PHYSICAL REVIEW A **81**, 042110 (2010)

All-versus-nothing proofs with n qubits distributed between m parties

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(Received 30 September 2009; revised manuscript received 28 February 2010; published 27 April 2010)

All-versus-nothing (AVN) proofs show the conflict between Einstein, Podolsky, and Rosen's elements of reality and the perfect correlations of some quantum states. Given an n-qubit state distributed between m parties, we provide a method with which to decide whether this distribution allows an m-partite AVN proof specific for this state using only single-qubit measurements. We apply this method to some recently obtained n-qubit m-particle states. In addition, we provide all inequivalent AVN proofs with less than nine qubits and a minimum number of parties.

DOI: 10.1103/PhysRevA.81.042110 PACS number(s): 03.65.Ud, 03.67.Mn, 42.50.Xa

I. INTRODUCTION

Einstein, Podolsky, and Rosen (EPR) showed that quantum mechanics is incomplete in the sense that not every element of reality has a counterpart inside the theory [1]. EPR proposed the following criterion to identify an element of reality: "If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity" [1]. In practice, nondisturbance can be guaranteed when the measurements are performed on distant systems. Predictions with certainty are possible for states having perfect correlations. A quantum state ρ has p perfect correlations when there are p different observables O_i such that $\langle O_i \rangle_\rho = 1$.

Thirty years after EPR's paper, Bell proved that there is a irresoluble conflict between EPR's elements of reality and quantum mechanics [2]. All-versus-nothing (AVN) proofs are the most direct way to reveal this conflict. An AVN proof is based on a set of s perfect correlations of a specific quantum state. The name "all-versus-nothing" [3] reflects one particular feature of these proofs: If one assumes EPR elements of reality, then $s - q$ of these perfect correlations lead to a conclusion that is the opposite of the one obtained from a subset of the other q perfect correlations. If all s correlations are essential to obtain a contradiction (i.e., if the contradiction vanishes when we remove one of them), then the AVN proof is said to be critical.

The first AVN proof was obtained by Heywood and Redhead [4]. However, the most famous AVN proof is Greenberger, Horne, and Zeilinger's (GHZ) [5–7]. The first bipartite AVN proof with qubits is in Refs. [8,9]. The first bipartite AVN proof with qubits and using only single-qubit measurements is in Refs. [10,11]. The interest of the case in which the parties are restricted to perform single-qubit measurements is motivated by the practical difficulty of making general N-qubit measurements ($N \geqslant 2$) when the qubits are encoded in different degrees of freedom of the same particle.

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Recently, several n-qubit m-particle states ($n > m$) having perfect correlations have been experimentally prepared, for instance, 4-qubit two-photon [12], 6-qubit two-photon [13,14], 6-qubit four-photon [15], 8-qubit four-photon [16], and 10-qubit five-photon graph states [16].

For these n-qubit m-particle states, a natural problem is the following: Consider m distant parties; party i can perform single-qubit measurements on particle i, and particle i contains $n_i \geqslant 1$ qubits ($\sum_{i=1}^{m} n_i = n$); which n-qubit m-particle states allow m-partite AVN proofs? This problem has been solved for the case of $m = 2$ particles or parties [17]. In this article, we address the problem for an arbitrary number m of particles or parties.

The article is organized as follows: In Sec. II, we show that there is an equivalence between pure states allowing AVN proofs and graph states. This will simplify the task of finding all inequivalent n-qubit m-partite AVN proofs.

An m-partite AVN proof is specific for an n-qubit m-particle graph state when there is no graph state with fewer qubits satisfying the same correlations. In Sec. III, we discuss the requirements of an m-partite AVN proof to be specific for an n-qubit m-particle graph state and describe a method to decide whether a given n-qubit m-particle graph state allows a specific m-partite AVN proof. We apply this method to decide whether some n-qubit m-particle graph states recently prepared in the laboratory allow m-partite AVN proofs. As supplementary material [18], we provide a computer program to decide whether a given n-qubit m-particle graph state allows a specific m-partite AVN proof.

In Sec. IV we solve the following problem: Given an n-qubit graph state, what is the minimum number m of parties that allows a specific m-partite AVN proof. As supplementary material [18], we provide a computer program to obtain, given an n-qubit graph state, all distributions between m parties and all distributions between a minimum number of parties which allow AVN proofs.

The solution of the previous problem allows us to obtain all inequivalent distributions allowing AVN proofs since any distribution obtained from one allowing a specific AVN proof by giving qubits that originally belong to the same party to new parties will also allow an AVN proof. As supplementary material, we provide all inequivalent distributions between a minimum number m of parties allowing specific m-partite AVN proofs for all n-qubit graph states of $n \leqslant 8$ qubits [19].

II. AVN PROOFS AND GRAPH STATES

A. AVN proofs and stabilizer states

An AVN proof requires an n-qubit quantum state distributed between m parties. This state has a set of perfect correlations between the results of single-qubit measurements. These correlations must satisfy two requirements. First, they must allow us to define m-partite EPR's elements of reality. This means that every single-qubit observable involved in the AVN proof must satisfy EPR's criterion of elements of reality (i.e., its value can be predicted with certainty using only the results of single-qubit measurements on distant particles). Second, they must lead to a contradiction when EPR's criterion of elements of reality is assumed. Therefore the conclusion of an AVN proof is that if the quantum predictions are correct, observables which satisfy EPR's condition cannot have predefined results since it is impossible to assign them values which simultaneously satisfy the perfect correlations predicted by quantum mechanics.

Perfect correlations are necessary to establish elements of reality and to prove that they are incompatible with quantum mechanics. Therefore the states which allow AVN proofs must be simultaneous eigenstates of a sufficient number of commuting n-fold tensor products of single-qubit operators. Indeed, the following observations lead us to the conclusion that without loss of generality, we can restrict our attention to a particular family of states.

Two different single-qubit operators A and B on the same qubit cannot commute. A necessary condition to make n-fold tensor products be commuting operators is to choose A and B to be anticommuting operators. Therefore, in an AVN proof, all single-qubit operators corresponding to the same qubit must be anticommuting operators. The maximum number of anticommuting single-qubit operators is three. Therefore, without loss of generality, we can restrict our attention to a specific set of three single-qubit anticommuting operators on each qubit, for example, the Pauli matrices $X = \sigma_x$, $Y = \sigma_y$, and $Z = \sigma_z$. This leads us to the concept of stabilizer state. An n-qubit stabilizer state is the simultaneous eigenstate with eigenvalue 1 of a set of n independent commuting elements of the Pauli group (i.e., the group, under matrix multiplication, of all n-fold tensor products of X, Y, Z and the identity $\mathbb{1}$). The n independent elements are called stabilizer generators and generate a maximally Abelian subgroup, the stabilizer group of the state [20]. The 2^n elements of the stabilizer group are the stabilizing operators and provide all the perfect correlations of the stabilizer state.

A further simplification is possible since any stabilizer state is local Clifford equivalent (i.e., equivalent under the local unitary operations that map the Pauli group to itself under conjugation) to a graph state [21]. Therefore the problem of which n-qubit pure states and distributions of qubits between the parties allow m-partite AVN proofs is reduced to the problem of which n-qubit graph states and distributions allow m-partite AVN proofs.

B. Graph states

A graph state [22] is a stabilizer state whose generators can be written with the help of a graph. $|G\rangle$ is the n-qubit state associated with the graph G, which gives a recipe both for preparing $|G\rangle$ and for obtaining n stabilizer generators that uniquely determine $|G\rangle$. On one hand, G is a set of n vertices (each representing a qubit) connected by edges (each representing an Ising interaction between the connected qubits). On the other hand, the stabilizer generator g_i is obtained by looking at the vertex i of G and the set $N(i)$ of vertices which are connected to i and is defined by

$$g_i = X_i \otimes_{j \in N(i)} Z_j, \tag{1}$$

where X_i, Y_i, and Z_i denote the Pauli matrices acting on the ith qubit. $|G\rangle$ is the only n-qubit state that fulfills

$$g_i|G\rangle = |G\rangle \quad \text{for} \quad i = 1, \ldots, n. \tag{2}$$

Therefore the stabilizer group is

$$S(|G\rangle) = \{s_j, j = 1, \ldots, 2^n\}, \quad s_j = \prod_{i \in I_j(G)} g_i, \tag{3}$$

where $I_j(G)$ denotes a subset of $\{g_i\}_{i=1}^N$. The stabilizing operators provide all the perfect correlations of $|G\rangle$:

$$\langle G|s_j|G\rangle = 1. \tag{4}$$

Graph states associated with connected graphs have been exhaustively classified. There is only 1 two-qubit graph state (equivalent to a Bell state), only 1 three-qubit graph state (equivalent to a GHZ state), and 2 four-qubit graph states (equivalent to a GHZ and a cluster state), 4 five-qubit graph states, 11 six-qubit graph states, 26 seven-qubit graph states [22], and 101 eight-qubit graph states [23].

III. n-QUBIT m-PARTITE AVN PROOFS

A. Specific m-partite AVN proofs

The perfect correlations of any graph state associated with a connected graph of three or more vertices lead to contradictions with the concept of elements of reality when each qubit is distributed to a different party [17,24–26]. However, the first problem consists of finding whether these contradictions are specific to a given distribution of a graph state or, on the contrary, they can be obtained with a graph state of fewer qubits.

For example, take the four-party AVN proof based on the following four perfect correlations of the distribution of the four-qubit fully connected graph state $|\text{FC}_4\rangle$ (a four-qubit GHZ state) in which each qubit belongs to a different party:

$$X_1 Z_2 Z_3 Z_4 = 1, \tag{5a}$$

$$Z_1 X_2 Z_3 Z_4 = 1, \tag{5b}$$

$$Z_1 Z_2 X_3 Z_4 = 1, \tag{5c}$$

$$-X_1 X_2 X_3 Z_4 = 1. \tag{5d}$$

This is an example of an AVN proof which is nonspecific for the state $|\text{FC}_4\rangle$, the reason being that neither the contradiction nor the definition of the elements of reality involved in this contradiction requires any choice from the party which has the fourth qubit. This party only has to measure Z_4 and broadcast the result. The only role of the result of Z_4 is to guarantee that X_1, Z_1, X_2, Z_2, X_3, and Z_3 are elements of reality in a four-party scenario. However, the contradiction occurs for any result of Z_4. It occurs because the following equations cannot be simultaneously satisfied:

$$X_1Z_2Z_3 = Z_1X_2Z_3 = Z_1Z_2X_3 = -X_1X_2X_3. \quad (6)$$

The particular value of Z_4 is irrelevant. The same contradiction can be obtained using the perfect correlations of a three-qubit fully connected graph state $|\text{FC}_3\rangle$ (a three-qubit GHZ state) distributed between three parties.

The next example illustrates that whether an AVN proof is specific can depend on the way in which the qubits are distributed between the parties. Consider the AVN proof based on the following four correlations of the four-qubit linear cluster state $|\text{LC}_4\rangle$ associated with the graph where qubit 1 is connected to qubit 2, which is connected to qubit 3, which is connected to qubit 4:

$$Y_1Y_2Z_3 = 1, \quad (7a)$$

$$Z_1X_2Z_3 = 1, \quad (7b)$$

$$Z_1Y_2Y_3Z_4 = 1, \quad (7c)$$

$$-Y_1X_2Y_3Z_4 = 1. \quad (7d)$$

If the qubits are distributed so that each qubit goes to a different party, then the AVN proof is not specific since the party who has the fourth qubit does not need to make any choice, neither for the contradiction nor for the definition of the elements of reality. The contradiction

$$Y_1Y_2Z_3 = Z_1X_2Z_3 = Z_1Y_2Y_3 = -Y_1X_2Y_3 \quad (8)$$

can be obtained from the perfect correlations of a three-qubit linear cluster state $|\text{LC}_3\rangle$ associated with the graph where qubit 1 is connected to qubit 2, which is connected to qubit 3.

However, if qubits 1 and 4 belong to Alice, and qubits 2 and 3 belong to Bob, then the only way to guarantee that, for example, X_2 is an element of reality in this scenario (i.e., that its result can be predicted using only the results of measurements on Alice's side) is by using the following perfect correlation of the $|\text{LC}_4\rangle$:

$$Z_1X_2X_4 = 1. \quad (9)$$

Therefore the party who has qubit 4 must choose between at least two measurements. To sum up, an AVN proof is specific for a given distribution of a graph state when at least two observables of all the qubits are involved.

Since the additional correlations needed to define the elements of reality can (together with those already used for the contradiction) involve additional contradictions, it is appropriate that the observables needed to guarantee that other observables are elements of reality (like X_4 and Z_4 in the previous example) are themselves elements of reality. Therefore hereinafter we will focus on AVN proofs in which at least two of the observables of all the qubits are elements of reality. It can be easily seen that when two Pauli observables, for example, X_i and Y_i, are elements of reality, then the third Pauli observable, Z_i, is also an element of reality. Therefore we shall focus only on those graph states and distributions in which the three Pauli observables of each and every one of the qubits are elements of reality.

B. When does a distribution allow a specific AVN proof?

The next problem is, given a distribution of an n-qubit graph state between m parties, how to decide whether it is one in which all single-qubit Pauli observables are elements of reality. For that purpose, it is useful to note that the 2^n perfect correlations (i.e., stabilizing operators) of an n-qubit graph state can be classified in four classes:

1. There are 2^{n-2} stabilizing operators (i.e., a quarter of the stabilizing operators of the graph state) that allow us to predict X_i from the results of measurements on other qubits: those that are products of the stabilizer generator g_i [defined in Eq. (1)], an even number (hereinafter "even" includes zero) of g_j with $j \in N(i)$, and an arbitrary number (hereinafter "arbitrary number" includes zero) of g_k with $k \neq i$ and $k \notin N(i)$.
2. There are 2^{n-2} stabilizing operators that allow us to predict Y_i from the results of measurements on other qubits: those that are products of g_i, an odd number of g_j with $j \in N(i)$, and an arbitrary number of g_k with $k \neq i$ and $k \notin N(i)$.
3. There are 2^{n-2} stabilizing operators that allow us to predict Z_i from the results of measurements on other qubits: those that are products of an odd number of g_j with $j \in N(i)$ and an arbitrary number of g_k with $k \neq i$ and $k \notin N(i)$.
4. There are 2^{n-2} stabilizing operators that contain $\mathbb{1}_i$: those that are products of an even number of g_j with $j \in N(i)$ and an arbitrary number of g_k with $k \neq i$ and $k \notin N(i)$.

Each particle can carry more than one qubit. It is therefore convenient to denote as $P(i)$ the set of qubits which are in the same particle as qubit i. The previous classification of the stabilizing operators is useful in the following sense: Given the distribution of an n-qubit graph state between m parties, X_i is an element of reality if and only if there exists a stabilizing operator of the graph state which satisfies the following two requirements: (1) It does not contain g_j for all $j \in P(i)$ but contains an even number of g_k with $k \in N(j)$ and (2) it contains g_i and an even number of g_l with $l \in N(i)$. For instance, consider the four-qubit linear cluster state $|\text{LC}_4\rangle$ associated with the graph where qubit 1 is connected to qubit 2, which is connected to qubit 3, which is connected to qubit 4, distributed such that Alice has qubits 1 and 4 and Bob has qubits 2 and 3. The question is, is X_1 an element of reality? This is equivalent to the question, is there a stabilizing operator such that it does not contain g_4 [since $P(1) = \{4\}$] but contains an even number (necessarily zero) of g_3 [since $N(4) = \{3\}$] and g_1 and an even number (necessarily zero) of g_2 [since $N(1) = \{2\}$]? The answer is yes; the only stabilizing operator with these properties is $g_1 = X_1Z_2$.

Similarly, Y_i is an element of reality if and only if there is a stabilizing operator satisfying (1) and the following condition: (3) It contains g_i and an odd number of g_l with $l \in N(i)$.

Finally, Z_i is an element of reality if and only if there is a stabilizing operator satisfying (1) and the following condition: (4) It does not contain g_i but contains an odd number of g_l with $l \in N(i)$.

To decide whether a specific distribution allows a specific AVN proof, we first focus on qubit i and test whether X_i and Y_i are elements of reality. If either is not an element of reality, then the distribution does not allow a specific AVN proof. If both are elements of reality, then we test whether X_j and Y_j of qubit j are elements of reality, and so on for all the qubits.

If all X_i and Y_i are elements of reality, then the distribution allows a specific AVN proof.

Indeed, there are simple cases where it can easily be seen that a distribution does not allow an AVN proof. For example, if more than $n/2$ qubits are carried by the same particle, for qubits of the particle with more than $n/2$ qubits, either requirement (1) is incompatible with (2), or (1) is incompatible with (3) and (4). An alternative proof will be provided in Sec. IV. If there is a qubit i such that $N(i) \in P(i)$ (i.e., if in the graph representing the state, qubit i is connected only to qubits of the same particle), requirement (1) is incompatible with requirements (3) and (4). As supplementary material [18], we provide a computer program to decide whether a given n-qubit m-particle graph state allows a specific m-partite AVN proof.

C. Examples

As an example of the application of these rules, it is interesting to discuss whether some recently prepared 6-qubit two- and four-particle states allow specific AVN proofs, assuming the natural scenario in which each party has one particle.

Figure 1 contains several possible distributions of a six-qubit linear cluster state $|\mathrm{LC}_6\rangle$. Figure 1(a) represents the four-photon $|\mathrm{LC}_6\rangle$ prepared in Ref. [15]. This distribution does not allow a specific AVN proof since qubit 1 is connected only to qubit 2 and qubit 6 is connected only to qubit 5.

Figure 1(b) represents the two-photon $|\mathrm{LC}_6\rangle$ prepared in Ref. [14]. This distribution satisfies all the requirements and thus allows a specific AVN proof. Indeed, Fig. 1(b) represents the only bipartite distribution of the six-qubit linear cluster state which allows a specific AVN proof [17]. Some distributions of $|\mathrm{LC}_6\rangle$ in four particles allowing AVN proofs can be trivially obtained from Fig. 1(b) by splitting qubits that belong to the same particle into several particles. For instance, a distribution allowing a specific AVN proof is illustrated in Fig. 1(c). It can be easily seen that there is no distribution in four particles which allows a specific AVN proof which cannot be obtained from the distribution in Fig. 1(b).

Figure 2 contains several possible distributions of a six-qubit Y-graph state $|Y_6\rangle$. Figure 2(a) represents the four-photon $|Y_6\rangle$ prepared in Ref. [15]. This distribution does not allow a specific AVN proof since qubit 1 is connected only to qubit 2 and qubit 5 is connected only to qubit 4. Figures 2(b)–2(d) represent distributions of $|Y_6\rangle$ between four particles allowing specific AVN proofs.

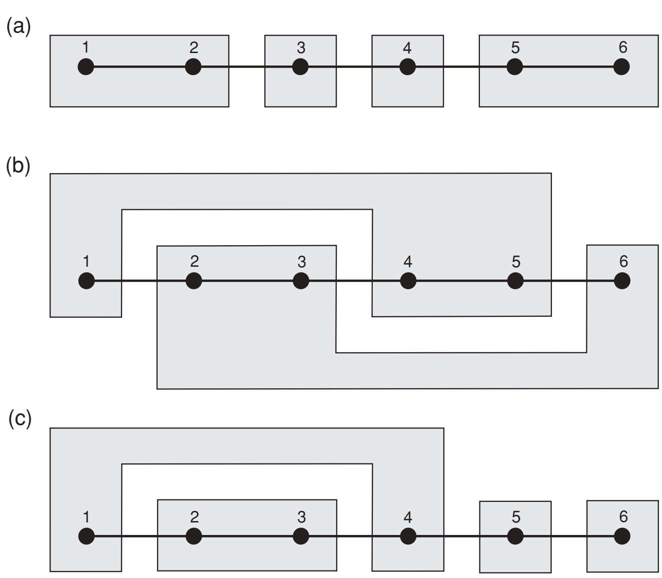

FIG. 1. Different distributions of a six-qubit linear cluster state $|\mathrm{LC}_6\rangle$ between two and four particles. Each gray area represents a particle. (a) Corresponds to the four-photon state prepared in Ref. [15]. (b) Corresponds to the two-photon state prepared in Ref. [14]. In (a), not all single-qubit Pauli observables are EPR elements of reality, and therefore no AVN proof is possible. (b) and (c) Allow AVN proofs.

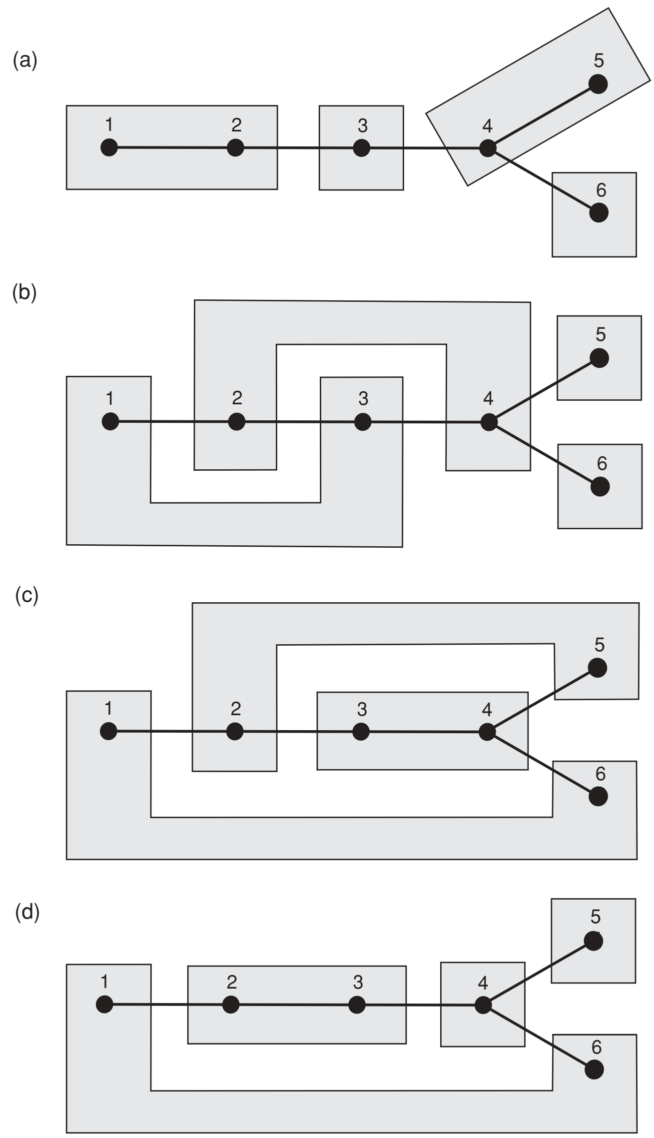

FIG. 2. Different distributions of the six-qubit Y-graph state between four particles. (a) Corresponds to the state prepared in Ref. [15] and does not allow a specific AVN proof. (b)–(d) Allow specific AVN proofs.

IV. AVN PROOFS WITH A MINIMUM NUMBER m OF PARTIES

A. Possible distributions between a minimum number of parties

In the previous section, we have seen that $|Y_6\rangle$ admits specific AVN proofs when their qubits are suitably distributed between four particles. The question is whether $|Y_6\rangle$ admits specific AVN proofs when it is distributed between three particles or less, or more generally speaking, the question is, given an n-qubit graph state, what is the minimum number of parties m which allows m-partite AVN proofs specific for this state?

The following definition will be useful for solving this problem. Let us define the reduced stabilizer of particle A's qubits as the one obtained from the stabilizer of the original state by replacing the observables on all other particles' qubits with identity matrices.

Lemma: A distribution of $n = n_{\max} + n_B + \cdots + n_m$ qubits between m parties such that $n_{\max} \geqslant n_B \geqslant \cdots \geqslant n_m$ allows m-partite elements of reality if and only if $n_{\max} \leqslant n_B + \cdots + n_m$.

Proof: Suppose that particle m_i carries qubits $1, \ldots, n_{\max}$, where $n_{\max}$ is the maximum number of qubits carried by any particle, and that particle m_j carries qubits $n_{\max} + 1, \ldots, n_{\max} + n_j$. If X_1, Y_1, Z_1, $X_2, \ldots, Z_{n_{\max}}$ are elements of reality, then the reduced stabilizer of m_i's qubits must contain

$$X_1 \otimes \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_{n_{\max}}, \tag{10a}$$
$$Y_1 \otimes \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_{n_{\max}}, \tag{10b}$$
$$Z_1 \otimes \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_{n_{\max}}, \tag{10c}$$
$$\mathbb{1}_1 \otimes X_2 \otimes \ldots \otimes \mathbb{1}_{n_{\max}}, \ldots, \tag{10d}$$
$$\mathbb{1}_1 \otimes \mathbb{1}_2 \otimes \ldots \otimes Z_{n_{\max}}. \tag{10e}$$

Moreover, the reduced stabilizer of m_i's qubits must contain all possible products of Eqs. (10a)–(10e), that is, all possible variations with repetition of the four elements $\mathbb{1}$, X, Y, and Z, choosing n_i, which are $4^{n_{\max}} = 2^{2n_{\max}}$. A similar reasoning applies to the three Pauli matrices of each and every one of m_j's qubits. Therefore the reduced stabilizer of m_j's qubits must also contain all possible products of

$$X_{n_{\max}+1} \otimes \mathbb{1}_{n_{\max}+2} \otimes \cdots \otimes \mathbb{1}_{n_{\max}+n_j}, \ldots, \tag{11a}$$
$$\mathbb{1}_{n_{\max}+1} \otimes \mathbb{1}_{n_{\max}+2} \otimes \cdots \otimes Z_{n_i+n_j}, \tag{11b}$$

which are $4^{n_j} = 2^{2n_j}$. However, the reduced stabilizer of the sum of the parties m_i and m_j has only $2^{n_{\max}+n_j}$ terms; therefore the only possibility is that $n_{\max} = n_j$. ■

Given an n-qubit graph state, $n_{\max}$ restricts the possible minimum numbers of particles and the possible numbers of qubits per particle. Given n, Table I presents the possible minimum numbers of particles and the corresponding possible distributions. Other possible distributions are already contained between those in Table I, but in those cases, the number of particles is not the minimum.

A corollary of the lemma is that there are no specific AVN proofs in which one particle has more than $n/2$ qubits (this result was used in Sec. III).

TABLE I. Possible distributions of an n-qubit graph state between a minimum number m of particles. For instance, (2,2,1) denotes a distribution of $n = 5$ qubits between $m = 3$ particles such that particles 1 and 2 have two qubits each and particle 3 has one qubit.

n	m	Distribution
2	2	(1,1)
3	3	(1,1,1)
4	2	(2,2)
	4	(1,1,1,1)
5	3	(2,2,1)
	5	(1,1,1,1,1)
6	2	(3,3)
	3	(2,2,2)
	4	(2,2,1,1)
	6	(1,1,1,1,1,1)
7	3	(3,3,1), (3,2,2)
	4	(2,2,2,1)
	5	(2,2,1,1,1)
	7	(1,1,1,1,1,1,1)
8	2	(4,4)
	3	(3,3,2)
	4	(3,3,1,1), (3,2,2,1), (2,2,2,2)
	5	(2,2,2,1,1)
	6	(2,2,1,1,1,1)
	8	(1,1,1,1,1,1,1,1)

B. AVN proofs with a minimum number of parties for any graph state

Equipped with these tools, we can obtain all possible distributions with a minimum number of particles allowing specific AVN proofs for any graph state. We have obtained all which are inequivalent under single-qubit unitary operations for all graph states up to $n = 8$ qubits. For this purpose, we used the classification of graph states up to $n = 7$ qubits proposed in Ref. [22] and the classification of eight-qubit graph states proposed in Ref. [23]. Given an n-qubit graph state, to obtain all the distributions between a minimum number of parties allowing specific AVN proofs, we can use Table I in the following way. Suppose that $n = 6$. We first test whether AVN proofs are possible for the simplest distributions permitted by Table I, that is, $m = 2$ parties with three qubits each. If no AVN proof is possible, then we test whether there are AVN proofs for the next possible distributions permitted by Table I, that is, $m = 3$ parties with two qubits each, and so on.

Applying this method, we have obtained all inequivalent distributions between a minimum number of particles for all graph states with up to eight qubits. In the supplementary material [19], we show all distributions between a minimum number of particles for the 19 classes of graph states with up to six qubits, the 26 classes of graph states with seven qubits, and the 101 classes of graph states with eight qubits. In addition, we provide as supplementary material [18] a computer program to obtain, given an n-qubit graph state, all distributions between m parties and all distributions between a minimum number of parties which allow AVN proofs.

V. CONCLUSIONS

We have developed tools with which to decide whether a distribution of n qubits between m parties allows an m-partite AVN proof specific for this distribution (i.e., which cannot be obtained using a state with fewer qubits). As a result, we have obtained all inequivalent m-partite AVN proofs using n-qubit m-particle quantum states with $n < 9$ qubits and a minimum number m of parties. This enables us to obtain all inequivalent m-partite AVN proofs using n-qubit m-particle quantum states with $n < 9$ qubits with an arbitrary number of parties.

The motivation of this work was to answer some natural questions raised by recent experimental developments allowing the preparation in the laboratory of graph states of several particles, each carrying several qubits. The results presented in this article provide tools to help experimentalists to design tests of new AVN proofs and new Bell inequalities based on these AVN proofs [7,10,27], similar to those reported in Refs. [12,14] for specific states but exploiting the possibility of experimentally preparing new classes of graph states.

ACKNOWLEDGMENTS

The authors thank W.-B. Gao, O. Gühne, and A. J. López Tarrida for their useful comments and M. Hein, J. Eisert, and H. J. Briegel for their permission for reproducing two figures of Ref. [20] in the supplementary material to this article. The authors acknowledge support from Project Nos. P06-FQM-02243 and FIS2008-05596.

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