections in many-body systems. For instance, it is generally accepted to introduce the order parameter for the description of phase transitions induced by local perturbation and furthermore to classify the diverse phase transitions by scaling the singularity of correlation functions with the universal critical exponents. This is the so-called Landau-Ginzburg-Wilson (LGW) paradigm [1]. However the situation becomes different for the strongly correlated electronic systems. The quantum Hall effect appearing in a two-dimensional electron gas with a high magnetic field shows distinct features not captured by the LGW paradigm. Instead, topological order is consequently defined to describe the underlying symmetry in quantum Hall systems, which is distinct from the notion of spontaneously broken symmetry [2].

Recently, extensive research on quantum entanglement in condensed matter systems has shown the potentiality that quantum entanglement would act as a universal description for many-body effects [3,4]. In particular, some general conclusions have been obtained about the connection between quantum entanglement and quantum phase transition in many-body systems. The concurrence, a measurement of two-party entanglement, has been shown to behave singularly at the critical point of the one-dimensional spin-half XY model, and the critical exponents can also be obtained by scaling this singularity [5,6]. Furthermore, the block entanglement entropy has been shown to display logarithmical divergency with the block size at critical points, and the scaling factor is directly related to the central charge of the conformal field theory [7]. Moreover, the universal area law for the entanglement entropy has also been constructed exactly in one-dimensional spin-chain systems [4], and similar behavior for single-copy entanglement is also found [8]. Recently, the entanglement spectrum has been defined to obtain general information about many-body systems [9–11]. As for quantum Hall systems, it is shown that the scaling behavior of entanglement entropy is directly related to the quantum number, which is used to characterize the topological order [12], and entanglement spectrum can also be used to detect the nonlocal features of quantum Hall systems [9].

Although important progress has been made, there are a few exceptions that lead to the suspicion of the validity of quantum entanglement as an universal description for many-body effects. Entanglement entropy sometimes provides ambiguous information about the phase transitions in higher dimensional many-body systems [4]. Even for one-dimensional systems, it cannot present complete information in some situations. As an example, recent studies have shown that the block entanglement entropy for the valence-bond-solid (VBS) state of integer spin seems insensible to the degeneracy manifested by the underlying topological symmetry and also does not display dependence on the parity of spin number s, but both can be manifested clearly by introducing a string order parameter [13]. As for quantum Hall systems, the entanglement entropy and entanglement spectrum have also been shown to have limited ability in identifying topological orders [11].

From the author's point of view, this defect would attribute to the trace-out of the superfluous degrees of freedom when one obtains the reduced density matrix, and some information on global features in many-body systems is inevitably lost. This point has been exemplified in a recent article of our group [14], in which geometric entanglement (GE) as a measure of multipartite entanglement is calculated for a VBS state. The interesting result in this article is that GE displays two different scaling behaviors dependent on the parity of spin number s, and the global GE is divergent linearly with the particle number.

Through this short introduction, it seems promising to measure multipartite entanglement in order to obtain complete information for many-body effects. Recently, some efforts have been made in this direction. The connection between multipartite entanglement and quantum phase transition has been discussed in some special models [14–17]. However, the crucial obstacle for further development is the absence of the unified understanding of the multipartite entanglement [18,19]. Whereas the maximally entangled state can be defined unambiguously for bipartite systems, what is the maximally entangled state for multipartite systems has been unclear until now [19]. Fortunately, it is well accepted that the fully separable state can be defined as

$$\rho^{\rm sep} = \sum_i p_i \rho_1^{(i)} \otimes \rho_2^{(i)} \otimes \cdots \otimes \rho_N^{(i)}, \tag{1}$$

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where N is the particle number and p_i denotes the common probability with which the single-particle state $\rho_n^{(i)}(n = 1,2,\ldots,N)$ happens. With respect to this point, GE is introduced first by Shimony for the pure bipartite state [20] and is later generalized to the multipartite case by Carteret *et al.* [21], Barnum and Linden [22], and Wei and Goldbart [23], and to the mixed state by Cao and Wang [24]. GE is a genuine multipartite entanglement measurement. The main idea of GE is to minimize the distance D between the state $|\Psi\rangle$ to be measured and the fully separable state $|\Phi\rangle$ in Hilbert space:

$$D = \min_{\{|\Phi\rangle\}}\{\||\Psi\rangle - |\Phi\rangle\|^2\}. \tag{2}$$

For the normalized $|\Psi\rangle$ and $|\Phi\rangle$, the evaluation of D is reduced to find the maximal overlap [23]:

$$\Lambda(\Psi) = \max_{\{|\Phi\rangle\}} |\langle\Phi|\Psi\rangle|. \tag{3}$$

Geometrically, $\Lambda(|\Psi\rangle)$ depicts the overlap angle between the vectors $|\Psi\rangle$ and $|\Phi\rangle$ in Hilbert space. Then, the larger $\Lambda(|\Psi\rangle)$ is, the shorter is the distance and the less entangled is $|\Psi\rangle$. But the optimum is in general a forbidden task, not spoken for a mixed state, and the analytical results can be obtained only for some very special cases [16,17,24]. Recently, many efforts have been devoted to the reduction of the optimum, and some interesting results have been obtained [25].

Given this difficulty, I introduce another different quantity in this article to capture the overall correlation in condensed matter systems, that is, the overlap with a special fully separable state. The starting point is still to find the minimal distance between the state to be measured and a special fully separable state defined in the next section. In contrast to GE, the optimum can be reduced by utilizing the geometric property of the overlap, and this overlap has very clear physical meaning, whether for a pure or mixed state. In Sec. II, the definition of this overlap is introduced, and the differences with several known similar definitions are clarified. Furthermore, we point out that our definition is connected intimately with the concept of the Anderson orthogonality catastrophe (AOC) [26,27]. As an illustration of the validity of our definition, the collective phase transition appearing in the Dicke model is discussed per this quantity in Sec. III. Multipartite entanglement in this model is also studied (Sec. IV) for displaying the potential connection between this overlap and multipartite entanglement. Finally, conclusions and further discussion are presented in Sec. V.

II. OVERLAP WITH FULLY SEPARABLE STATE

Similar to the introduction of GE, our starting point is also to find the minimal distance D between the fully separable state ρ^{sep} and the state ρ to be measured:

$$D = \min_{\{\rho^{\text{sep}}\}}\{\|\rho - \rho^{\text{sep}}\|^2\}. \tag{4}$$

Generally, this minimal distance is still decided mainly by the maximal overlap

$$\Lambda = \max_{\{\rho^{\text{sep}}\}} \text{Tr}[\rho\rho^{\text{sep}}]. \tag{5}$$

The density matrix can also be written as the Bloch form

$$\rho = \left(I + \sum_{i=1}^{d^2-1} r_i\lambda_i\right)\Big/ d, \tag{6}$$

where d denotes the dimension, λ_i is the generator of the SU(d) group, and $\{r_i\}$ is the so-called Bloch vector [28]. Thus Λ has clear geometric meaning which depicts the minimal overlap angle θ between the Bloch vectors $\{r_i\}$ and $\{r_i\}_{\text{sep}}$ in the Bloch vector space, that is,

$$\max_{\{\rho^{\text{sep}}\}} \text{Tr}[\rho\rho^{\text{sep}}] = \frac{1}{d}(1 + |\{r_i\}||\{r_i\}_{\text{sep}}| \cos[\min_{\{\theta\}} \theta]). \tag{7}$$

Two limit cases are beneficial to the understanding of the physical meaning of θ. For $\cos\theta = 1$, the overlap is maximal, and ρ and ρ^{sep} share the same physical characters since Bloch vector $\{r_i\}$ is the reflection of the intrinsic symmetry in the systems [28], whereas for $\cos\theta = -1$, one has minimal overlap, and ρ^{sep} shows distinct properties from ρ.

In contrast to the Bures fidelity [29], the overlap Λ has clear geometric meaning whether for a pure or mixed state, as shown earlier. Furthermore, with this geometric meaning, the optimal procession can be reduced to find the fully separable state ρ^{sep} sharing the same physical properties with ρ (see Appendix A for a proof). Moreover, this definition is more popular than Eq. (3). First Λ comes back to the form Eq. (3) for pure states. Second, Eq. (5) includes the case when one state is pure and the other is mixed. This situation always happens as exemplified in Ref. [24] but is not covered in the original discussion [23]. Third, for a mixed state, the geometric characteristics of GE become ambiguous because of the *convex roof* construction [23], whereas the geometric meaning of Λ is clear whether for a pure or mixed state.

With these advantages, the evaluation of Λ, however, is difficult for mixed-state ρ^{sep} since there are infinite possibilities of the decomposition of ρ^{sep}. Recently we note a popular concept in condensed matter physics, the AOC [26,27], which refers to the vanishing of the overlap between the many-body ground states with and without the potential as a power law in the number of particles in the systems. AOC is defined as

$$\Delta = |\langle\Phi|\Phi^p\rangle|^2, \tag{8}$$

where $|\Phi^p\rangle$ and $|\Phi\rangle$ correspond respectively to the many-body states with the potential and the state described entirely in terms of free plane waves, including the ground state of the unperturbed system [26]. Anderson proved that the overlap Δ approached zero under the thermodynamic limit $N \to \infty$ even for a very weak potential, which means that $|\Phi\rangle$ is orthogonal to $|\Phi^p\rangle$ [26] and the transition between the two states is forbidden. It is the physical meaning of *catastrophe* [27]. As claimed in Ref. [26], it becomes impossible because of the appearance of catastrophe to find the characters for many-body systems by adiabatically imposing the potential and observing the response since the significant changes in many-body systems can be induced even for infinitesimal perturbation. AOC presents an understanding of a number of Fermi-edge singularities, for example, in the Kondo effect [30] or in the X-ray edge problem [31], for which the local singularity has an overall effect on the property of the many-body systems.

With these points, AOC manifestly shows that the correlation in many-body systems can be constructed simultaneously whenever the interaction appears and thus can be used to give a full description of correlation in many-body systems. Furthermore, the important feature is that this prohibition can be conquered by the symmetry-breaking process, as exemplified by the observation of the X-ray absorption in the electron gas [27], which means that AOC can also be used to characterize the phase transitions induced by the symmetry breaking process. In a word, AOC presents a comprehensive description of the many-body effects.

This crucial observation forces us to define the following fully separable state for N parties:

$$\rho^s = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_N, \tag{9}$$

which represents the many-body state without potential, compared to the state $|\Phi\rangle$ in Eq. (8). Then we can define the overlap with fully separable state ρ^s to capture the overall correlation in many-body systems:

$$\Delta = \max_{\{\rho^s\}} \mathrm{Tr}\,[\rho\rho^s]. \tag{10}$$

This definition is the main contribution of this article and has some distinct advantages, summarized as follows: (1) Δ has clear geometric meaning, which depicts the minimal overlap angle between the Bloch vectors $\{r_i\}$ and $\{r_i\}_s$, and for pure states, it returns to the original definition Eq. (3) of GE; (2) by this geometric meaning, the optimal process in Δ can be reduced to find the fully separable state ρ^s sharing the same physical features with ρ, and (3) Δ can be regarded as the generalization of AOC to the mixed state and can faithfully reflect the overall correlation in many-body systems. It should be emphasized that this definition does not try to present a complete measurement of the multipartite entanglement. Instead, my purpose is to find a universal method to characterize the overall correlation in many-body systems, whether quantum or classical. However, this definition is also meaningful for finding the unified understanding of multipartite entanglement in many-body systems. As shown in the next section, Δ indeed presents the interesting information for the phase transition in the Dicke model. Moreover, the connection between Δ and multipartite entanglement in the Dicke model has also been discussed in Sec. III. Additionally, in contrast to the recent interest in fidelity for many-body systems [32], Δ does not serve for the state discrimination.

III. EXEMPLIFICATION: PHASE TRANSITION IN THE DICKE MODEL

In order to demonstrate the generality of this definition, the phase transition in the Dicke model is discussed by Δ in this section. The Dicke model describes the dynamics of N independently identical to two-level atoms coupling to the same quantized electromagnetic field [33]. Owing to the presence of dipole-dipole force between atoms, the Dicke model shows the normal-superradiant transition [34].

The Dicke model is related to many fundamental issues in quantum optics, quantum mechanics, and condensed matter physics such as the coherent spontaneous radiation [34], the dissipation of the quantum system [35], quantum chaos [36], and atomic self-organization in a cavity [37]. The normal-superradiant transition was first observed with Rydberg atoms [38] and recently in a superfluid gas coupled to an optical cavity [37] and nuclear spin ensemble surrounding a single photon emitter [39]. Quantum entanglement in the Dicke model has also been discussed extensively in [40,41]. Furthermore, the Dicke model is also related to the issues of how the opened multipartite system is affected by the environment and the robustness of multipartite entanglement [42].

The Hamiltonian for a single-model Dicke model reads

$$\begin{aligned} H &= \omega a^\dagger a + \frac{\omega_0}{2}\sum_{i=1}^{N}\sigma_i^z + \frac{\lambda}{\sqrt{N}}\sum_{i=1}^{N}(\sigma_i^+ + \sigma_i^-)(a^\dagger + a) \\ &= \omega_0 J_z + \omega a^\dagger a + \frac{\lambda}{\sqrt{N}}(a^\dagger + a)(J_+ + J_-), \end{aligned} \tag{11}$$

where $J_z = \sum_{i=1}^N \sigma_i^z/2$ and $J_\pm = \sum_{i=1}^N \sigma_i^\pm$ are the collective angular momentum operators. At zero temperature, the normal-superradiant transition happens when $\lambda = \lambda_c = \sqrt{\omega\omega_0}/2$. For finite temperature, the critical temperature is decided by the relation [43]

$$\beta_c = \frac{\omega_0}{2\lambda^2}\frac{\tanh(\beta_c\omega/2)}{\tanh(\beta_c\omega_0/2)}. \tag{12}$$

An intrinsic property of the Dicke model is the parity symmetry

$$\begin{aligned} [H,\Pi] &= 0, \\ \Pi &= e^{i\pi\left(a^\dagger a + J_z + \frac{N}{2}\right)}. \end{aligned} \tag{13}$$

Moreover, Eq. (11) is obviously made permutationally invariant through the exchange of any two atoms.

With this information, the overlap Δ for the Dicke model is studied explicitly for zero and finite temperatures in the following two subsections. Some interesting features of Δ are displayed.

A. Zero temperature

With respect to the permutation invariance of atoms in Eq. (11), it is convenient to introduce the Holstein-Primakoff (HP) transformation:

$$\begin{aligned} J_z &= b^\dagger b - \frac{N}{2}, \\ J_+ &= b^\dagger\sqrt{N - b^\dagger b}, \\ J_- &= \sqrt{N - b^\dagger b}\,b, \end{aligned} \tag{14}$$

with bosonic operator $b^{(\dagger)}$. Semiclassically, there is a ground state with $J_z = -N/2$ for the Dicke model under the thermodynamic limit $N \to \infty$. Hence it is reasonable to adopt the low-excitation approximation at zero temperature, and then one obtains two effective Hamiltonians for different regions of λ (refer to [44] for details):

$$H^{(1)} = \omega a^\dagger a + \omega_0 b^\dagger b + \lambda(a^\dagger + a)(b^\dagger + b) - \frac{N}{2}\omega_0, \quad \lambda < \lambda_c,$$

$$H^{(2)} = \omega a^\dagger a + \left[\omega_0 + \frac{2}{\omega}\left(\lambda^2 - \lambda_c^2\right)\right] b^\dagger b$$

$$+\frac{(\lambda^2-\lambda_c^2)(3\lambda^2+\lambda_c^2)}{2\omega(\lambda^2+\lambda_c^2)}(b+b^\dagger)^2 + \frac{\sqrt{2}\lambda_c^2}{\sqrt{\lambda^2+\lambda_c^2}}(a^\dagger+a)(b^\dagger+b)+\text{const.},\quad \lambda>\lambda_c. \tag{15}$$

$H^{(1)}$ and $H^{(2)}$ can be diagonalized readily by transforming them into phase space, and then one has the diagonalized form [44]

$$H=\omega_1 c_1^\dagger c_1+\omega_2 c_2^\dagger c_2, \tag{16}$$

where the forms of $\omega_{1(2)}$ and $c_{1(2)}$ are dependent on $\lambda>\lambda_c$ or $\lambda<\lambda_c$ [44]. Then the ground state can be written as

$$|g\rangle=|g\rangle_1\otimes|g\rangle_2, \tag{17}$$

where $|g\rangle_{1(2)}$ denotes the vacuum state for mode $\omega_{1(2)}$. Furthermore, the average spin along the z direction per atom shows distinct values across the phase transition point,

$$\frac{\langle J_z\rangle}{N}=\begin{cases}-\frac{1}{2}, & \lambda<\lambda_c,\\ -\frac{\lambda_c^2}{2\lambda^2}, & \lambda>\lambda_c,\end{cases}. \tag{18}$$

which then can act as the order parameter. It is obvious that a macroscopic number of atoms are excited for $\lambda>\lambda_c$, which is the so-called superradiant phase, while for $\lambda<\lambda_c$, it is the normal phase.

With this information, we are ready to evaluate Δ. Our focus is mainly on the atomic system. The crucial step is to decide the fully separable state ρ^s for the atomic system. As mentioned in Sec. II and proved in Appendix A, the optimum process in Eq. (4) can be reduced to find ρ^s sharing the same global features with the ground state [Eq. (17)]. First, with the requirement of the permutation invariance of atoms in the Dicke model, the single atomic state should have the same form in ρ^s, that is, $\rho_i=\varrho, i=1,2,\ldots,N$, and then

$$\rho^s=\varrho^{\otimes N}. \tag{19}$$

Second, the parity symmetry for the Dicke model is reduced for a single atom state ρ as

$$[e^{i\pi J_z},\rho^s]=0, \quad \Rightarrow [e^{i\pi\sigma_z},\varrho]=0. \tag{20}$$

Thus one has under σ_z representation

$$\varrho=\begin{pmatrix} a & 0\\ 0 & 1-a\end{pmatrix}. \tag{21}$$

Finally, with the requirement of Eq. (18), $a=1/2+(\langle J_z\rangle/N)$. Thus ρ^s can be uniquely determined as

$$\rho^s=\begin{pmatrix} 1/2+\frac{\langle J_z\rangle}{N} & 0\\ 0 & 1/2-\frac{\langle J_z\rangle}{N}\end{pmatrix}^{\otimes N}. \tag{22}$$

I should point out that the procedure for the determination of ρ^s is popular whether for zero or finite temperature.

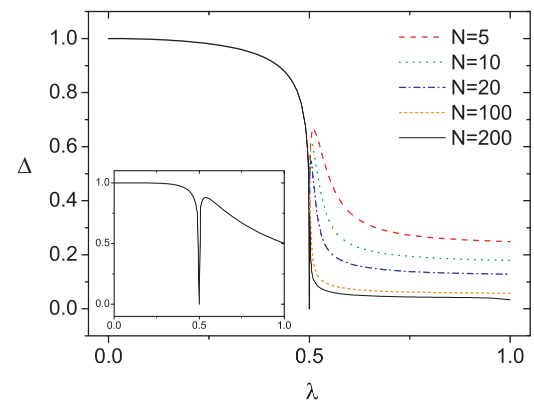

FIG. 1. (Color online) The overlap Δ with fully separable state ρ^s vs. the coupling λ at zero temperature; $\omega_0=\omega=\hbar=1$ has been chosen for this plot, and the critical point is $\lambda_c=0.5$ in this case. The inset is a plot of the purity of the reduced density of atomic freedom under $N\to\infty$.

For the evaluation of the overlap Δ, it should be noted that ρ^s can be rewritten as the following contract form under the J_z representation:

$$\rho^s=\sum_{n=1}^{N} {}_NC_n a^n(1-a)^{N-n}\left|n-\frac{N}{2}\right\rangle\left\langle n-\frac{N}{2}\right|, \tag{23}$$

where ${}_NC_k$ denotes the binomial function, and $|n-(N/2)\rangle$ presents the state for which n particles are spun up and the others are spun down. Together with the HP transformation [Eq. (14)], it is obvious that

$$b^\dagger b\left|n-\frac{N}{2}\right\rangle=n\left|n-\frac{N}{2}\right\rangle. \tag{24}$$

Then Δ can be evaluated easily under this representation.

In Fig. 1, the overlap Δ with ρ^s is plotted. At normal phase ($\lambda<\lambda_c$), one has $\langle J_z\rangle/N=-0.5$, $a=0$, and then ρ^s is the fully separable pure state. Δ is determined mainly by the first diagonal element of the reduced density matrix of the atomic system in this special case. While for $\lambda>\lambda_c$, Δ shows a sudden rise, it then decreases with increments of λ and tends to be steady with $\lambda\to\infty$. Moreover, under $N\to\infty$, Δ tends to be vanishing. Then two different phases can be clearly identified by evaluating Δ.

Some intricate features of the phase transition can be disclosed by Δ. For normal phase $\lambda<\lambda_c$, it is known that the atomic system becomes entangled with the electromagnetic field and attains the maximal value at the critical point [40]. The entanglement leads the state of the atomic system to be mixed, and the purity of its reduced density is decreased as shown by the inset in Fig. 1. At the same time, the pairwise entanglement between any two atoms is also raised, mediated by their couplings to the electromagnetic field, and the atoms become correlated with each other [40]. These intrinsic properties can be captured by Δ at the same time. For normal phase, ρ^s is pure and fully separable. Thus the decrement of Δ reflects the fact that the atoms become correlated with each other and attains the maximal correlation at the critical point, at which Δ has minimal value. Furthermore, since there is no interaction among atoms, the only reason for the construction of correlation in atoms is the couplings to the same electromagnetic field, which just induce the state for the atomic system to be mixed. This feature can also be manifested by the decrement of Δ with respect to a ρ^s that is pure.

For superradiant phase $\lambda > \lambda_c$, it is known that the entanglement between the atoms and electromagnetic field decreases monotonously to a steady value with the increment of λ, while the pairwise entanglement in atoms disappears asymptotically [40]. Contrastably, the purity for the state of the atomic system has a sudden increase closed to λ_c and then decreases to a steady value, as shown by the inset of Fig. 1. The two different behaviors can also be captured by Δ. Similar to the behavior of the purity of the state for the atomic system, Δ also has a sudden rise close to λ_c and then decreases to a steady value with increments of λ. Given that ρ^s is mixed in this case and its purity is monotonically decreasing with increments of λ, the abrupt increment of Δ means that the sudden recovery of the purity of the atomic system is at the expense of the reduction of a correlation between atoms. It is obvious from Fig. 1 that Δ tends to be zero with the increment of N for large λ. However, the vanishing of Δ cannot be attributed to the mixedness of ρ^s since the steady value of Δ for finite N is always greater than the maximal mixedness $1/N$, manifested by Fig. 1. This feature means that the correlation in atoms still exists. Since the pairwise entanglement of atoms is known to be vanishing in this limit [40], the correlation between atoms must be global.

The scaling behavior of Δ near the critical point shows some interesting features. At the normal phase ($\lambda < \lambda_c$), one has for $\omega = \omega_0 = 1$

$$\Delta = \frac{2^{3/2}(1-4\lambda^2)^{1/4}}{[1+3\sqrt{1-4\lambda^2}+0.5(\sqrt{1+2\lambda}+\sqrt{1-2\lambda})^3]}. \tag{25}$$

Similar to the method in Ref. [17], one can define the globe overlap $-\ln\Delta$ to measure the atomic correlation in the Dicke model. It is obvious that the global overlap is mainly determined by $(1-4\lambda^2)^{1/4}$ near $\lambda_c = 1/2$, and then

$$-\ln\Delta \sim -\frac{1}{4}\ln\left(1-\frac{\lambda}{\lambda_c}\right), \tag{26}$$

which is identical to the scaling behavior of multipartite entanglement in the Lipkin-Meshkov-Glick (LMG) model [17]. This result is not strange since the Dicke model and the LMG model belong to the same universality class. However, it strongly implies that Δ could be correlated directly with the multipartite entanglement in the Dicke model. As shown in Sec. IV, the atomic system indeed displays the multipartite entanglement in this case.

B. Finite temperature

At finite temperature, the phase transition is induced by thermal fluctuation. To determine the critical temperature, the general method is to evaluate the partition function z. In Ref. [43], z has been obtained analytically,

$$z = \frac{\sqrt{1/2\pi}}{1-e^{-\beta\omega}}\int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2}} \times \left\{2\cosh\left[\beta\sqrt{\frac{\omega_0^2}{4}+\frac{x^2\lambda^2}{N}\coth\frac{\beta\omega}{2}}\right]\right\}^N, \tag{27}$$

and the critical temperature is determined by Eq. (12). For $\omega = \omega_0$, it is reduced to $T_c = 2\lambda^2/k_B\omega_0$. With the same trick used in [43], the overlap Δ can also be written analytically as (see Appendix B for the details of calculation)

$$\begin{aligned}\Delta = {} & \frac{1}{z}\frac{\sqrt{1/2\pi}}{1-e^{-\beta\omega}}\int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2}} \\ & \times \left\{2\cosh\left[\beta\sqrt{\frac{\omega_0^2}{4}+\frac{x^2\lambda^2}{N}\coth\frac{\beta\omega}{2}}\right]\right. \\ & + \frac{\omega_0(1-2a)/2}{\sqrt{\frac{\omega_0^2}{4}+\frac{x^2\lambda^2}{N}\coth\frac{\beta\omega}{2}}}\sinh \\ & \left.\times\left[\beta\sqrt{\frac{\omega_0^2}{4}+\frac{x^2\lambda^2}{N}\coth\frac{\beta\omega}{2}}\right]\right\}^N.\end{aligned} \tag{28}$$

As shown in Fig. 2, Δ can clearly detect the phase transition by its abrupt variance close to the critical line. Given that ρ^s is mixed and fully separable in this case, Δ reflects that the correlation between atoms exists even for finite temperature. However, this type of correlation is obviously induced by the thermal fluctuation and thus is incoherent in contrast to zero temperature. This difference will become clear if one focuses on the multipartite entanglement of atoms in the next section.

IV. MULTIPARTITE ENTANGLEMENT IN THE DICKE MODEL

Another interesting aspect of the Dicke model is the multipartite entanglement in atoms. Since all atoms simultaneously couple isotropically to the same electromagnetic field, it is expected that the multipartite correlation of atoms could be readily constructed in this case.

However, the measure of multipartite entanglement is a difficult task in general, especially for the mixed state. An indirect way of resolving this difficulty is to find the characters uniquely belonging to the fully separable state [Eq. (1)], and the violation of these properties implies the appearance of multipartite entanglement. Spin squeezing is one of the most successful approaches to the multipartite entanglement in this way [45]. Recently, Tóth *et al.* [46] provided a series of inequalities about spin squeezing to identify the multipartite entanglement in collective models:

$$\langle J_x^2\rangle + \langle J_y^2\rangle + \langle J_z^2\rangle \leqslant \frac{N(N+2)}{4}, \tag{29a}$$

$$\Delta^2 J_x + \Delta^2 J_y + \Delta^2 J_z \geqslant \frac{N}{2}, \tag{29b}$$

$$\langle J_\alpha^2\rangle + \langle J_\beta^2\rangle - \frac{N}{2} \leqslant (N-1)\Delta^2 J_\gamma, \tag{29c}$$

$$(N-1)[\Delta^2 J_\alpha + \Delta^2 J_\beta] \geqslant \langle J_\gamma^2\rangle + \frac{N(N-2)}{4}, \tag{29d}$$

where α,β,γ adopt all permutations of x,y,z and $\Delta^2 J_\alpha = \langle J_\alpha^2\rangle - \langle J_\alpha\rangle^2$. The violation of any one of these inequalities

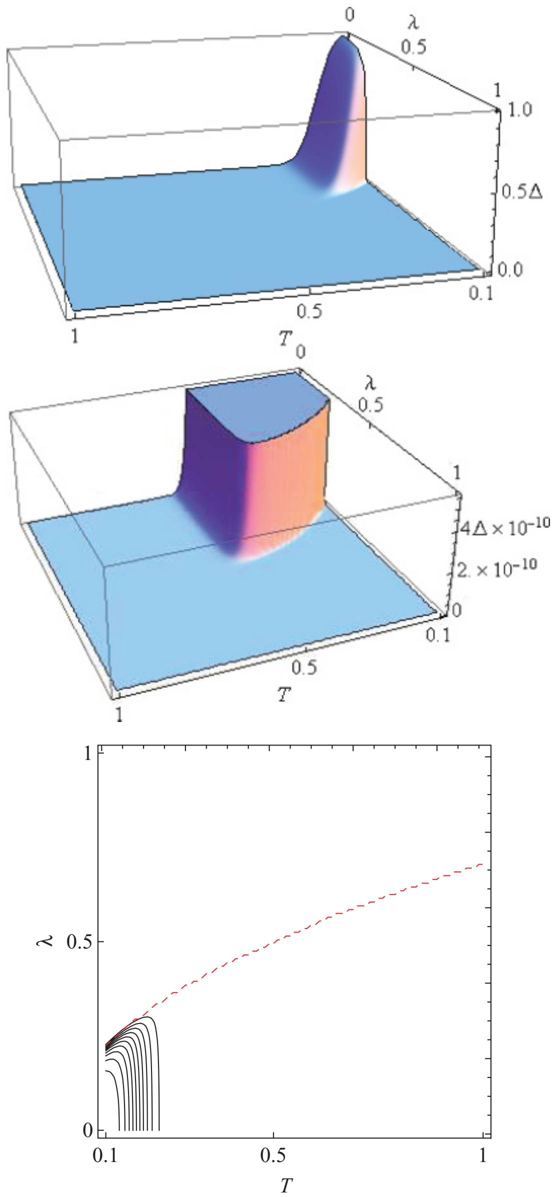

FIG. 2. (Color online) Overlap Δ with fully separable state ρ^s vs. the coupling λ and temperature T; $\omega_0=\omega=k_B=\hbar=1$ and $N=100$ have been chosen for this plot. The two three-dimensional figures are the same plot with different plot ranges for clarity. In the contour plot, the red dashed line corresponds to the critical line $T_c=2\lambda^2/k_B\omega_0$, and because of the rapid decay of Δ, only a finite range of its values is shown for this contour plot.

implies the appearance of entanglement [46]. With respect to the limit large N, these inequalities can be rewritten as

$$\frac{1}{N^2}\left(\langle J_x^2\rangle+\langle J_y^2\rangle+\langle J_z^2\rangle\right)\leqslant\frac{1}{4}, \tag{30a}$$

$$\frac{1}{N^2}(\Delta^2J_x+\Delta^2J_y+\Delta^2J_z)-\frac{1}{2N}\geqslant 0, \tag{30b}$$

$$\frac{\Delta^2J_\gamma}{N}-\frac{1}{N^2}\left(\langle J_\alpha^2\rangle+\langle J_\beta^2\rangle\right)+\frac{1}{2N}\geqslant 0, \tag{30c}$$

$$\frac{1}{N}(\Delta^2J_\alpha+\Delta^2J_\beta)-\frac{\langle J_\gamma^2\rangle}{N^2}-\frac{1}{4}\geqslant 0, \tag{30d}$$

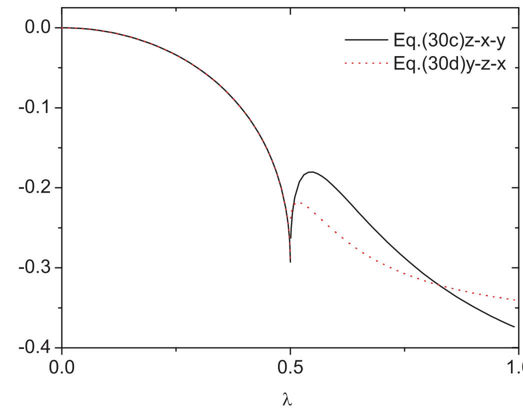

FIG. 3. (Color online) Equations (30a)–(30d) vs. the coupling λ at zero temperature; $\omega_0=\omega=\hbar=1$ and $N=100$ have been chosen for this plot. The labels z-x-y and y-z-x denote the sequence and values of α-β-γ in corresponding inequalities.

in which $1/N^2\langle J_\alpha^2\rangle$ and $1/N^2\Delta^2J_\alpha$ are equivalent for evaluating the average $\langle(J_\alpha/N)^2\rangle$ and $\Delta^2(J_\alpha/N)=\langle(J_\alpha/N)^2\rangle-\langle J_\alpha/N\rangle^2$. For large N, these inequalities have nontrivial results since the average magnetization per particle and its fluctuation still have nonvanishing values. It should be pointed out that Eq. (30a) is obviously satisfied for an arbitrary state, so the following discussion mainly concerns Eqs. (30b)–(30d).

A. Zero temperature

The evaluations of $\langle J_\alpha/N\rangle$ and $\langle(J_\alpha/N)^2\rangle$ can be implemented readily through the Bogoliubov transformation [44]. Our calculations show that Eq. (30b) is always satisfied at both normal and superradiant phases. In Fig. 3, several situations for Eqs. (30a)–(30d) have been plotted with limit $N\to\infty$, and the others can be proved to be bigger than zero. The violation implies that the atoms should be entangled. Moreover, since the pairwise entanglement between atoms is known to be vanishing with increments of λ [40], this entanglement is sure to be multipartite. Furthermore, there is also a sudden increment closed to the critical point, similar to the behavior of Δ shown in Fig. 1. This feature means that there is a sudden reduction of the correlation of atoms, and Δ can also be used to detect the entanglement of atoms in the Dicke model at zero temperature.

B. Finite temperature

At finite temperature, the evaluations of $\langle J_\alpha/N\rangle$ and $\langle(J_\alpha/N)^2\rangle$ can adopt the same trick used in Ref. [43] (also shown in Appendix B). In Fig. 4, Eqs. (30b)–(30d) have been plotted with all possible permutations of x,y,z. It is obvious that all inequalities are satisfied simultaneously, and one can conclude that there is no quantum entanglement of atoms in this case. This result is not surprising since the thermal fluctuation is dominant at finite temperature and is considered to be incoherent. Despite the absence of quantum correlation, the correlation induced by thermal fluctuation predominates, as shown in Fig. 2, by Δ, which means that Δ can also be used to detect the thermal correlation.

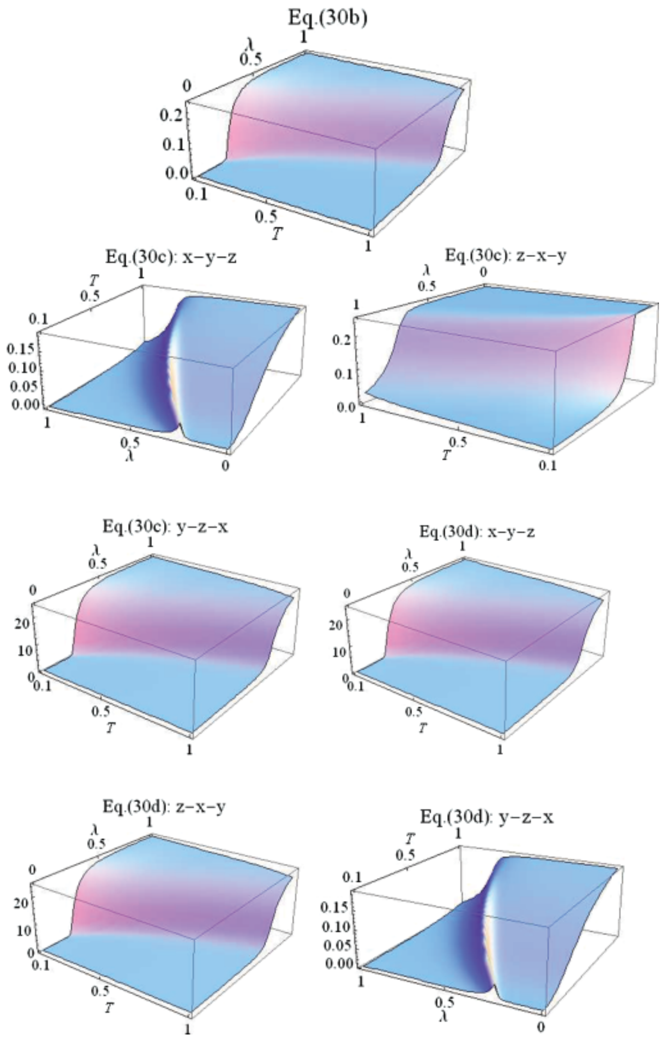

FIG. 4. (Color online) Equations (30a)–(30d) vs. the coupling λ and temperature T; $\omega_0 = \omega = k_B = \hbar = 1$ and $N = 100$ have been chosen for these plots. The labels x-y-z, z-x-y, and y-z-x denote the sequence and values of α-β-γ in corresponding inequalities. The similarity among several plots arises because their corresponding inequalities would become closed under limit $N \to \infty$.

V. CONCLUSIONS AND FURTHER DISCUSSION

In this article, the overlap Δ with a special fully separable state defined in Eq. (10) is introduced to capture the overall correlation in many-body systems, whether quantum or classical. Δ has clear geometric and physical meaning, as shown in Sec. II. With these features, the optimum process in the definition of Δ can be reduced to find the fully separable state ρ^s, defined in Eq. (9), which shares the same physical properties with the state to be measured. Importantly, Δ can be considered as the generalization of the concept of Anderson's orthogonality catastrophe [26,27], which is critical for the understanding of some effects in condensed matter physics. This important connection shows the popularity of Δ for the detection of the global correlation in many-body systems. And as an exemplification, the phase transition in the Dicke model has been discussed by Δ.

As shown in Sec. III, Δ unambiguously depicts the phase transition features and the global correlation in the Dicke model, whether for zero or finite temperatures. At zero temperature, Δ displays the distinct behaviors across the critical point. Furthermore, with ρ^s, Δ predicts the appearance of the multipartite entanglement in the atomic system, as verified in Sec. IV A.

As for finite temperature, Δ can still be used to mark the phase transition in the Dicke model, as shown in Fig. 2. It displays the sudden variance at the critical line decided by the temperature T and the coupling λ. An intricate feature appears when $T \to \infty$. It is believed that all atoms would become independent in this case and could be considered the ideal system [47]. Quantum mechanically, the state in this case can be described by a fully separable state, whereas Δ approaches zero, as shown in Fig. 2, and the nonzero Δ appears only at intermediate temperature, shown in Fig. 2. This phenomenon implies that the correlation in atoms would exist even under high temperature. Moreover, under the J_z representation, the dimension is proportional to the atomic number N, and the value of the overlap shown in Fig. 2 has exceeded greatly the limit by N. Thus this phenomenon cannot be attributed to the mixedness of the state for the atomic system. Unfortunately, we do not know how to understand these two different features.

Although Δ cannot present a complete measurement of multipartite entanglement, its intimate connection to quantum entanglement in some special cases has been shown, such as with the Dicke model at zero temperature described in this article. From this discussion, Δ presents a complete description for the global correlation in many-body systems, whether quantum or classical. Therefore it is not surprising that Δ can be used to identify the quantum entanglement in some special cases. However, it is difficult to describe the general relation between Δ and quantum entanglement in the absence of a unified understanding of multipartite entanglement. This point will be studied in a future article.

APPENDIX A: FIND THE NEAREST ρ^s FOR A DEFINITE ρ

Two arbitrary density matrices ρ_1 and ρ_2 always have the following simultaneous decompositions:

$$\rho_1 = \sum_n p_n^{(1)} |n\rangle_{11}\langle n|, \qquad \rho_2 = \sum_m p_m^{(2)} |m\rangle_{22}\langle m|, \tag{A1}$$

where $p_{n(m)}^{1(2)}$ denotes the probability that the system is in the state $|n(m)\rangle_{1(2)}$. It should be emphasized that it is *unnecessary* for the states labeled by different n or m to be orthogonal with each other. Thus the preceding decompositions can always be realized at the same time. Then the overlap between ρ_1 and ρ_2 reads

$$\mathrm{Tr}\,[\rho_1\rho_2] = \sum_{m,n} p_n^{(1)} p_m^{(2)} |_2\langle m|n\rangle_1|^2. \tag{A2}$$

Obviously, the maximization of overlap is dependent on the inner product $|_2\langle m|n\rangle_1|^2$. It is well known that for two *different* states $|v\rangle$ and $|w\rangle$, their inner product is bounded by the Cachy-Schwartz (CS) inequality, that is,

$$|\langle v|w\rangle|^2 \leqslant \langle v|v\rangle\langle w|w\rangle, \tag{A3}$$

where the equality occurs if and only if the two vectors $|v\rangle$ and $|w\rangle$ in the Hilbert state are linearly related, that is, $|v\rangle = c|w\rangle$ for some scalar c. The important point for this condition is that c is *not* necessarily a constant for which $|v\rangle$ and $|w\rangle$ become physically identical, and the CS inequality has a trivial consequence. Thus

$$\mathrm{Tr}\,[\rho_1\rho_2] \leqslant \sum_{m,n} p_n^{(1)} p_m^{(2)}{}_1\langle n|n\rangle_{12}\langle m|m\rangle_2, \tag{A4}$$

where the equality occurs if and only if arbitrary $|n\rangle_1$ and $|m\rangle_2$ are still linearly related. But in this case, the scalar c has to be dependent on both n and m, that is, $c = c_{mn}$, which means that any $|n\rangle_1$ have to be linearly related to all $|m\rangle_2$. An interesting consequence for this condition is $[\rho_1, \rho_2] = 0$, which means that ρ_1 and ρ_2 share the same set of eigenvectors, and thus they share the same global symmetry and belong to the same space.

This conclusion is not strange if one notes that the overlap between two matrices is mainly determined by the inclusion relation of the spaces decided by the matrices. As an example, let us consider two matrices belonging to two completely different spaces. The overlap must be zero since mathematically, the intersection of the two spaces is null, and there is no crossing items between the two matrices. Comparably, if one space is the subspace or equivalent to the other space, the overlap, then, is nontrivial generally since the two matrices belong to the same space. Hence, in order to find the maximal overlap between two matrices, it is also *necessary* for the two matrices to be in the same space. From a physical standpoint, this means that the two operators are *necessarily* commutative. Furthermore, it is easy to understand why the maximal GE for a pure entangled state always happens for a purely separable state.

As for the determination of ρ^s in Eq. (10), it is required for ρ^s to be commutative to ρ, that is, $[\rho, \rho^s] = 0$, which means that ρ^s shares the same global symmetry with ρ. With this point, one can determine ϱ as Eq. (21). Furthermore, since ρ^s is diagonal under the collective basis $\{|n - \frac{N}{2}\rangle, n = 0, 1, \ldots, N\}$,

$$\mathrm{Tr}\,[\rho\rho^s] = \sum_n \rho_{nn}\rho_{nn}^s \leqslant \sum_n \frac{\rho_{nn}^2 + \left(\rho_{nn}^s\right)^2}{2}, \tag{A5}$$

where ρ_{nn} and ρ_{nn}^s denote the diagonal elements of ρ and ρ^s, respectively. Obviously, the second equality occurs if and only if $\rho_{nn} = \rho_{nn}^s$, which means that $\langle J_z\rangle$ has the same value for both ρ and ρ^s. Then a can be determined in Eq. (21).

As for the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1010\rangle + |0101\rangle), \tag{A6}$$

the preceding discussion is also applicable. We should emphasize our point clearly in this place that $|\psi\rangle$ is not really translational invariance. Actually, when one talks of the translational invariance of a system, this means

$$DHD^\dagger = H, \tag{A7}$$

in which D is the translation operator and H is the Hamiltonian for this system. Hence that one speaks of the translational invariance for a state is meaningless without specifying the Hamiltonian. Our discussion about the Dicke model manifests this point clearly. So the crucial point is to find the Hamiltonian for which $|\psi\rangle$ is one of its eigenvectors. It seems that one can construct the following Hamiltonian:

$$H = \sum_i \sigma_i^z\sigma_{i+1}^z, \tag{A8}$$

for which $ket\psi$ is seemingly one of the degenerate ground states. If the translational invariance is required for this system, one must have the periodic boundary condition $\sigma_{N+1}^z = \sigma_1^z$, where N is the total particle number. However, $|\psi\rangle$ tells us that for one particle, its neighbored particles always have an opposite state, which obviously does not satisfy this periodic boundary condition. So we argue in this place that the translational invariance for $|\psi\rangle$ is only occasional because of its special form.

Instead, $|\psi\rangle$ is the true ground state for the Hamiltonian

$$H = -\sum_i \sigma_i^z\sigma_{i+2}^z, \tag{A9}$$

with the boundary condition $\sigma_{N+2}^z = \sigma_2^z$. It means that the particle always has the same state as its next neighbored particle. Thus this could explain naturally why the maximal overlap with $|\psi\rangle$ happens for the fully separable states $|1010\rangle$ and $|0101\rangle$, which obviously satisfy this boundary condition and also are the ground states for this Hamiltonian.

One can also find a state which seemingly satisfies the requirement of the "translational invariance" defined by $|\psi\rangle$, that is,

$$\rho' = \tfrac{1}{2}(|0101\rangle\langle 0101| + |1010\rangle\langle 1010|), \tag{A10}$$

which obviously maximizes the overlap with $|\psi\rangle$. These features demonstrate again that $|\psi\rangle$ is not truly translationally invariant since ρ' is the incoherent superposition of the two degenerate ground states for Eq. (A9).

APPENDIX B: DERIVATION OF EQ. (28)

Set

$$\begin{aligned} H_0 &= \omega a^\dagger a, \\ H_I &= \omega_0 J_z + \frac{2\lambda}{\sqrt{N}}(a^\dagger + a). \end{aligned} \tag{B1}$$

Under $\beta = \frac{1}{k_B T} \ll 1$, the partition function can be approximated as [43]

$$\begin{aligned} z &= \mathrm{Tr}\,[e^{-\beta(H_0+H_I)}], \\ &= \mathrm{Tr}\,[e^{-\beta H_0/2}e^{-\beta H_I/2}e^{-\beta H_0/2} + O(\beta^3)], \\ &\simeq \mathrm{Tr}\,[e^{-\beta H_0}e^{-\beta H_I}]. \end{aligned} \tag{B2}$$

Given

$$\rho^s = \sum_{n=1}^{N} {}_N C_n a^n (1-a)^{N-n} \left|n - \frac{N}{2}\right\rangle\left\langle n - \frac{N}{2}\right|, \tag{B3}$$

then

$$\begin{aligned} \Delta &= \frac{1}{z}\mathrm{Tr}\,[\rho\rho^s] \\ &= \frac{1}{z}\mathrm{Tr}\left[\sum_{k=1}^{N} {}_N C_n a^n (1-a)^{N-n} \right. \\ &\quad \left. \times \left\langle n - \frac{N}{2}\right| e^{-\beta H_0} e^{-\beta H_I} \left| n - \frac{N}{2}\right\rangle\right], \end{aligned} \tag{B4}$$

for which $[\rho^s, H_0] = 0$ is applied. The tricky thing for the tracing in Eq. (B4) is noting that $|N/2; n-(N/2)\rangle$ denotes the state in which n particles are spun up and the others are spun down. And then

$$\begin{aligned}
&\left\langle n-\frac{N}{2}\right| e^{-\beta H_0} e^{-\beta H_I} \left| n-\frac{N}{2}\right\rangle \\
&= e^{-\beta\omega a^\dagger a}\left\langle n-\frac{N}{2}\right| \prod_{i=1}^{N} e^{-\beta[\frac{\omega_0}{2}\sigma_i^z + \frac{\lambda}{\sqrt{N}}(a^\dagger + a)\sigma_i^x]} \left| n-\frac{N}{2}\right\rangle \\
&= e^{-\beta\omega a^\dagger a}\left\langle n-\frac{N}{2}\right| \otimes_{i=1}^{N} \sum_{k=0}^{\infty}\frac{\beta^{2k}}{(2k)!}\left[\frac{\omega_0^2}{4} + \frac{\lambda^2}{N}(a^\dagger + a)^2\right]^k \left\{1 - \frac{\beta}{2k+1}\left[\frac{\omega_0}{2}\sigma_i^z + \frac{\lambda}{\sqrt{N}}(a^\dagger + a)\sigma_i^x\right]\right\} \left| n-\frac{N}{2}\right\rangle \\
&= e^{-\beta\omega a^\dagger a}\left\{\sum_{k=0}^{\infty}\frac{\beta^{2k}}{(2k)!}\left[\frac{\omega_0^2}{4} + \frac{\lambda^2}{N}(a^\dagger + a)^2\right]^k \left(1 - \frac{\beta}{2k+1}\frac{\omega_0}{2}\right)\right\}^n \left\{\sum_{k=0}^{\infty}\frac{\beta^{2k}}{(2k)!}\left[\frac{\omega_0^2}{4} + \frac{\lambda^2}{N}(a^\dagger + a)^2\right]^k \left(1 + \frac{\beta}{2k+1}\frac{\omega_0}{2}\right)\right\}^{N-n}.
\end{aligned} \tag{B5}$$

Thus

$$\Delta = \frac{1}{z}\mathrm{Tr}\left[e^{-\beta\omega a^\dagger a}\left\{\sum_{k=0}^{\infty}\frac{\beta^{2k}}{(2k)!}\left[\frac{\omega_0^2}{4} + \frac{\lambda^2}{N}(a^\dagger + a)^2\right]^k \times\left[1 + \frac{\beta}{2k+1}\frac{\omega_0}{2}(1-2a)\right]\right\}^N\right]. \tag{B6}$$

Expanding the item in the corbeil bracket,

$$\begin{aligned}
\Rightarrow &\sum_{k_1=0;k_2=0\cdots k_N=0}^{\infty}\left(\prod_{i=1}^{N}\frac{\beta^{2k_i}}{(2k_i)!}\left[1 + \frac{\beta}{2k+1}\frac{\omega_0}{2}(1-2a)\right]\right) \\
&\times \sum_{q=0}^{K=k_1+k_2+\cdots+k_N}\frac{K!}{q!(K-q)!}\left(\frac{\omega_0}{2}\right)^{2(K-q)} \\
&\times \left(\frac{\lambda}{\sqrt{N}}\right)^{2q}(a^\dagger + a)^{2q}.
\end{aligned} \tag{B7}$$

Define $a^\dagger a|m\rangle = m|m\rangle$, and then

$$\begin{aligned}
\Delta = \frac{1}{z} &\sum_{k_1=0;k_2=0\cdots k_N=0}^{\infty}\left(\prod_{i=1}^{N}\frac{\beta^{2k_i}}{(2k_i)!}\left[1 + \frac{\beta}{2k+1}\frac{\omega_0}{2}(1-2a)\right]\right) \\
&\times \sum_{q=0}^{K=k_1+k_2+\cdots+k_N}\frac{K!}{q!(K-q)!}\left(\frac{\omega_0}{2}\right)^{2(K-q)}\left(\frac{\lambda}{\sqrt{N}}\right)^{2q} \\
&\times \frac{d^{2q}}{d\eta^{2q}} e^{\frac{\eta^2}{2}}\sum_{m=0}^{\infty} e^{-m\beta\omega}L_m(-\eta^2)\Big|_{\eta=0},
\end{aligned} \tag{B8}$$

where $L_m(x)$ is the mth Laguerre polynomial and the following relation is used:

$$\begin{aligned}
\langle m|(a^\dagger + a)^{2q}|m\rangle &= \frac{d^{2q}}{d\eta^{2q}}\langle m|e^{\eta(a^\dagger + a)}|m\rangle\Big|_{\eta=0} \\
&= \frac{d^{2q}}{d\eta^{2q}}\left[e^{\frac{\eta^2}{2}}L_m(-\eta^2)\right]\Big|_{\eta=0}.
\end{aligned} \tag{B9}$$

Apply the relation

$$\sum_{m=0}^{\infty} e^{-m\beta\omega}L_m(-\eta^2) = \frac{1}{1-e^{-\beta\omega}}\exp\left[\eta^2\frac{1}{e^{\beta\omega}-1}\right]; \tag{B10}$$

then

$$\begin{aligned}
\Delta = \frac{1}{z}\frac{1}{1-e^{-\beta\omega}} &\sum_{k_1=0;k_2=0\cdots k_N=0}^{\infty} \\
&\times \left(\prod_{i=1}^{N}\frac{\beta^{2k_i}}{(2k_i)!}\left[1 + \frac{\beta}{2k+1}\frac{\omega_0}{2}(1-2a)\right]\right) \\
&\times \sum_{q=0}^{K=k_1+k_2+\cdots+k_N}\frac{K!}{q!(K-q)!}\left(\frac{\omega_0}{2}\right)^{2(K-q)}\left(\frac{\lambda}{\sqrt{N}}\right)^{2q} \\
&\times \frac{d^{2q}}{d\eta^{2q}} e^{\frac{\eta^2}{2}\coth\frac{\beta\omega}{2}}\Big|_{\eta=0}.
\end{aligned} \tag{B11}$$

With the relations

$$\begin{aligned}
\frac{d^{2q}}{d\eta^{2q}} e^{\frac{\eta^2}{2}\coth\frac{\beta\omega}{2}}\Big|_{\eta=0} &= (2q-1)!!\coth^q\frac{\beta\omega}{2}, \\
(2q-1)!! &= \sqrt{\frac{A}{\pi}}2^p A^p \int_{-\infty}^{\infty} dx e^{-Ax^2}x^{2p},
\end{aligned} \tag{B12}$$

set $A = 1/2$:

$$\begin{aligned}
\Delta = \frac{1}{z}\frac{\sqrt{1/2\pi}}{1-e^{-\beta\omega}}\int_{-\infty}^{\infty} dx e^{-x^2/2} &\sum_{k_1=0;k_2=0\cdots k_N=0}^{\infty} \\
&\times \left(\prod_{i=1}^{N}\frac{\beta^{2k_i}}{(2k_i)!}\left[1 + \frac{\beta}{2k+1}\frac{\omega_0}{2}(1-2a)\right]\right) \\
&\times \sum_{q=0}^{K=k_1+k_2+\cdots+k_N}\frac{K!}{q!(K-q)!}\left(\frac{\omega_0}{2}\right)^{2(K-q)} \\
&\times \left(\frac{x^2\lambda^2}{N}\coth\frac{\beta\omega}{2}\right)^q.
\end{aligned} \tag{B13}$$

Finally, inverse the procedure from Eqs. (B6) and (B7) for the sum item and apply relations $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$; one then obtains Eq. (28).

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